Comparing Shapes for Higher Structures

Brandon Shapiro

Cornell University

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Outline

- What are cell structures?
- How can they describe higher categories?
Simplicial Sets

The simplex category $\Delta$ is the category of finite nonempty ordinals and order preserving functions:

$\Delta := \{0\} \to \{0, 1\} \to \{0, 1, 2\} \to \ldots$

$d_0$ skips $i$, $s_i$ repeats $i$, and these generate all maps in $\Delta$

A simplicial set $X$ is a functor $\Delta^{op} \to \text{Set}$:

$X = X_0 \to X_1 \to X_2 \to \ldots$

Shapiro Cell Shapes
The simplex category $\Delta$ is the category of finite nonempty ordinals and order preserving functions:

A simplicial set $X$ is a functor $\Delta^{\text{op}} \to \text{Set}$: $X = X_0 \to X_1 \to X_2 \to \cdots$.
The simplex category $\Delta$ is the category of finite nonempty ordinals and order preserving functions:

$$\Delta := \{0\} \xleftarrow{s^0} \{0, 1\} \xrightarrow{d^0} \{0, 1, 2\} \xleftarrow{s^1} \{0, 1, 2\} \xrightarrow{d^1} \{0, 1, 2\} \xleftarrow{s^2} \{0, 1, 2\} \xrightarrow{d^2} \ldots$$

$d^i$ skips $i$, $s^i$ repeats $i$, and these generate all maps in $\Delta$.
The simplex category $\Delta$ is the category of finite nonempty ordinals and order preserving functions:

$$
\begin{align*}
\Delta := \{0\} & \xrightarrow{s^0} \{0, 1\} & \xrightarrow{d^0} \{0, 1, 2\} & \ldots \\
\{0\} & \xleftarrow{d^1} \{0, 1\} & \xleftarrow{s^0} \{0, 1, 2\} & \xleftarrow{d^1} \{0, 1, 2, 3\} & \ldots \\
\{0, 1\} & \xrightarrow{s^1} \{0, 1, 2\} & \xrightarrow{d^1} \{0, 1, 2, 3\} & \ldots
\end{align*}
$$

$d^i$ skips $i$, $s^i$ repeats $i$, and these generate all maps in $\Delta$

A simplicial set $X$ is a functor $\Delta^{op} \to \text{Set}$:
The simplex category $\Delta$ is the category of finite nonempty ordinals and order preserving functions:

$$\Delta := \{0\} \quad \xymatrix{ & d^0 \ar[l] & \{0, 1\} \ar[r] & \{0, 1, 2\} \ar[r] & \cdots \ar[l] & s^0 \ar[l] & d^1 \ar[l] & s^1 \ar[l] & d^2 \ar[l] & }$$

$d^i$ skips $i$, $s^i$ repeats $i$, and these generate all maps in $\Delta$

A simplicial set $X$ is a functor $\Delta^{op} \to \text{Set}$:

$$X = X_0 \xymatrix{ & d_0 \ar[l] & X_1 \ar[r] & X_2 \ar[r] & \cdots \ar[l] & s_0 \ar[l] & d_1 \ar[l] & s_1 \ar[l] & d_2 \ar[l] & }$$
The simplex category $\Delta$ is the category of finite nonempty ordinals and order preserving functions:

$$
\Delta := \{0\} \xrightarrow{s^0} \{0, 1\} \xleftarrow{s^0} \{0, 1\} \xrightarrow{d^1} \{0, 1, 2\} \xrightarrow{d^1} \cdots
$$

$d^i$ skips $i$, $s^i$ repeats $i$, and these generate all maps in $\Delta$.
Simplicial Sets

The simplex category $\Delta$ is the category of finite nonempty ordinals and order preserving functions:

$$\Delta := \{0\} \xleftarrow{s^0} \{0, 1\} \xrightarrow{d^1} \{0, 1, 2\} \xrightarrow{d^0} \ldots$$

$d^i$ skips $i$, $s^i$ repeats $i$, and these generate all maps in $\Delta$

There is a functor $\Delta \to \text{Top}$:
The simplex category $\Delta$ is the category of finite nonempty ordinals and order preserving functions:

$$\Delta := \{0\} \xleftarrow{s^0} \{0, 1\} \xrightarrow{d^0} \{0, 1, 2\} \xrightarrow{d^1} \{0, 1, 2, 3\} \ldots$$

$d^i$ skips $i$, $s^i$ repeats $i$, and these generate all maps in $\Delta$

There is a functor $\Delta \to \text{Top}$:

where the maps act on the vertices as in $\Delta$ and extend linearly
Simplicial Sets

Idea:

\[ X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \cdots \]

\[ \parallel \parallel \parallel \parallel \]

\[ \{ \text{0-simplices} \} \rightarrow \{ \text{1-simplices} \} \rightarrow \{ \text{2-simplices} \} \rightarrow \{ \text{3-simplices} \} \cdots \]

'face maps' \[ d_i \], 'degeneracy maps' \[ s_i \]
Simplicial Sets

Idea:

\[ X = X_0 \leftrightarrow X_1 \leftrightarrow X_2 \leftrightarrow X_3 \leftrightarrow \cdots \]

\{0-simplices\} \quad \{1-simplices\} \quad \{2-simplices\} \quad \{3-simplices\} \quad \cdots

'face maps' \( d_i \), 'degeneracy maps' \( s_i \)
Simplicial Sets

Example: Simplicial Circle

\[ X = X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots \]

\[ \{ \mathcal{b} \} \rightarrow \{ \ell, s_0 \mathcal{b} \} \rightarrow \{ s_0 \ell, s_1 \ell \} \rightarrow \cdots \]

Shapiro Cell Shapes
Example: Simplicial Circle

\[ X = \]

\[ X_0 \leftrightarrow X_1 \leftrightarrow X_2 \leftrightarrow \cdots \]

\[ \{ b \} \leftrightarrow \{ \ell, s_0 b \} \leftrightarrow \{ sb, s_0 \ell, s_1 \ell \} \leftrightarrow \cdots \]
Simplicial Sets

Example: Simplicial Circle

\[ X = X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \ldots \]

\[ \{ b \} \rightarrow \{ l, s_0 b \} \rightarrow \{ s_b, s_0 l, s_1 l \} \rightarrow \ldots \]

Shapiro Cell Shapes
The cube category □ is the category:

□ := 0 1 2 3 \ldots

A cubical set is a functor □^{op} \rightarrow \text{Set}.

Shapiro Cell Shapes
Cubical Sets

The cube category $\square$ is the category:

$\square := 0 \xrightarrow{\cdot} 1 \xleftarrow{\cdot} 2 \xrightarrow{\cdot} 3 \xleftarrow{\cdot} \cdots$

with composition described via a functor $\square \to \text{Top}$:

where the maps are two face inclusions and one projection in each dimension
The cube category $\square$ is the category:

$\square := 0 \xleftrightarrow{} 1 \xleftrightarrow{} 2 \xleftrightarrow{} 3 \xleftrightarrow{} \cdots$

with composition described via a functor $\square \to \text{Top}$:

where the maps are two face inclusions and one projection in each dimension

A cubical set is a functor $\square^{op} \to \text{Set}$
The glob(e) category $G$ is the category:

$$G := 0 \xrightarrow{s} 1 \xrightarrow{s} 2 \xrightarrow{s} 3 \xrightarrow{s} \cdots$$

with $s \circ s = t \circ s$ and $s \circ t = t \circ t$
Globular Sets

The glob(e) category $G$ is the category:

$$G := 0 \xrightarrow{\bar{s}} 1 \xrightarrow{\bar{s}} 2 \xrightarrow{\bar{s}} 3 \xrightarrow{} \ldots$$

with $\bar{s} \circ \bar{s} = \bar{t} \circ \bar{s}$ and $\bar{s} \circ \bar{t} = \bar{t} \circ \bar{t}$

$G$ can be realized in $Top$ by lemons:
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$$G := 0 \xrightarrow{\bar{s}} 1 \xrightarrow{\bar{s}} 2 \xrightarrow{\bar{s}} 3 \xrightarrow{} \cdots$$

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$G$ can be realized in $Top$ by lemons:

but these don’t capture the directionality
The glob(e) category $G$ is the category:

$$G := 0 \xrightarrow{\bar{s}} 1 \xrightarrow{\bar{s}} 2 \xrightarrow{\bar{s}} 3 \xrightarrow{} \ldots$$

with $\bar{s} \circ \bar{s} = \bar{t} \circ \bar{s}$ and $\bar{s} \circ \bar{t} = \bar{t} \circ \bar{t}$
The glob(e) category $G$ is the category:

$$G := 0 \xrightarrow{\tilde{s}} 1 \xrightarrow{\tilde{s}} 2 \xrightarrow{\tilde{s}} 3 \xrightarrow{} \cdots$$

with $\tilde{s} \circ \tilde{s} = \tilde{t} \circ \tilde{s}$ and $\tilde{s} \circ \tilde{t} = \tilde{t} \circ \tilde{t}$

Instead, think of globular cells as arrows between arrows (glob(e)s):
The glob(e) category $G$ is the category:

$$G := 0 \xrightarrow{\bar{s}} 1 \xrightarrow{\bar{s}} 2 \xrightarrow{\bar{s}} 3 \cdots$$

with $\bar{s} \circ \bar{s} = \bar{t} \circ \bar{s}$ and $\bar{s} \circ \bar{t} = \bar{t} \circ \bar{t}$

Instead, think of globular cells as arrows between arrows (glob(e)s):

$$\vdots$$
Globular Sets

A globular set $X$ is a functor $G \op \to \Set$ where $s \circ t = s \circ t$ and $t \circ s = t \circ t$.

"a collection of things in each dimension having source and target with fixed boundary"

Shapiro

Cell Shapes
A globular set $X$ is a functor $G^{op} \to \text{Set}$

$$X = X_0 \xleftarrow{s \circ s} X_1 \xleftarrow{s \circ t} X_2 \xleftarrow{s \circ t} X_3 \ldots$$

where $s \circ s = s \circ t$ and $t \circ s = t \circ t$
A globular set $X$ is a functor $G^{op} \rightarrow \text{Set}$

$$X = X_0 \xleftarrow{s} \xrightarrow{t} X_1 \xleftarrow{s} \xrightarrow{t} X_2 \xleftarrow{s} \xrightarrow{t} X_3 \cdots$$

where $s \circ s = s \circ t$ and $t \circ s = t \circ t$

"a collection of things in each dimension having source and target with fixed boundary"
Algebraic Composition

- Where are the categories?
Where are the categories?

Things with source and target are ripe for composition!
Where are the categories?

Things with source and target are ripe for composition!

How does this work with higher dimensions?
Algebraic Composition

Dimension 1:
Dimension 1:

\[ x \xrightarrow{f} y \xrightarrow{g} z \sim x \xrightarrow{f;g} z \]
Dimension 1:

\[ x \xrightarrow{f} y \xrightarrow{g} z \rightsquigarrow x \xrightarrow{f;g} z \]

Like a category!
Dimension 1:

\[ x \xrightarrow{f} y \xrightarrow{g} z \leadsto x \xrightarrow{f;g} z \]

Like a category!

Identities added either as algebraic structure or to cell structure:
Dimension 1:

\[ x \xrightarrow{f} y \xrightarrow{g} z \rightsquigarrow x \xrightarrow{f;g} z \]

Like a category!

Identities added either as algebraic structure or to cell structure:

\[ G' := 0 \xleftarrow{\bar{s}} 1 \xleftarrow{\bar{i}} 2 \xleftarrow{\bar{s}} 3 \xleftarrow{\bar{i}} \ldots \]
Dimension 1:

\[ x \xrightarrow{f} y \xrightarrow{g} z \xrightleftharpoons{} x \xrightarrow{f;g} z \]

Like a category!

Identities added either as algebraic structure or to cell structure:

\[
\begin{align*}
G' &:= 0 \xleftarrow{i} 1 \xleftarrow{i} 2 \xleftarrow{i} 3 \ldots \\
X &= X_0 \xrightarrow{i} X_1 \xrightarrow{i} X_2 \xrightarrow{i} X_3 \ldots
\end{align*}
\]
Dimension 1:

\[ x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{\sim} x \xrightarrow{f;g} z \]

Like a category!
Dimension 1:

\[ x \xrightarrow{f} y \xrightarrow{g} z \rightsquigarrow x \xrightarrow{f;g} z \]

Like a category!

Associativity can come in many forms:
Algebraic Composition

Dimension 1:
\[ x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{\text{strict}} x \xrightarrow{f;g} z \]

Like a category!

Associativity can come in many forms:

- **strict**: \((f;g);h\)
- **weak**: \((f;g);h\)
- **weaker**: \((f;g);h\)

\(f;(g;h)\)
Algebraic Composition

Dimension 1:

\[ x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{f;g} z \]

Like a category!

Associativity can come in many forms:

**strict**

\[ (f;g);h \]

**weak**

\[ (f;g);h \]

**weaker**

\[ (f;g);h \]

Similar choices for unit laws

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Algebraic Composition

Dimension 2:

Like a 2-category!
Algebraic Composition

Dimension 2:

Like a 2-category!

Associativity, identity conditions can similarly be strict or (various forms of) weak

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Dimension 2:

$$\begin{align*}
\bullet & \xrightarrow{\alpha} \bullet \\
\downarrow g & \quad \downarrow \beta \\
\bigcirc h & \\

\therefore \quad \sim \sim \quad \therefore \\

\bullet & \xrightarrow{\alpha;\beta} \bullet \\
\downarrow \bigcirc h \\

\bullet & \xrightarrow{\alpha} \bullet \\
\downarrow h & \quad \downarrow \beta \\
\bigcirc k & \\

\therefore \quad \sim \sim \quad \therefore \\

\bullet & \xrightarrow{\alpha\cdot\beta} \bullet \\
\downarrow \bigcirc h;k \\

\end{align*}$$
Dimension 2:

Like a 2-category!
Dimension 2:

\[ f \xrightarrow{g} h \xrightarrow{\alpha} \bullet \]
\[ \downarrow \beta \]
\[ \bullet \xrightarrow{f} \bullet \]

\[ f \xrightarrow{h} \bullet \]
\[ \alpha ; \beta \]

Like a 2-category!

Associativity, identity conditions can similarly be strict or (various forms of) weak
Higher dimensions even more complicated, especially weak versions. This is all algebraic structure on an underlying globular set. A globular set with (some sort of) algebraic composition structure is called (some version of) an $\infty$-category. (Often called an $(\infty,\infty)$-category.) An $(\infty,n)$-category is an $\infty$-category in which all cells of dimension $>n$ have (some kind of) inverses.
Higher dimensions even more complicated, especially weak versions
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(Often called an $(\infty, \infty)$-category)
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A globular set with (some sort of) algebraic composition structure is called (some version of) an $\infty$-category

(Often called an $(\infty, \infty)$-category)

An $(\infty, n)$-category is an $\infty$-category in which all cells of dimension $\geq n$ have (some kind of) inverses
Algebraic Composition

A globular set with (some sort of) algebraic composition structure is called (some version of) an $\infty$-category.

An $(\infty, n)$-category is an $\infty$-category in which all cells of dimension $> n$ have (some kind of) inverses.

Example: $\text{Top}_0 = \text{(some nice set of) spaces}$

$\text{Top}_1 = \text{continuous functions}$

$\text{Top}_2 = \text{homotopies}$

$\text{Top}_3 = \text{homotopies of homotopies}$

and so on.
Algebraic Composition

A globular set with (some sort of) algebraic composition structure is called (some version of) an $\infty$-category.

An $(\infty, n)$-category is an $\infty$-category in which all cells of dimension $> n$ have (some kind of) inverses.

Example: $\text{Top}$
Algebraic Composition

A globular set with (some sort of) algebraic composition structure is called (some version of) an $\infty$-category.

An $(\infty, n)$-category is an $\infty$-category in which all cells of dimension $> n$ have (some kind of) inverses.

Example: $\text{Top}$

$\text{Top}_0 = \text{(some nice set of) spaces}$

$\text{Top}_1 = \text{continuous functions}$

$\text{Top}_2 = \text{homotopies}$

$\text{Top}_3 = \text{homotopies of homotopies}$
Geometric Composition

Algebraic composition structure is tough to work with.

What about simplicial sets? Plus similar 'filling' conditions in higher dimensions to give associativity.

These 'quasicategories' are 'equivalent' to $(\infty, 1)$-categories.

Simplicial sets can only model $(\infty, 1)$ in this way.
Algebraic composition structure is tough to work with
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Algebraic composition structure is tough to work with

What about simplicial sets?

\[
\begin{array}{c}
\text{plus similar ‘filling’ conditions in higher dimensions to give associativity}
\end{array}
\]

These ‘quasicategories’ are ‘equivalent’ to \((\infty, 1)\)-categories

Simplicial sets can only model \((\infty, 1)\) in this way
Thanks!
The End