Appendix A

Notions from Topology

A.1 Covering Spaces

Definition A.1.1. Let $\pi : \bar{X} \to X$ be a continuous map. An open set $U \subseteq X$ is evenly covered by π if the preimage $\pi^{-1}(U)$ is a disjoint union of open sets $U_i' \subseteq \bar{X}$, called sheets, such that

$$\pi\,|_{U_i'} : U_i' \to U$$

is a homeomorphism for each i.

The map $\pi : \bar{X} \to X$ is a covering projection if every point in X has a neighborhood that is evenly covered. The space $\bar{X}$ together with the map π is a covering space for X if $\bar{X}$ is path connected and π is a covering projection.

Observation A.1.2. *Every covering projection is an open surjection. In particular, any that space has a covering space is path connected.* **q.e.d.**

Remark A.1.3. Every covering projection is a local homeomorphism. The converse is not true.

Exercise A.1.4. Give an example of a local homeomorphism that is not a covering projection.

Example A.1.5. Let $\bar{X}$ be a path connected topological space and G be a group that acts topologically free on $\bar{X}$, i.e., every point has an open neighborhood U such that $gU \cap U = \emptyset$ for all $g \in G - \{1\}$. Then

$$\pi : \bar{X} \to X := {}_{G}\backslash^{\bar{X}}$$

turns $\bar{X}$ into a covering space for the quotient space X.

Exercise A.1.6. Prove the statement of example (A.1.5)

A.1.1 Lifting Maps

Lemma and Definition A.1.7. *Let $\pi : \bar{X} \to X$ be a covering space. For every path $p : \mathbb{I} \to X$ in X issuing from $x = p(0)$ and every point $\bar{x} \in \pi^{-1}(x)$ in the fiber over x there is a unique <u>lift</u>, i.e., a map $\bar{p} : \mathbb{I} \to \bar{X}$ that makes the diagram*

$$\begin{tikzcd} \{0\} \arrow[r, "0 \mapsto \bar{x}"] \arrow[d, "0 \mapsto 0"'] & \bar{X} \arrow[d, "\pi"] \\ \mathbb{I} = [0,1] \arrow[r, "p"'] \arrow[ur, "\bar{p}"] & X \end{tikzcd}$$

commute.

Proof. We show uniqueness first. Let $\bar{p}_0$ and $\bar{p}_1$ be two lifts of p both starting at $\bar{x}$. Then the set

$$M := \{t \in \mathbb{I} \mid \bar{p}_0(t) = \bar{p}_1(t)\}$$

is visibly closed and non-empty as it contains 0. However, it is also open: Suppose $\bar{p}_0(t) = \bar{p}_1(t)$. Let $U \subseteq X$ be an open neighborhood of $p(t)$ that is evenly covered. Let U' be the sheet above U that contains $\bar{p}_0(t) = \bar{p}_1(t)$. Since $\pi : U' \to U$ is a homeomorphism, we see that the lifts $\bar{p}_0$ and $\bar{p}_1$ agree in an open interval around t. Being an open, closed, and non-empty subset of $\mathbb{I}$, the set M equals $\mathbb{I}$, whence $\bar{p}_0 = \bar{p}_1$.

Now, we turn to existence. Subdivide $\mathbb{I}$ by points

$$0 = t_0 < t_1 < t_2 < \cdots < t_{r-1} < t_r = 1$$

such that for each $i \in \{1, \ldots, r\}$, there is an evenly covered open set U_i in X containing the path segment $p\,|_{[t_{i-1}, t_i]}$. Since $\bar{x}$ determines a sheet U_0' above U_0, it is easy to lift the path up to t_1. However, now U_2 takes over, and since t_1 has already been lifted, we are given a sheet above U_2 to continue the construction. Iterating the procedure, we clearly arrive at a lift of p. **q.e.d.**

Corollary A.1.8. *Let Y be a topological space. Suppose the diagram*

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "\bar{f}_0"] \arrow[d] & \bar{X} \arrow[d, "\pi"] \\
Y \times [0,1] \arrow[r, "f"'] & X
\end{tikzcd}$$

commutes. Then there exists a unique continuous map

$$\bar{f} : Y \times [0,1] \to \bar{X}$$

such that

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "\bar{f}_0"] \arrow[d] & \bar{X} \arrow[d, "\pi"] \\
Y \times [0,1] \arrow[r, "f"'] \arrow[ur, "\bar{f}"] & X
\end{tikzcd}$$

commutes.

Proof. For any fixed $y \in Y$, we have the path $p_y := f(y, -)$. Since these paths have unique lifts, we see that there is a unique function $\bar{f} : Y \times [0,1] \to \bar{X}$ making the diagram commute. The only task is to show that this function is continuous. This is left as an exercise. **q.e.d.**

Exercise A.1.9. Prove that the map $\bar{f}$ is continuous.

Exercise A.1.10 (Covering Homotopy Lemma). Let $\bar{X}$ be a covering space for X with covering projection $\pi : \bar{X} \to X$. Let x_0 and x_1 be two points in X connected by two paths $p, q : \mathbb{I} \to X$ starting at x_0 and ending at x_1 that are homotopic relative to their endpoints. Furthermore, let $\bar{x}_0$ be a point in the fiber over x_0. Consider the two lifts $\bar{p}$ and $\bar{q}$ starting at $\bar{x}_0$. Prove that $\bar{p}(1) = \bar{q}(1)$ and that the paths $\bar{p}$ and $\bar{q}$ are homotopic relative to their endpoints.

Corollary A.1.11. *Every lift of a homotopically trivial loop in any covering space is a homotopically trivial loop.* **q.e.d.**

A.1.2 The Fundamental Group

Notation A.1.12. If $p : \mathbb{I} \to X$ and $q : \mathbb{I} \to X$ are paths such that $p(1) = q(0)$, we can concatenate these paths:

$$\begin{aligned} pq : \mathbb{I} &\to X \\ t &\mapsto \begin{cases} p(2t) & \text{for } 0 \leq t \leq \frac{1}{2} \\ q(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1. \end{cases} \end{aligned}$$

Definition A.1.13. Let X be a topological space with base point $\underline{\mathtt{x}}$. The <u>fundamental group</u> $\pi_1(X, \underline{\mathtt{x}})$ of X based at $\underline{\mathtt{x}}$ is the set of homotopy classes of paths starting and ending at $\underline{\mathtt{x}}$ relative to their endpoints, i.e:

$$\pi_1(X, \underline{\mathtt{x}}) := \{p : \mathbb{I} \to X \mid p(0) = p(1) = \underline{\mathtt{x}}\} \Big/ \begin{array}{l} p = q \text{ if } p \text{ is homotopic to } q \\ \text{relative to both endpoints.} \end{array}$$

Since all these paths are closed loops, we can concatenate these loops and it is clear that concatenation of loops descends to a well defined multiplication on homotopy classes. This way, we define a multiplication in $\pi_1(X, \underline{\mathtt{x}})$. It is easy to check that this multiplication turns the set $\pi_1(X, \underline{\mathtt{x}})$ into a group.

Definition A.1.14. If $f:(X,\underline{\mathtt{x}})\to(Y,\underline{\mathtt{y}})$ is a map of base pointed spaces, then there is an induced map

$$\pi_1(f):\pi_1(X,\underline{\mathtt{x}})\to\pi_1(Y,\underline{\mathtt{y}})$$

defined by pushing loops in X to Y via f.

Observation A.1.15. *If $h:(Y,\underline{y})\to(Z,\underline{z})$ is another map of base pointed spaces, then*

$$\pi_1(h\circ f)=\pi_1(h)\circ\pi_1(f)\,.$$

In other words, the fundamental group is a covariant functor.

Example A.1.16. Let $(\bar{X},\bar{\underline{\mathtt{x}}})$ be a covering space of the base pointed space $(X,\underline{\mathtt{x}})$ with base point preserving covering projection $\pi:\bar{X}\to X$. Then the induced homomorphism

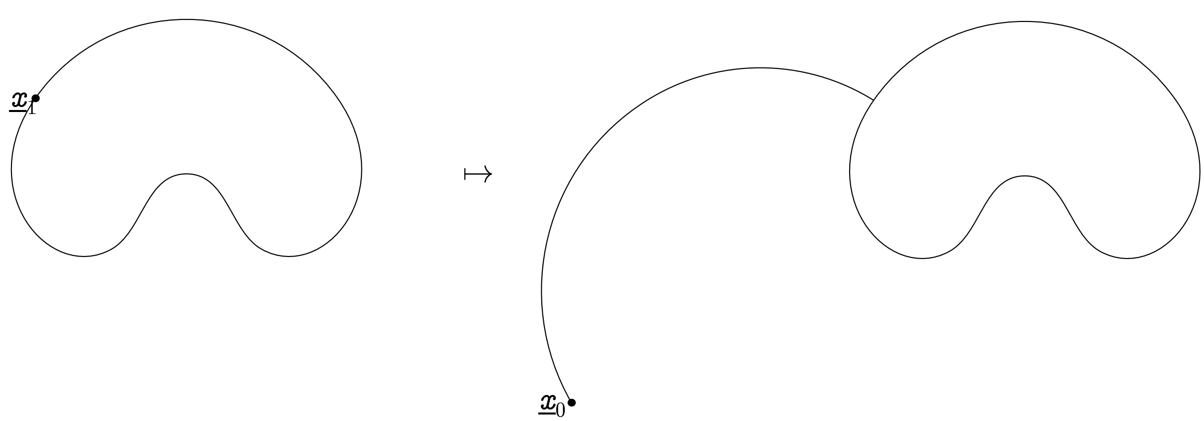
$$\pi_1(\pi):\pi_1\big(\bar{X},\bar{\underline{\mathtt{x}}}\big)\to\pi_1(X,\underline{\mathtt{x}})$$

is injective because of (A.1.11).

Observation A.1.17. *Let $\underline{x}_0$ and $\underline{x}_1$ be two points in X and fix a path p from $\underline{x}_0$ to $\underline{x}_1$. Then p induces an isomorphism*

$$\Phi_p:\pi_1(X,\underline{x}_1)\to\pi_1(X,\underline{x}_0)$$

by means of the lollipop construction:



A.1.3 Morphisms of Covering Spaces

Proposition A.1.18. *Let* $(Y, \underline{y})$ *be* 1*-connected and locally path connected and let* $\bar{X}$ *be a covering space with covering projection* $\pi : \bar{X} \to X$*. For each map* $f : (Y, \underline{y}) \to (X, \underline{x})$ *of base pointed spaces and each choice of a lift* $\underline{\bar{x}} \in \pi^{-1}(\underline{x})$*, there is a unique continuous* *<u>lift</u>*

$$\bar{f} : (Y, \underline{y}) \to (\bar{X}, \underline{\bar{x}}).$$

Proof. Since Y is path-connected, we know precisely how the lift $\bar{f}$ has to be defined: any point $y \in Y$ is the endpoint of some path p_y in Y starting at $\underline{y}$. The path $f \circ p_y$ is a path in X starting at $\underline{x}$ and has a unique lift in $\bar{X}$ starting at $\underline{\bar{x}}$. The endpoint of that lift has to be $\bar{f}(y)$.

Since Y is simply connected, this value $\bar{f}(y)$ is well defined: Any two paths p_y and q_y connecting $\underline{y}$ to y are homotopic relative to their endpoints in Y. The same holds true for $f \circ p_y$ and $f \circ q_y$. Thus the corresponding lifts have identical endpoints by (A.1.10).

It remains to show that $\bar{f}$ is continuous. We will prove that $\bar{f}$ is continuous everywhere. So let y be any point in Y. We find an open neighborhood U in X around $f(y)$ that is evenly covered. Let U' be the sheet above U that contains the lift $\bar{f}(y)$. Let $\zeta : U \to U'$ be the local homeomorphism inverse to π. Since f is open, the preimage $f^{-1}(U)$ is an open neighborhood V around y. It would be nice, if we could argue that

$$\bar{f}\mid_V = \zeta \circ f \mid_V$$

since this would visibly imply that $\bar{f}$ is continuous near y. However, life is not that easy. The problem is that in evaluating $\bar{f}$ at, say, $y' \in V$, we will want to use a path from y to y' that we can push to U via f and lift via ζ. Unfortunately there might be no path from y to y' that stays inside V.

To fix this problem, we use the hypotheses that Y is locally path connected. Thus, we can shrink V to $V' \subseteq V$ such that any point $y' \in V'$ can be connected to y by a path in V. Now, these paths will be pushed into U via f and therefore lift nicely by means of ζ. Thus

$$\bar{f}\mid_{V'} = \zeta \circ f \mid_{V'}$$

which also shows $\bar{f}$ to be continuous near y. **q.e.d.**

Definition A.1.19. Let $\bar{X}_0$ and $\bar{X}_1$ be two covering spaces over X with projections $\pi_i : \bar{X}_i \to X$. A map

$$f : \bar{X}_0 \to \bar{X}_1$$

is a <u>X-map</u> if the diagram

$$\begin{array}{ccc} \bar{X}_0 & \xrightarrow{f} & \bar{X}_1 \\ \big\downarrow \pi_0 & & \big\downarrow \pi_1 \\ X & = & X \end{array}$$

commutes. Note that compositions of X-maps are X-maps.

The two covering spaces $\bar{X}_0$ and $\bar{X}_1$ are <u>equivalent</u> if there are X-maps $f : \bar{X}_0 \to \bar{X}_1$ and $h : \bar{X}_1 \to \bar{X}_0$ such that $f \circ h = \mathrm{id}_{\bar{X}_1}$ and $h \circ f = \mathrm{id}_{\bar{X}_0}$.

For any covering space $\bar{X}$, let

$$\mathrm{Cov}_X(\bar{X})$$

denote the group of self-X-maps of $\bar{X}$.

Observation A.1.20. *Every X-map is a covering projection. To see this, consider an open subset U' of $\bar{X}_1$ that small enough so that its image in X is evenly covered. Then $f^{-1}(U')$ is contained in ${\pi_0}^{-1}(\pi_1(U'))$. It follows that U' is evenly covered.*

Definition A.1.21. A simply connected base pointed covering space $(\tilde{X}, \underline{\tilde{x}})$ with covering projection $\tilde{\pi} : (\tilde{X}, \underline{\tilde{x}}) \to (X, \underline{x})$ is called a <u>universal cover</u> of X.

Observation A.1.22. *Let* $(\bar{X}, \underline{\bar{x}})$ *be a covering space for the base pointed space* $(X, \underline{x})$ *with covering projection* $\pi : (\bar{X}, \bar{X}) \to (X, \underline{x})$ *be a covering space. Then there is a unique* X*-map* $\tilde{X} \to \bar{X}$ *that takes* $\underline{\tilde{x}}$ *to* $\underline{\bar{x}}$. *This follows immediately from (A.1.18).* **q.e.d.**

Corollary A.1.23. *Up to equivalence, there is at most one universal cover.* **q.e.d.**

Theorem A.1.24. *If* $(\tilde{X}, \underline{\tilde{x}})$ *is a universal cover for a locally path connected space* $(X, \underline{x})$, *then*

$$\mathrm{Cov}_X\left(\tilde{X}\right) = \pi_1(X, \underline{x}).$$

Proof. We give a pair of mutually inverse homomorphisms. The homomorphism

$$\mathrm{Cov}_X\left(\tilde{X}\right) \to \pi_1(X, \underline{\mathrm{x}})$$

is given as follows: Let $f : \tilde{X} \to \tilde{X}$ be a deck transformation. There is path from $\underline{\tilde{\mathrm{x}}}$ to $f(\underline{\tilde{\mathrm{x}}})$. This path projects down to a loop in X centered at $\underline{\mathrm{x}}$. This construction yields a well defined map since any two paths in $\tilde{X}$ connecting $\underline{\tilde{\mathrm{x}}}$ to $f(\underline{\tilde{\mathrm{x}}})$ are homotopic relative to their endpoints. The map is visibly a homomorphism: Composition of deck transformation turns into concatenation of paths and then concatenation of loops.

For the other direction

$$\pi_1(X, \underline{\mathrm{x}}) \to \mathrm{Cov}_X\left(\tilde{X}\right)$$

we start with a loop S in X based at $\underline{\mathrm{x}}$. It lifts to a unique path p in $\tilde{X}$ starting at $\underline{\tilde{\mathrm{x}}}$. By (A.1.18), there is a unique lift f of $\tilde{\pi}$ that takes $\underline{\tilde{\mathrm{x}}}$ to the endpoint of p. This construction is easily seen to yield a homomorphism.

It is routine to check that these two homomorphisms are mutually inverse. **q.e.d.**

Corollary A.1.25. $\pi_1(\mathbb{S}^1) = C_\infty$. **q.e.d.**

Definition A.1.26. A space X is semilocally simply connected if every point has a neighborhood such that every closed path in that neighborhood is homotopically trivial in X.

Theorem A.1.27 (Existence of the Universal Cover). *Every connected, locally path connected, semilocally simply connected space* $(X, \underline{x})$ *has a universal cover. Indeed, the quotient*

$$\tilde{X} := \{p : \mathbb{I} \to X \mid p \textit{ is continuous and } p(0) = \underline{x}\}/\sim$$

is a universal cover, where the equivalence relation $p \sim q$ *holds if and only if* p *and* q *are homotopic relative endpoints. The topology on* $\mathcal{P}_{\underline{x}}(X) := \{p : \mathbb{I} \to X \mid p \textit{ is continuous and } p(0) = \underline{x}\}$ *is defined by the basic open sets* $U_{p,V}$ *where* V *is an open neighborhood of the endpoint* $p(1)$*. The basic open set* $U_{p,V}$ *is defined as*

$$U_{p,V} := \{pq \mid q \textit{ is a path in } V \textit{ starting at } p(1)\}.$$

The base point of $\tilde{X}$ *is given as the homotopy class of the constant path that stays at* $\underline{x}$*.*

The covering projection is induced by the endpoint map $p \mapsto p(1)$.

Proof. Let us see that $\tilde{X}$ is simply connected. So let

$$\begin{array}{rcl} p : \mathbb{I} & \to & \mathcal{P}_{\underline{\mathbf{x}}}(X) / \sim \\ t & \mapsto & p_t \end{array}$$

be a closed path in $\tilde{X}$. By (A.1.28), any representative of the class p_t is homotopic relative endpoints to the path

$$\begin{array}{rcl} q_t : \mathbb{I} & \to & X \\ s & \mapsto & p_t(ts). \end{array}$$

In particular, q_t is a representative of the class p_t. Moreover, q_t is just an initial segment of q_1. Since $p_1 = p_0$, the path q_1 is homotopic to the constant path. A contracting homotopy for q_1 restricts to all the initial segments q_t and, therefore, induces a contracting homotopy of p in $\tilde{X}$. **q.e.d.**

Exercise A.1.28. Let

$$\begin{aligned} p : \mathbb{I} &\to \mathcal{P}_{\underline{\mathtt{x}}}(X) \,/ \sim \\ t &\mapsto p_t \end{aligned}$$

be a path. Prove that any representative of the class p_1 is homotopic relative endpoints to the path

$$\begin{aligned} q : \mathbb{I} &\to X \\ t &\mapsto p_t(1)\,. \end{aligned}$$

Exercise A.1.29. Prove that the endpoint map $p \mapsto p(1)$ induces a covering projection $(\mathcal{P}_{\underline{\mathtt{x}}}(X) \,/ \sim) \to X$.