Math 661 – Geometric Topology (homework 1, due Sep 13)

Exercise 1.1. Prove that there are precisely two homeomorphism of the circle $\mathbb{S}^1$ up to isotopy. Prove that these two are not homotopic.

Exercise 1.2. Let X be a locally path-connected space, i.e., the path-components of open subsets in X are open sets. Show that the following are equivalent:

1. X is connected.
2. X is path connected.
3. Every locally constant function on X is globally constant.

Moreover, show that any two points in a connected open subset of $\mathbb{E}^m$ or $\mathbb{S}^m$ can be connected by a broken geodesic, i.e., a path that consists of finitely many geodesic segments. (In $\mathbb{E}^m$, this is called a polygonal arc.)

Exercise 1.3. Let $\mathrm{H}^0(U)$ be the $\mathbb{R}$-vector space of all locally constant $\mathbb{R}$-valued functions on U. Prove that $\mathrm{H}^0(U)$ is the dual vector space of $\mathrm{H}_0(U)$.

Exercise 1.4. Show that the Mayer-Vietoris sequence is exact at the second and fifth term.

Definition. A space X is called simply connected if every loop is homotopic to a constant loop. Equivalently, X is simply connected if every map $f : \mathbb{S}^1 \to X$ can be extended to a map $\tilde{f} : \mathbb{B}^2 \to X$.

A space is called 1-connected if it is non-empty, path-connected and simply connected.

Exercise 1.5. Suppose $X = U \cup V$ where U and V are open, and suppose U, V, and $U \cap V$ are all 1-connected. Prove that X is 1-connected.

Exercise 1.6. Call a planar rectangle *admissible* if at least one of its sides has an integer length, i.e., the height or width or both are integers. Show that only admissible rectangles can be tiled by admissible rectangles.

Each problem is worth 5 points, but you can earn at most 20 point with this assignment.

Late homework will not be accepted.