# Math 661 – Geometric Topology (homework 7, due Oct 25)

**Exercise 7.1 (Correction).** Let $\mathcal{P}_{\underline{\mathrm{x}}}(X)$ denote the set

$$\mathcal{P}_{\underline{\mathrm{x}}}(X) := \{p : \mathbb{I} \to X \mid p \text{ is continuous and } p(0) = \underline{\mathrm{x}}\}$$

and let $\sim$ be the equivalence relation of paths being homotopic relative to endpoints. Let the topology on $\mathcal{P}_{\underline{\mathrm{x}}}(X) \,/\sim$ be defined by basic open sets

$$U_{p,V} := \{p \twoheadrightarrow q \mid q \text{ is a path in } V \text{ starting at } p(1)\}$$

where $p : \mathbb{I} \to X$ is a path and $V$ is an open neighborhood of the endpoint $p_1$.

Let

$$\begin{array}{rcl} p : \mathbb{I} & \to & \mathcal{P}_{\underline{\mathrm{x}}}(X) \,/\sim \\ t & \mapsto & p_t \end{array}$$

be a path **starting at the base point of $\mathcal{P}_{\underline{\mathrm{x}}}(X) \,/\sim$ which is class of the constant path** $p_0 : s \mapsto \underline{\mathrm{x}}$. Prove that any representative of the class $p_1$ is homotopic relative endpoints to the path

$$\begin{array}{rcl} q : \mathbb{I} & \to & X \\ t & \mapsto & p_t(1)\,. \end{array}$$

**Exercise 7.2.** Show that the isometry groups of Euclidean spaces are linear. That is, show that for every $m$, the group $\mathrm{Isom}(\mathbb{E}^m)$ is isomorphic to a subgroup of $\mathrm{GL}_n(\mathbb{R})$ for some $n$.

**Exercise 7.3.** Let $X$ and $Y$ be topological spaces. Assume $X$ is discrete. Then, every function from $X$ to $Y$ is continuous. Thus in the realm of sets, we have the identity

$$\mathrm{C}\,(X, Y) = \mathrm{Map}(X, Y)\,.$$

However, in the realm of topological spaces, the left hand carries the compact-open topology whereas the right hand comes with the product topology—recall that $\mathrm{Map}(X, Y)$ is just a product of "$X$-many" copies of $Y$. Show that the identity above holds in the realm of topological spaces, i.e., show that the compact open topology agrees with the product topology.

**Exercise 7.4.** Consider the action of $\mathrm{GL}_2(\mathbb{R})$ on the real projective line $\mathbb{RP}_1$. Let $M \in \mathrm{GL}_2(\mathbb{R})$ be a matrix with $\mathrm{tr}(M) > 2$. Prove that there are precisely two fixed points $x_-$ and $x_+$ in $\mathbb{RP}_1$. Moreover show that the dynamics of $M$ is as follows: For any pair of open neighborhoods $U_i$ of $x_i$, there is a power $n$ such that

$$M^n(x) \in U_+ \text{ for any point } x \notin U_-.$$

Thus, $x_+$ is attracting and $x_-$ is repelling.

Each problem is worth 5 points, but you can earn at most 15 points with this assignment.

Late homework will not be accepted.