Math 661 – Geometric Topology (homework 11, due Nov 25)

Exercise 11.1. Prove: A polygon diagram describes an orientable surface if and only if, for each edge-color a, the two edges of color a are oriented oppositely in the boundary circle of the polygon diagram.

Exercise 11.2. Show that any non-orientable surface has a one-vertex-diagram whose boundary reads the colors

$$\bullet \xrightarrow{a_1} \bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \bullet \xrightarrow{a_2} \bullet \xrightarrow{a_3} \bullet \xrightarrow{a_3} \bullet \cdots \bullet \xrightarrow{a_g} \bullet \xrightarrow{a_g}$$

for some $g \geq 0$.

Exercise 11.3. Prove: In a closed surface with a fixed hyperbolic structure, every closed curve is freely homotopic to a unique closed geodesic – here, a closed geodesic need not be simple.

Defintion. Let G be a group with a fixed generating system Σ. The <u>Cayley graph</u> $\Gamma_\Sigma(G)$ is a directed graph whose vertices are the elements of G. For each vertex g and each generator $x \in \Sigma$, there is an edge from g to gx. We ignore the orientation of these edges and define a metric on the vertex set by declaring all edges to have length 1: The metric

$$d_\Sigma : G \times G \to \mathbb{R}$$

is then given by shortest paths – note that $\Gamma(G)$ is connected since Σ generated G.

Exercise 11.4. Let G and H be groups generated by the *finite* generating sets Σ and Ξ, respectively. Let $\varphi : G \to H$ be a group homomorphism. Show that there is a constant C such that for all $g, h \in G$,

$$d_\Xi(\varphi(g)\,, \varphi(h)) \leq C d_\Sigma(g, h)\,.$$

Definition. Two metric space X and Y are called <u>quasi-isometric</u> if there exist two non-negative constants K and C and a function

$$\varphi : X \to Y$$

such that:

1. For all $x, y \in X$,

$$\frac{1}{C} d_X(x, y) - K \leq d_Y(\varphi(x)\,, \varphi(y)) \leq C d_X(x, y) + K.$$

2. Every point in Y is within distance K of the image of φ.

Exercise 11.5. Show that quasi-isometry is an equivalence relation on the class of metric spaces.

Exercise 11.6. Let Σ be a closed oriented surface with negative Euler characteristic. Show that the Cayley graph of $\pi_1(\Sigma)$ with respect to any finite generating set is quasi-isometric to $\mathbb{H}^2$.

Each problem is worth 5 points, but you can earn at most 20 points with this assignment.

Late homework will not be accepted.