Math 661 – Geometric Topology (homework 12, due Dec 02)

Exercise 12.1. Recall that every choice of a point $\underline{\tilde{\mathrm{P}}}$ in the fiber above $\underline{\mathrm{P}}$ defines an isomorphism

$$\pi_1(\Sigma, \underline{\mathrm{P}}) \to \mathrm{Cov}\left(\tilde{\Sigma}/\Sigma\right).$$

Show that, independent of the choice of $\underline{\tilde{\mathrm{P}}}$, we obtain a well defined isomorphism

$$\Phi : \mathrm{Out}(\pi_1(\Sigma, \underline{\mathrm{P}})) \to \mathrm{Out}\Big(\mathrm{Cov}\left(\tilde{\Sigma}/\Sigma\right)\Big)$$

and that this isomorphism makes the diagram

$$\begin{tikzcd}
 & \mathrm{Out}(\pi_1(\Sigma, \underline{\mathrm{P}})) \arrow[d, "\Phi"] \\
M(\Sigma) \arrow[ur, "\nu"] \arrow[dr, "\tilde{\nu}"'] & \\
 & \mathrm{Out}\Big(\mathrm{Cov}\left(\tilde{\Sigma}/\Sigma\right)\Big)
\end{tikzcd}$$

commute.

Exercise 12.2. Let Σ be a closed oriented surface of Euler characteristic χ. Show that $\pi_1(\Sigma)^{\mathrm{ab}} = \mathbb{Z}^{2-\chi}$.

Exercise 12.3. Show that the fundamental group of any non-compact surface is free. (Hint: First, consider the case of a punctured surface Σ, i.e., a closed surface with some discrete set of points removed. Show that there is a graph inside the surface onto which Σ deformation retracts. Then, the fundamental group of the surface is the fundamental group of the graph and hence free.

For the general case, consider a triangulation of the surface Σ. Show that there is a graph Γ inside the 1-skeleton of the triangulation whose complementary components are all infinite, simply-connected, and one-ended: A space X is called one-ended if every compact subset is contained in another compact subset that has a connected complement. Show that Σ deformation retracts onto Γ.)

Exercise 12.4. Let Σ be a closed, non-orientable surface. Prove that $\pi_1(\Sigma)^{\mathrm{ab}}$ contains an element of order 2.

Definition. A set $\{\gamma_1, \gamma_2, \ldots\}$ of simple closed curves on a closed oriented surface Σ of negative Euler characteristic is called admissible if it does not contain a null-homotopic loop and if all its loops are pairwise disjoint and non-homotopic.

The set of all non-empty admissible sets over Σ is ordered by inclusion. Like any partially ordered set, it gives rise to a simplicial complex, which, in this case, is called curve complex: The vertices of curve complex are the finite admissible sets over Σ. Simplices in curve complex are given by chains of admissible sets, i.e., a set of vertices (i.e., a set of admissible sets) is a simplex if it is totally ordered with respect to inclusion.

Exercise 12.5. Show that every admissible set of loops in a closed oriented surface of negative Euler characteristic is contained in a maximal admissible set. This set is finite and if you cut the surface open along its curves, it decomposes into finitely many pieces each of which is homeomorphic to a pair of pants.

Exercise 12.6. Show that the curve complex of a closed oriented surface of negative Euler characteristic is connected.

Each problem is worth 5 points, but you can earn at most 20 points with this assignment.

Late homework will not be accepted.