1. Devise a notion of rotation such that the arguments in this section extend to the spherical setting.

2. Show that the results do not hold for the unpunctured sphere. Find a 1-cycle in the sphere that is bounded by two disjoint 2-chains.

2.2.5 Concluding Remarks on Geodesic Homology

Geodesic homology of open sets in $\mathbb{E}^2$ or $\mathbb{S}^2$ is defined in terms of geodesically embedded finite graphs. The reason to require these graphs to be geodesically embedded comes from our need to add them. Since two finite graphs that are just topologically embedded may intersect in infinitely many points, we might be in trouble when adding those. Singular homology is especially designed to handle the technicalities that arise from considering topologically embedded graphs.

Avoiding technical issues comes at a price. We do not know whether our homology groups $\mathrm{H}_1(U)$ are invariants of U considered as a topological space, or whether these groups depend on the particular embedding $U \hookrightarrow \mathbb{E}^2$ or $U \hookrightarrow \mathbb{S}^2$. Even more down to earth problems are still open: The stereographic projection induces a map

$$\iota : \mathbb{E}^2 \to \mathbb{S}^2$$

which we can use to identify open subsets of $\mathbb{E}^2$ with open subsets of the (punctured) sphere. That gives us two ways of looking at a planar open set U. We can either regard it as an open subset of $\mathbb{E}^2$ and compute its homology this way, or we can identify it with an open subset of $\mathbb{S}^2$ via the stereographic projection and compute its homology groups that way. A priori, there can be a difference since the stereographic projection does not identify planar geodesics with spherical geodesics. In fact, however, there is no real difference:

Exercise 2.39. Let $\iota : \mathbb{E}^2 \to \mathbb{S}^2$ denote the stereographic projection. Show that for each open subset $U \subseteq \mathbb{E}^2$,

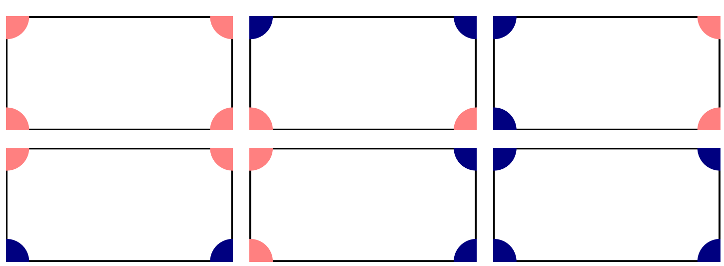
$$\mathrm{H}_1^{\mathbb{E}^2}(U) = \mathrm{H}_1^{\mathbb{S}^2}(\iota(U)).$$

One way of attacking the problem is to introduce a second homology theory. Geodesic homology is based on geodesically embedded graphs. We could, however, allow for more general embeddings as follows: A graph embedded in $\mathbb{E}^2$ or $\mathbb{S}^2$ is <u>weakly circle-like</u>, if all of its edges edges are either geodesic segments or circular arcs. It is clear, that two weakly circle-like graphs intersect in only finitely many points. Thus, we can construct homology groups $\overline{\mathrm{H}}_0(U)$ and $\overline{\mathrm{H}}_1(U)$ for planar and spherical open sets. As the stereographic projection takes planar geodesics/circles to spherical geodesics/circles, it is easy to solve exercise 2.39 for these homology theories. That, however leaves the problem to prove

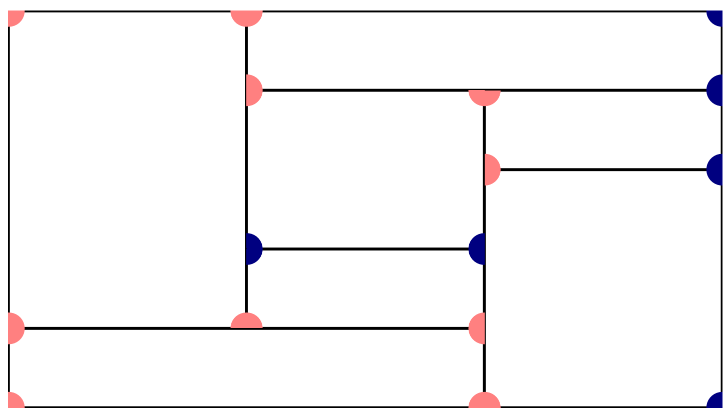
Exercise 2.40. Show that for a planar open set $U \subseteq \mathbb{E}^2$ and a spherical open set $V \subseteq \mathbb{S}^2$,

$$\begin{aligned} \mathrm{H}_1^{\mathbb{E}^2}(U) &= \overline{\mathrm{H}}_1(U) \\ \mathrm{H}_1^{\mathbb{S}^2}(V) &= \overline{\mathrm{H}}_1(V). \end{aligned}$$

Exercise 2.41. Consider a rectangle that is tiled by subrectangles. Each of the subrectangles has is corners colored by red and blue according to one of the following six patterns:



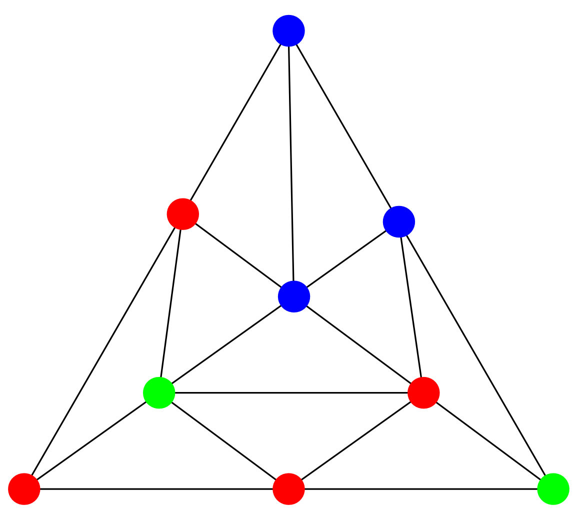
Prove that the induced coloring of the four corners of the big rectangle also fits one of these patterns provided that in the interior the colors around vertices that belong to different rectangles match. Here is an example:



Exercise 2.42. This is a similar coloring problem concerning triangles. Assume, you have a triangle which is triangulated, i.e., it is tiled by triangles that intersect only in vertices and edges such that a vertex of one triangle does not lie in the interior of an edge of another triangle. Assume that the vertices are colored in red, blue, and green such that the following conditions are satisfied:

1. The three vertices of the big triangle are given pairwise different colors.
2. Along each of the three edges of the big triangle, only two colors are used for the vertices in this edge.

Prove that you will always find a tricolored triangle in the subdivision.



2.3 Proof of the Curve Theorem

Theorem 2.43. *The complement of an embedded broken geodesic arc A in $\mathbb{S}^2$ is acyclic.*

Proof. Let P be a vertex in the interior of the arc. Since the arc is embedded, P splits the arc into two halves $A_{\mathbf{L}}$ and $A_{\mathbf{R}}$ that intersect in $\{P\}$. We put

$$\begin{aligned} X &:= \mathbb{S} - \{P\} \\ U &:= \mathbb{S} - A_{\mathbf{L}} \\ V &:= \mathbb{S} - A_{\mathbf{R}} \\ Y &:= \mathbb{S} - A \end{aligned}$$

and it is apparent that $X = U \cup V$ and $Y = U \cap V$. Thus, from the Mayer-Vietoris sequence, we have:

$$0 = \mathrm{H}_1(X) \to \mathrm{H}_0(Y) \to \mathrm{H}_0(U) \oplus \mathrm{H}_0(V).$$

Thus the map

$$\mathrm{H}_0(Y) \to \mathrm{H}_0(U) \oplus \mathrm{H}_0(V)$$

is injective. By the spherical analogue of (2.36) the map

$$\mathrm{H}_1(Y) \to \mathrm{H}_1(U) \oplus \mathrm{H}_0(V)$$

is also injective.

Thus equipped, we can argue by contradiction. Suppose ζ was a non-trivial cycle in $\mathrm{H}_i(Y)$ then it would remain non-trivial in at least one of the vector spaces $\mathrm{H}_i(U)$ or $\mathrm{H}_i(V)$. Thus, if we have a witness for the complement of A not being acyclic, the same witness would prove one of the pieces $A_{\mathbf{L}}$ or $A_{\mathbf{R}}$ to have a non-acyclic complement.

Now we finish the proof by induction: The cut vertex P can be chosen so that $A_{\mathbf{L}}$ and $A_{\mathbf{R}}$ both have fewer segments than A. Since we already know that one-segment arcs have acyclic complements, the claim follows. **q.e.d.**

Essentially the same proof yields the stronger statement:

Theorem 2.44. *The complement of an embedded topological arc A in $\mathbb{S}^2$ is acyclic.*

Proof. The proof above applies almost literally, the only difficulty is the induction part at the end: we might keep subdividing our arc, getting smaller and smaller pieces, but the process never stops. The point is that, morally, the process converges to a point which gives the start of the induction as a single point has an acyclic complement.

To make this precise let us assume that the cycle ζ is a witness for $\mathbb{S}^2 - A$ not being acyclic. We argued above that, given any subdivision of A into two pieces $A_{\mathbf{L}}$ and $A_{\mathbf{R}}$ that intersect in a single point, the witness ζ will either prove $\mathbb{S}^2 - A_{\mathbf{L}}$ or $\mathbb{S}^2 - A_{\mathbf{R}}$ to be non-acyclic. We choose the evil part an keep subdividing. This way, we obtain a sequence of subarcs $A = A_0 \supset A_1 \supset A_2 \supset \cdots$ that converges to a single point $P = \bigcap A_i$.

We know, that the complement of P is acyclic. So ζ is a boundary in $\mathbb{S}^2 - \{P\}$. The chain that allows us to write ζ as a boundary, however, has a compact carrier that avoids P and thus an open neighborhood of P. This open neighborhood already contains almost all A_i . Thus, ζ is a boundary in $\mathbb{S}^2 - A_i$ for sufficiently large i; but that contradicts the way we constructed the sequence A_i keeping ζ a witness for all these arcs to be evil. **q.e.d.**

Corollary 2.45 (Jordan's Curve Theorem). *The complement of a embedded topological loop S in $\mathbb{S}^2$ has two components.*

Proof. We write the loop as the union $S = A_{\mathbf{L}} \cup A_{\mathbf{R}}$ of two arcs that intersect in the two point set $\{P, Q\}$. Put

$$\begin{aligned} X &:= \mathbb{S} - \{P, Q\} \\ U &:= \mathbb{S} - A_{\mathbf{L}} \\ V &:= \mathbb{S} - A_{\mathbf{R}} \\ Y &:= \mathbb{S} - S \end{aligned}$$

and write down the Mayer-Vietoris sequence:

$$\mathrm{H}_1(U) \oplus \mathrm{H}_1(V) \to \mathrm{H}_1(X) \to \tilde{\mathrm{H}}_0(Y) \to \tilde{\mathrm{H}}_0(U) \oplus \tilde{\mathrm{H}}_0(Y).$$

We know:

$$\begin{aligned} \mathrm{H}_1(U) \oplus \mathrm{H}_1(V) &= 0 \\ \mathrm{H}_1(X) &= \mathbb{R} \\ \mathrm{H}_0(U) \oplus \tilde{\mathrm{H}}_0(Y) &= 0. \end{aligned}$$

Thus, $\mathrm{H}_0(Y) = \mathbb{R}$ which implies that the loop complement Y has precisely two components. **q.e.d.**

Exercise 2.46. Let C be a compact set in the plane $\mathbb{E}^2$. Suppose the complement has precisely two components and each point in C is a r c w i s e a c c e s s i b l e from both components. (A point in C is arcwise accessible from a given complementary component if it is the endpoint of an arc that is, away from this endpoint, completely contained in a complementary component.) Show that any two point subset of C disconnects C.

2.4 Schönflies' Theorem

2.4.1 The Schönflies Theorem for Polygons

In this section, we follow [Mois77].

Definition 2.47. Let $\mathcal{T}$ be a triangulation of a planar compact region whose boundary curve is a closed polygon P. A triangle $\Delta \in \mathcal{T}$ is called <u>free</u> if $\partial(\Delta) \cap P$ consists of one or two edges of Δ.

Lemma 2.48. *Let $\mathcal{T}$ be a triangulation of a planar compact region whose boundary curve is a closed polygon P. Suppose that $\mathcal{T}$ consists of two or more triangles. Then at least two of them are free.*

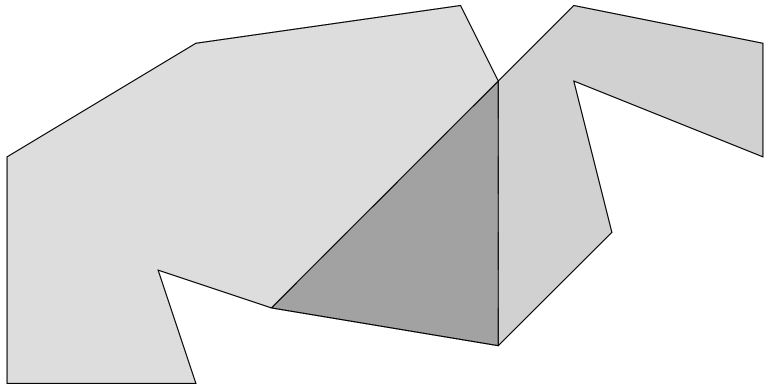


Table 5: Splitting a polygon

Proof. We proceed by induction. The case $|\mathcal{T}| = 2$ is clear: Both triangles are free.

Since P has more than three edges, there are at least two triangles in $\mathcal{T}$ that have an edge in P. If both of them are free, we are done. Otherwise, there is a triangle Δ that has an edge in P but is not free. In this case, the picture 5 captures the essence of the situation: The triangulation can be split into two parts both of which have fewer triangles and contain the triangle Δ as a free triangle. By induction, both parts contain free triangles that will be free in $\mathcal{T}$ too. **q.e.d.**

Exercise 2.49. Let P be a simple closed polygon, which cuts the plane into two regions. The closure of the bounded region can be triangulated and has P as its boundary.

Theorem 2.50. *Let P be a simple closed polygon, which cuts the plane into two regions. There is a homeomorphism of the plane that takes the closure of the bounded region B to a triangle.*

Proof. First, we find a triangulation of the plane that has the polygon P in its 1-skeleton. This is possible since P has only finitely many edges and the lines containing these segments chop up the plane into finitely many convex regions which in turn can be

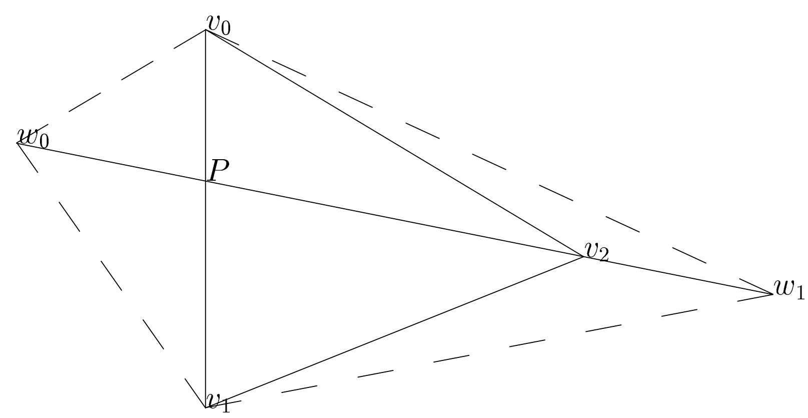


Table 6: Neighborhood of a free triangle

subdivided into triangles. Obviously, we can arrange things so that the interior region cut out by P covers only finitely many triangles. Thus, let $\mathcal{T}$ be a finite triangulation of this interior region.

If $|\mathcal{T}| = 1$, we are done. Otherwise, we will find a homeomorphism of the plane that removes a free triangle from $\mathcal{T}$. Then we find the next homeomorphism, and we continue in this fashion until we transformed P into a triangle.

It remains to present the homeomorphism that gets rid of a free triangle $\Delta \in \mathcal{T}$. Free triangles come in two flavors:

Δ has two edges in P: Let v_0, v_1, and v_2 be the vertices of Δ. Suppose that the edge v_0v_1 is inner. Then you can find two points w_0 and w_1 collinear with v_2 such that:

1. The point w_0 is inner.
2. The line segments w_0v_2 and v_0v_1 intersect. Let P be the point of intersection.
3. The point w_1 is out.

Thus, we have the picture shown in figure 6 By affine extension, we find a homeomorphism that leaves the outer quadrilateral of this figure fixed pointwise and takes v_2 to P. We extend this to the plane. This homeomorphism visibly kills the triangle.

<u>Δ has one edge in P</u>: In this case, we assume that v_2 is the inner vertex of Δ. Here everything works as above, except that w_0 is out and w_1 is inner. The final homeomorphism also works the other way: it takes P to v_2. **q.e.d.**

Remark 2.51. From the proof, it is clear that we can construct the homeomorphism so that it is the identity outside a previously chosen small open neighborhood of B.

Corollary 2.52. *Every simple closed polygon in the plane is the boundary of an embedded disc.* **q.e.d.**

2.4.2 The General Case

Theorem 2.53 (Schönflies). *Let S be a simple closed curve in the plane. Then the closure of the bounded complementary region is a disc.*

Lemma and Definition 2.54. *Let S be a simple closed curve in the plane with interior (bounded complementary region) I, and let A be an open arc in S. Then there is a line segment vw with $v \in A$ and $vw - \{v\} \subset I$.*

In this case, we say that v is <u>linearly accessible</u> from I and vw is a <u>driveway</u> for v.

Proof. Let v_0 be any point in A and let w be a point in I that close to v_0 such that the closed disc centered at v_0 through w does not contain any points of S outside A. Draw the line segment v_0w and let v be the last point on that segment that belongs to S. Observe that $v \in A$. **q.e.d.**

Lemma 2.55. *Let S be a simple closed curve in the plane with interior region I and let A be a closed arc in S with linearly* *accessible endpoints $v_{\mathbf{L}}$ and $v_{\mathbf{R}}$. Let $v_{\mathbf{L}}w_{\mathbf{L}}$ and $v_{\mathbf{R}}w_{\mathbf{R}}$ be driveways for $v_{\mathbf{L}}$ and $v_{\mathbf{R}}$. For any positive number ε, there is a broken line joining the segment $v_{\mathbf{L}}w_{\mathbf{L}}$ to $v_{\mathbf{R}}w_{\mathbf{R}}$ that is contained in I, stays within ε-distance to A, and intersects each of the two driveways in precisely one point.*

Proof. We can always find a broken line β connecting the two driveways inside I satisfying all conditions but ε-closeness. Let $w'_{\mathbf{L}}$ and $w'_{\mathbf{R}}$ be the points where β meets the driveways. Now, if β is not ε-close to A, then we can shrink ε and assume that β actually misses the ε-neighborhood of A altogether and moreover lies in its unbounded complementary region.

We draw a square grid whose square are that small that none of them contains two points of distance ε. Moreover, we do not want the grid to have a vertex on one of the driveways. Let Q be the union of those squares that meet A. Note that Q is connected and ε-close to A. Thus β lies in the unbounded complementary region of Q. See figure 7.

The part of the boundary of Q that borders the unbounded complementary region is a simple closed polygon because Q is *gallery connected*: any two squares in Q can be joined by a chain of squares in which neighboring squares share an edge.

Exercise 2.56. Justify this way of reasoning.

Let P be this simple closed polygon bordering the unbounded complementary region. We consider the simple closed curve

$$S' := v_{\mathbf{L}}w'_{\mathbf{L}} \cup \beta \cup v_{\mathbf{R}}w'_{\mathbf{R}} \cup A$$

and observe that its interior I intersects P in a finite number of broken lines. We will show that one of them connects the driveways.

So suppose none of the parts of $\overline{I} \cap P$ connects the two driveways. Consider the polygon P. Its interior contains A whereas β lies in the exterior. Thus P separates A and β. Thus

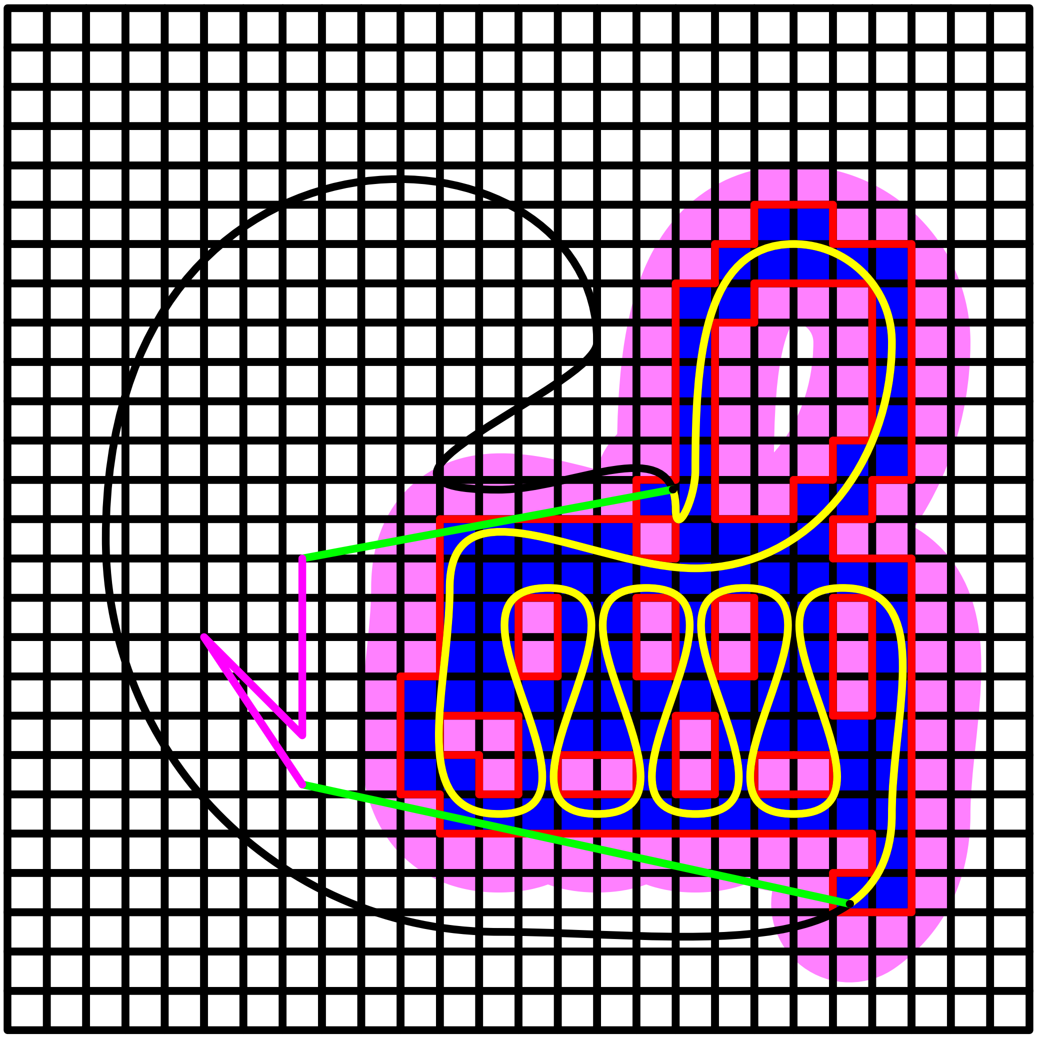


Table 7: The outer red boundary is P

$\overline{I} \cap P$ separates A and β inside $\overline{I}$. Hence any broken line π from β to A will intersect at least one of the pieces in $\overline{I} \cap P$. So pick an π with the minimum number of intersection points. We claim that this particular π does not intersect $\overline{I} \cap P$ at all, which is a contradiction.

To see that, we consider the first point on π that belongs to P. The particular arc in $\overline{I} \cap P$ does not connect the two driveways. Thus both endpoints of this arc are on the same driveway. It follows that there is another intersection of this arc

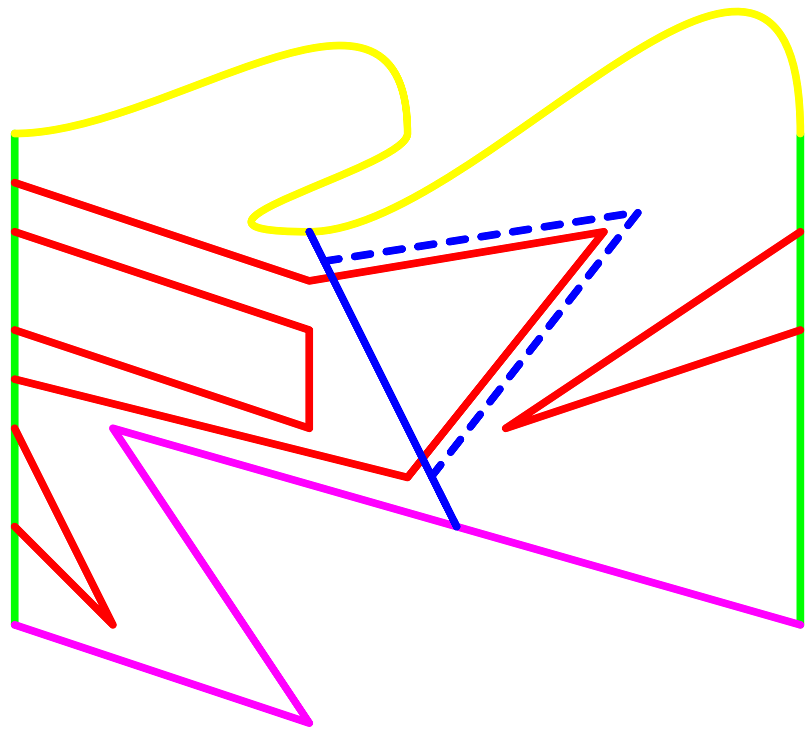


Table 8: Eliminating two intersections

with π. We now modify π as indicated in figure 8 to get rid of two intersections. This proves the claim. **q.e.d.**

Exercise 2.57. Let $f : \mathbb{S}^1 \to \mathbb{E}^2$ be continuous and injective. Show that f is a homeomorphism from $\mathbb{S}^1$ onto its image.

Proof of Schönflies' Theorem.

Claim A. There is a sequence $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \ldots$ such that:

- Each $\mathcal{P}_i$ is a finite decomposition of S into arcs intersecting in their endpoints.
- These endpoints are all linearly accessible.
- Each arc has diameter $\leq \frac{1}{i}$.
- $\mathcal{P}_{i+1}$ is a refinement of $\mathcal{P}_i$, i.e., The arcs in $\mathcal{P}_i$ are unions of consecutive arcs in $\mathcal{P}_{i+1}$. For technical reasons, we

require that each arc in $\mathcal{P}_i$ really is subdivided in the next step. Thus, the number of arcs in the decompositions at least doubles in each step.

PROOF. By exercise 2.57, we can pretend that S is a circle in the topology induced by the metric inherited from $\mathbb{E}^2$. Since each open arc contains linearly accessible points, these points are dense in S. Now it is easy to find the first partition and then refine it successively. $\square$

Claim B. The is a sequence $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}_2, \dots$ such that

- Each $\mathcal{D}_i$ is a collection of driveways, containing precisely one driveway leading to each of the partition points in $\mathcal{P}_i$.
- No two driveways in $\mathcal{D}_i$ intersect.
- $\mathcal{D}_i \subset \mathcal{D}_{i+1}$.

PROOF. Since $\mathcal{P}_1$ is finite, we can construct $\mathcal{D}_1$. Since the $\mathcal{P}_i$ stay finite, we can continue. $\square$

Now, we use Lemma 2.55 to construct a polygonal disc C_0 and a sequence of annuli $R_1, R_2, \dots$ where R_i is decomposed into polygonal discs $C_i^{(j)}$. We proceed as follows:

1. For every arc $A \in \mathcal{P}_0$, we create broken line β_A connecting the two driveways that stays within distance 1 of A. With only little extra effort, we arrange that two broken lines of our collection do not intersect away from the driveways and that two adjacent broken lines hit their intermediate driveway in the same point. Thus the broken lines form a simple closed polygon. By the polygonal Schönflies Theorem, its interior region is a disc C_0.

2. Similarly, for every $A' \in \mathcal{P}_1$, we create a broken line $\beta_{A'}$ that connects the driveways for A' and stays within distance $\frac{1}{2}$ of A'. Again, we avoid that these broken lines have unwanted

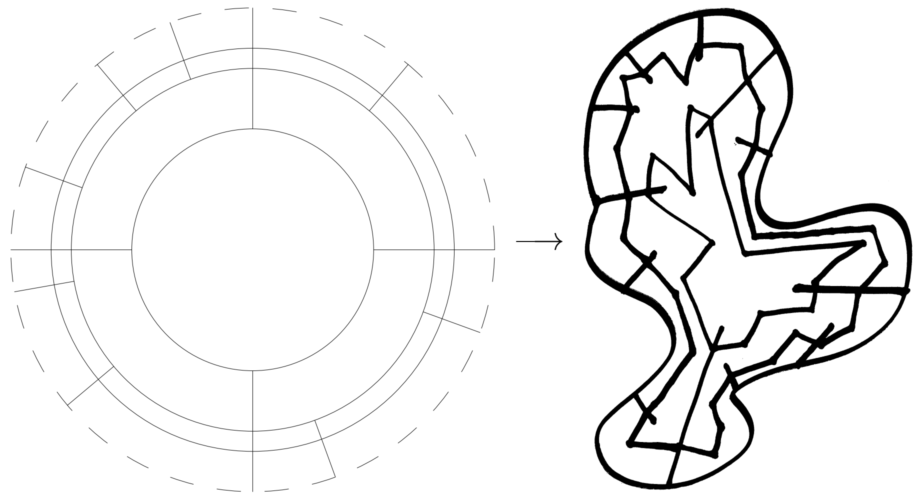


Table 9: !!! still needs to be done in metapost !!!

intersections among each other or with the driveways. We also avoid intersections with C_0 which we can easily achieve by creating these lines even closer to their arcs. These broken lines form a new simple closed polygon. The disc cut out by this polygon contains the middle disc C_0. Thus we have constructed an annulus R_1. The driveways for $\mathcal{P}_0$ chop this annulus into discs $C_1^{(j)}$. Here again, we use the Schönflies Theorem for polygonal curves to see that the regions $C_1^{(j)}$ are, in fact, discs.

3. Now we turn to $\mathcal{P}_2$. Here we will have the broken lines within distance $\frac{1}{3}$ of their corresponding arcs. We continue the process for all partitions $\mathcal{P}_i$ using a maximum distance of $\frac{1}{i}$. Thus, we create annuli R_i chopped up into discs $C_i^{(j)}$ by the driveways in $\mathcal{D}_{i-1}$. Observe that the disc $C_i^{(j)}$ is contained in an $\frac{1}{i}$-neighborhood of its arc, which in turn has diameter $\leq \frac{1}{i}$.

We copy this picture into the unit disc where we subdivide each arc of one generation evenly within the next generation. Thus we have a middle disc $\tilde{C}_0$ surrounded by annuli $\tilde{R}_1, \tilde{R}_2, \ldots$ chopped into discs $\tilde{C}_i^{(j)}$. See figure 9

We can easily define a homeomorphism on the 1-skeleton of these pictures that matches corresponding arcs. This homeomorphism extends to the 2-skeleta since any homeomorphism between the boundaries of two discs extends to their interiors. Thus, we have a homeomorphism

$$\tilde{C}_0 \cup \bigcup_i^j \tilde{C}_i^{(j)} \to C_0 \cup \bigcup_i^j C_i^{(j)}.$$

It remains to see that this homeomorphism extend continuously to the boundary. This, however, follows from the fact that an "ith-generation" disc is small and close to the boundary: The discs $\tilde{C}_i^{(j)}$ are small since their number per annulus at least doubles each step. The disc $C_i^{(j)}$ in the image is small by construction: It is within $\frac{1}{i}$-distance from its corresponding arc. This arc, in turn has diameter $\leq \frac{1}{i}$. **q.e.d.**