3.5.1 The Dehn-Nielsen Theorem for the Torus

We fix a base point $\underline{\mathtt{t}}$ on the torus. Let $\zeta : T \to T$ be a homeomorphism, and let $p : \mathbb{I}$ be a path from $\underline{\mathtt{t}}$ to $\zeta(\underline{\mathtt{t}})$. Then, we have two group homomorphisms:

$$\begin{aligned} \zeta_* : \pi_1(T, \underline{\mathtt{t}}) &\to \pi_1(T, \zeta(\underline{\mathtt{t}})) \\ [\gamma] &\mapsto [\zeta \circ \gamma] \end{aligned}$$

and

$$\begin{aligned} p_* : \pi_1(T, \zeta(\underline{\mathtt{t}})) &\to \pi_1(T, \underline{\mathtt{t}}) \\ [\gamma] &\mapsto [p \rightarrowtail \zeta \circ \gamma \rightarrowtail p^{\mathrm{rev}}]\,. \end{aligned}$$

Observation 3.5.4. *If q is another path from $\underline{t}$ to $\zeta(\underline{t})$, the two homomorphisms p_* and q_* differ by an inner automorphism of $\pi_1(T, \underline{t})$ given by the loop $p \rightarrowtail q^{\mathrm{rev}}$.* **q.e.d.**

Observation 3.5.5. *If $\xi : T \to T$ is a homeomorphism homotopic to ζ via a homotopy $\Phi : T \times \mathbb{I} \to T$, then*

$$p_* \circ \zeta_* = (p \rightarrowtail q)_* \circ \xi_*$$

where q is the path from $\zeta \underline{t}$ to $\xi(\underline{t})$ given by $\Phi(\underline{t}, -)$. **q.e.d.**

Thus, we obtain a well defined map

- $\nu : \mathcal{M}(T) \to \mathrm{Out}\,(\pi_1(T, \underline{\mathtt{t}}))$.

Theorem 3.5.6 (Dehn-Nielsen). *The map ν is an isomorphism of groups.*

Proof. First, let us check that ν is a homomorphism of groups. So let ζ and ξ be two homeomorphism of the torus T. We choose paths p and q from $\underline{\mathtt{t}}$ to $\zeta(\underline{\mathtt{t}})$ and $\xi(\underline{\mathtt{t}})$, respectively. Then

$$\begin{aligned} \nu(\zeta)\,\nu(\xi) &= [p_* \circ \zeta_*]\,[q_* \circ \xi_*] \\ &= [(p \rightarrowtail \zeta \circ q)_*\,(\zeta \circ \xi)_*] \\ &= \nu(\zeta \circ \xi)\,. \end{aligned}$$

Now, we show that ν is injective. So let $\zeta : T \to T$ be a homeomorphism with $\nu([\zeta]) = 1$. So, for any path p from $\underline{\mathrm{t}}$ to $\zeta(\underline{\mathrm{t}})$, the homomorphism $p_* \circ \zeta_*$ is an inner automorphism of $\pi_1(T, \underline{\mathrm{t}})$. We have to show that ζ is homotopic to the identity.

Let γ be a loop based at $\underline{\mathrm{t}}$ such that $p_* \circ \zeta_*$ is conjugation by γ. This is to say that

$$\gamma_* = p_* \circ \zeta_*.$$

Put $q := \gamma^{\mathrm{rev}} \rightarrowtail p$. Then

$$q_* \circ \zeta_* = \gamma_*^{\mathrm{rev}} \circ p_* \circ \zeta_* = 1.$$

Thus, for any loop γ', the curve $q \rightarrowtail \zeta \circ \gamma' \rightarrowtail q^{\mathrm{rev}}$ is homotopic to γ'. We apply this result to the two fundamental curves γ_1 and γ_2 on T. We obtain the following map on the surface of a cube:

On the front, we have the standard identification map $\mathbb{I}^2 \to T$. In the back, we have the composition $\mathbb{I}^2 \to T \xrightarrow{\zeta} T$. The four faces in the boundary annulus are filled by the homotopies

$$\gamma_i \sim q \rightarrowtail \zeta \circ \gamma_i \rightarrowtail q^{\mathrm{rev}}.$$

This is a map defined on the two-dimensional sphere $\mathbb{S}^2 \to T$. Since T is aspherical, it extends to a map on the ball. Moreover, note that opposite faces along the boudary annulus are mapped identically, we actually can make face identifications and obtain a map

$$T \times \mathbb{I} \to T$$

that visibly gives a homotopy from the identity (front) to ζ (back).

Finally, we observe that ν is onto. We know that $\mathrm{Out}\,(\pi_1(T, \underline{\mathrm{t}})) = \mathrm{GL}_2(\mathbb{Z})$. The action of $\mathrm{GL}_2(\mathbb{Z})$ on the plane $\mathbb{R}^2$ immediately descends to an action on the torus by homeomorphisms. This gives an inverse to ν. **q.e.d.**

Lemma 3.5.7. *The torus is aspherical.*

Proof. The sphere is 1-connected. Hence any map to the torus lifts to the universal cover, which is the plane. The lift extends to a map on the ball, and so does the original map. **q.e.d.**

Corollary 3.5.8. *Let $\tilde{\zeta} : \tilde{T} \to \tilde{T}$ be a homeomorphism that commutes with all deck transformations, i.e., the following diagram commutes for all deck transformations $\tau : \tilde{T} \to \tilde{T}$:*

$$\begin{tikzcd}
\tilde{T} \arrow[r, "\tilde{\zeta}"] & \tilde{T} \\
\tilde{T} \arrow[u, "\tau"] \arrow[r, "\tilde{\zeta}"] & \tilde{T} \arrow[u, "\tau"']
\end{tikzcd}$$

Then $\tilde{\zeta}$ induces a homeomorphism $\zeta : T \to T$, which is homotopic to the identity.

Proof. It is easy to see that $\tilde{\zeta}$ induces a homeomorphism ζ of T. We will only show that ζ is homotopic to the identity. By the Dehn-Nielsen Theorem (3.5.6), it suffices to prove that $\nu(\zeta)$ is the class of inner automorphisms of $\pi_1(T)$.

Fix a path $\tilde{p}$ in $\tilde{T}$ from the base point $\underline{\tilde{\mathrm{t}}}$ to $\tilde{\zeta}(\underline{\tilde{\mathrm{t}}})$. For any loop γ in T based at $\underline{\mathrm{t}} = \pi(\underline{\tilde{\mathrm{t}}})$, let $\tilde{\gamma}$ be the lift of γ based at $\underline{\tilde{\mathrm{t}}}$. This lift is a path from $\underline{\tilde{\mathrm{t}}}$ to $\tau_\gamma(\underline{\tilde{\mathrm{t}}})$ where τ_γ is the deck transformation corresponding to γ.

From

$$\left(\tilde{\zeta} \circ \tau_\gamma\right)(\underline{\tilde{\mathrm{t}}}) = \left(\tau_\gamma \circ \tilde{\zeta}\right)(\underline{\tilde{\mathrm{t}}}),$$

it follows that

$$\tilde{\gamma} \rightarrowtail \tau_\gamma \circ \tilde{p} \rightarrowtail \left(\tilde{\zeta} \circ \tilde{\gamma}\right)^{\mathrm{rev}} \rightarrowtail \tilde{p}^{\mathrm{rev}}$$

is a closed path in $\tilde{T}$. See figure 3.1. Thus,

$$\gamma \sim p \rightarrowtail \zeta \circ \gamma \rightarrowtail \gamma^{\mathrm{rev}}.$$

Thus, $p_* \circ \zeta_*$ is the identity automorphism of $\pi_1(T, \underline{\mathrm{t}})$. **q.e.d.**

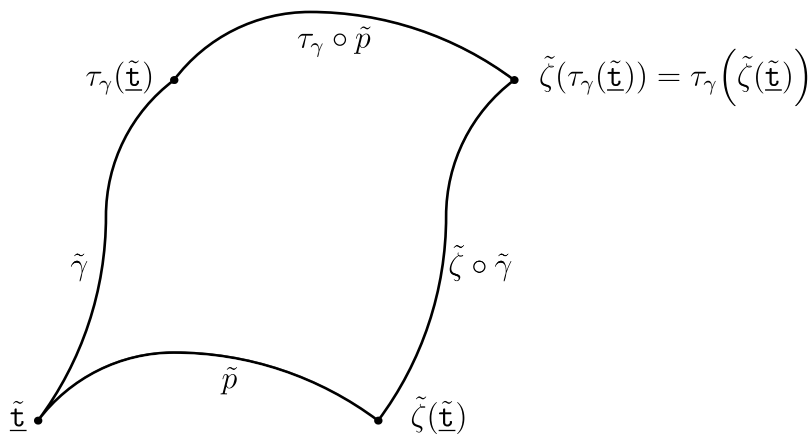

Figure 3.1: A closed path in $\tilde{T}$

3.5.2 Calculating Teichmüller Space

Theorem 3.5.9. *The map*

$$\begin{aligned} \Psi : \mathcal{T}_T &\rightarrow \mathcal{D}_T \\ [\mathcal{E}] &\mapsto \left[\eta^{\delta}_{\mathcal{E}}\right] \end{aligned}$$

is a bijection.

Proof of Injectitivity. Suppose we have two Euclidean structures $\mathcal{E}_1$ and $\mathcal{E}_2$ on T such that

$$\left[\eta^{\delta_1}_{\mathcal{E}_1}\right] = \left[\eta^{\delta_2}_{\mathcal{E}_2}\right].$$

Then there is a similarity $\sigma : \mathbb{E}^2 \rightarrow \mathbb{E}^2$ such that, for each loop γ, the following diagram commutes:

$$\begin{tikzcd}
 & \tilde{T} \arrow[r, "\delta_1"] & \mathbb{E}^2 \arrow[r, "\sigma"] & \mathbb{E}^2 & \tilde{T} \arrow[l, "\delta_2"'] \\
\tilde{T} \arrow[ur, "\tau_\gamma"] \arrow[r, "\delta_1"] \arrow[d, "\pi"] & \mathbb{E}^2 \arrow[ur, "\eta^{\delta_1}_{\mathcal{E}_1}(\gamma)"] \arrow[r, "\sigma"] & \mathbb{E}^2 \arrow[ur, "\eta^{\delta_1}_{\mathcal{E}_1}(\gamma)"] & \tilde{T} \arrow[l, "\delta_2"'] \arrow[ur, "\tau_\gamma"] \arrow[d, "\pi"] & \\
T & & & T &
\end{tikzcd}$$

By (3.5.8), it follows that $\delta_2^{-1} \circ \sigma \circ \delta_1$ induces a homeomorphism $\zeta : T \rightarrow T$ that is homotopic to the identity. It is easy to check that all these diagrams add up to:

$$\sigma\mathcal{E}_1\zeta = \mathcal{E}_2$$

Thus, $[\mathcal{E}_1] = [\mathcal{E}_2]$. **q.e.d.**

Exercise 3.5.10. Let T_1 and T_2 be two tori, and let $\phi : \mathrm{Cov}(T_1) \to \mathrm{Cov}(T_2)$ be an isomorphism. Show that there exists a homeomorphism

$$\tilde{\zeta} : \tilde{T}_1 \to \tilde{T}_2$$

of the universal covers that makes the following diagram commute for each deck transformation $\tau \in \mathrm{Cov}(T_1)$:

$$\begin{tikzcd} & \tilde{T}_1 \arrow[r, "\tilde{\zeta}"] & \tilde{T}_2 \\ \tilde{T}_1 \arrow[ur, "\tau"] \arrow[r, "\tilde{\zeta}"] & \tilde{T}_2 \arrow[ur, "\phi(\tau)"'] & \end{tikzcd}$$

Proof of Surjectivity. Let

$$\eta : \pi_1(T) \to \mathrm{Isom}\big(\mathbb{E}^2\big)$$

be a discrete, injective homomorphism. We have seen already that η factors through the group of translations. Thus, the image $G := (\eta)$ is a free abelian group generated by two linearly independend translations that acts on $\mathbb{E}^2$ topologically freely. Hence the quotient $G\backslash\mathbb{E}^2$ is a torus. This torus comes with a Euclidean structure. The idea is, of course, to transfer this structure to T.

By (3.5.10), there is a homeomorphism

$$\tilde{\zeta} : \tilde{T} \to \mathbb{E}^2$$

such that

$$\begin{tikzcd} & \tilde{T} \arrow[r, "\tilde{\zeta}"] & \mathbb{E}^2 \\ \tilde{T} \arrow[ur, "\tau_\gamma"] \arrow[r, "\tilde{\zeta}"] & \mathbb{E}^2 \arrow[ur, "\eta(\gamma)"'] & \end{tikzcd}$$

commutes for any loop γ. Thus, we can use $\tilde{\zeta}$ to define a Euclidean structure on $\tilde{T}$ which actually descends to a Euclidean Structure on T. Using $\tilde{\zeta}$ as our developing map, we see that this structure induces the holonomy η. **q.e.d.**

3.5.3 Classification of Homeomorphisms

Let us consider orientation preserving homeomorphisms of the torus up to homotopy. They form the group

$$\mathrm{SL}_2(\mathbb{Z}).$$

A matrix $M \in \mathrm{SL}_2(\mathbb{Z})$ can be studied by looking at its trace.

Definition 3.5.11. M is elliptic if $|\mathrm{tr}(M)| < 2$.
M is parabolic if $|\mathrm{tr}(M)| = 2$.
M is hyperbolic if $|\mathrm{tr}(M)| > 2$.

The significance lies in the fact that the characteristic polynomial of M is given by:

$$\mathrm{Det}(M) - \lambda\,\mathrm{tr}(M) + \lambda^2.$$

Thus we have:

M is elliptic: In this case, we have two complex conjugate eigenvalues λ_1, λ_2. There are only three possibilities:

$\mathrm{tr}(M) = 0$: We find $\lambda_1 = \mathrm{i}$ and $\lambda_2 = \mathrm{i}$. Thus, the matrix has order four.

$\mathrm{tr}(M) = 1$: Here we find λ_i is a third root of unity, and M has order three.

$\mathrm{tr}(M) = -1$: Finally λ_i is a sixth root of unity and M has order six.

Thus, elliptic elements are periodic. They have finite order.

Moreover, since one the eigenvalues lies in the upper half plane, there is a fixed point in Teichmüller space.