Chapter 4

Higher Genus Surfaces

4.1 The Main Result

We will outline two proofs of the main theorem:

Theorem 4.1.1. *Let Σ be a closed oriented surface of genus $g > 1$. Then every homotopy class of homeomorphisms has a representative $\zeta : \Sigma \to \Sigma$ satisfying one of the following conditions:*

elliptic case: *The homeomorphism has finite order, i.e., $\zeta^k = \mathrm{id}_\Sigma$.*

hyperbolic case: *The homeomorphism leaves a pair of geodesic laminations on Σ invaraint.*

parabolic case: *There is a non-empty collection of simple closed cuves on Σ that is left invariant as a subset of Σ. In this case, a power of ζ fixes the curves point wise.*

Definition 4.1.2. For a closed oriented surface of genus $g > 1$, Teichmüller space is defined as

$$\mathcal{T}_\Sigma = {\{\text{hyperbolic structures on } \Sigma\}}\Big/{\mathrm{Homeo}_1(\Sigma)} \; .$$

The main problem to overcome in both proofs is that the action of $M(\Sigma)$ on $\mathcal{T}_\Sigma$ is not cocompact. There are two main strategies to overcome this obstacle:

- Restrict your attention to a cocompact subspace of $\mathcal{T}_\Sigma$.
- Compactify $\mathcal{T}_\Sigma$ so that the action of $M(\Sigma)$ extends to the compactification.

4.1.1 First Proof: Cutting off Infinity

Promise 4.1.3 *There is a metric on Teichmüller space $\mathcal{T}_\Sigma$ such that:*

1. *$\mathcal{T}_\Sigma$ is a geodesic metric space.*
2. *Geodesics are unique.*
3. *Local geodesics are global.*
4. *The action of $M(\Sigma)$ on $\mathcal{T}_\Sigma$ is by isometries.*

Thus, $\mathcal{T}_\Sigma$ is a proper metric space and uniquely geodesic.

Definition 4.1.4. Let X be a metric space and $\lambda : X \to X$ be an isometry. The displacement function of λ is

$$\begin{aligned} D_\lambda : X &\to \mathbb{R} \\ x &\mapsto d_X(x, \lambda(x)) \,. \end{aligned}$$

The displacement of λ is

$$D(\lambda) := \inf_{x \in X} D_\lambda(x) \,.$$

The displacement is realized if there is a point $x \in X$ such that

$$D(\lambda) = D_\lambda(x) \,.$$

Fix a homeomorphism

$$\zeta : \Sigma \to \Sigma,$$

which induces an isometry λ_ζ on Teichmüller space by

$$\lambda_\zeta : [\mathcal{H}] \mapsto [\mathcal{H}\zeta] \,.$$

There are three cases:

- The displacement is realized and equals 0.
- The displacement is realized and strictly positive.
- The displacement is not realized.

The Displacement is Realized and Equals 0

Let $\mathcal{H}$ be a hyperbolic structure on Σ such that $[\mathcal{H}] \in \mathcal{T}_\Sigma$ realizes the displacement 0. Note that this point is a fixed point of ζ:

$$[\mathcal{H}] = [\mathcal{H}\zeta] .$$

Thus there is a homeomorphism $\xi : \Sigma \to \Sigma$ homotopic to the identity such that

$$\mathcal{H}\xi = \mathcal{H}\zeta.$$

Therefore, $\zeta \circ \xi^{-1}$ is an isometry of $(\Sigma, \mathcal{H})$. Since ξ is homotopic to the identity, we conclude that ζ is homotopic to an isometry of $(\Sigma, \mathcal{H})$. This isometry has finite order:

Promise 4.1.5 *Any isometry of an oriented closed hyperbolic surface has finite order.*

The Displacement is realized and Strictly Positive

Our first goal is to construct a geodesic that is fixed by λ_ζ:

Lemma 4.1.6. *Let X be a geodesic metric space and $\lambda : X \to X$ be an isometry whose displacement is strictly positive and realized at a point $x \in X$. Then*

$$l := \bigcup_{k \in \mathbb{Z}} \left[X, \lambda^k(x)\right] \lambda^{k+1}(x) = \bigcup_{k \in \mathbb{Z}} \lambda^k \left[X, x\right] \lambda(x)$$

is locally a geodesic.

Proof. We know that l is geodesic at all points in the interior of $[x, \lambda(x)]$. Since λ preserves being locally geodesic, it suffices to show that l is geodesic at $\lambda(x)$.

Consider the midpoint y of $[x, \lambda(x)]$. Observe that

$$D(\lambda) \leq d(y, \lambda(y)) \leq d(y, \lambda(x)) + d(\lambda(x), \lambda(y)) \leq d(x, \lambda(x)) = D(\lambda).$$

Thus l is geodesic at $\lambda(x)$. **q.e.d.**

This construction applies to Teichmüller space and yields are global bi-infinite geodesic C by (**??**.3). Note that this geodesic is invariant with respect to λ_ζ.

This is the hyperbolic case:

Promise 4.1.7 *Every geodesic in Teichmüller space $\mathcal{T}_\Sigma$ gives rise to a pair of transverse geodesic laminations.*

The Displacement is Not Realized

Definition 4.1.8. A metric space is <u>proper</u> if closed balls are compact.

Exercise 4.1.9. Show that a metric space is proper if an only if:

$$\text{compact} \iff \text{closed and bounded}$$

Exercise 4.1.10. Show that a geodesic metric space is proper if it is complete and locally compact.

Definition 4.1.11. A group G acts <u>properly discontinuously</u> on a topological space X if for every compact subset $C \subseteq X$, the set

$$\{g \in G \mid gC \cap C \neq \emptyset\}$$

is finite.

Remark 4.1.12. A properly discontinuous action is a topological analogue of an action with finite stabilizers.

We already know that the mapping class group does not act freely on Teichmüller space.

Promise 4.1.13 *Teichmüller space is a complete, locally compact, proper metric space, and the action of the mapping class group acts properly discontinuously on Teichmüller space.*

We need a big theorem. For any $\varepsilon > 0$ let $\mathcal{T}_\varepsilon$ be the subset of $\mathcal{T}_\Sigma$ of those hyperbolic structures for which the length of all closed geodesics in Σ are bounded from below by ε. Note that $\mathcal{T}_\varepsilon$ is $M(\Sigma)$-invariant.

Promise 4.1.14 (Mumford's Compactness Theorem) *For each $\varepsilon > 0$, there is a compact subset $C_\varepsilon \subset \mathcal{T}_\Sigma$ such that*

$$\mathcal{T}_\varepsilon = C_\varepsilon M(\Sigma)\,.$$

In fact, C_ε can be taken to be a fundamental domain for the action.

Let us choose a sequence of hyperbolic structures $(\mathcal{H}_i)$ such that

$$d([\mathcal{H}_i]\,,[\mathcal{H}_i\zeta]) \to D\lambda_\zeta \qquad \text{as } i \to \infty.$$

Lemma 4.1.15. *There is no $\varepsilon > 0$ such that $[\mathcal{H}_i] \in \mathcal{T}_\varepsilon$ for all i.*

Proof. We argue by contradiction. So suppose $[\mathcal{H}_i] \in \mathcal{T}_\varepsilon$ for all i. Then we can find a sequence $\xi_i \in M(\Sigma)$ such that

$$[\mathcal{H}_i\xi_i] \in C_\varepsilon.$$

Note that the sequence

$$d([\mathcal{H}_i]\,,[\mathcal{H}_i\zeta]) = d([\mathcal{H}_i\xi_i]\,,[\mathcal{H}_i\zeta\xi_i])$$

is bounded. Thus the points

$$[\mathcal{H}_i\zeta\xi_i] = \left[\mathcal{H}_i\xi_i \circ \xi_i^{-1} \circ \zeta \circ \xi_i\right]$$

stays within bounded distance from the compact set C_ε. Thus we can pass to a subsequence such that simultaneously

$$[\mathcal{H}_i\xi_i] \to \mathcal{H}_+$$

and

$$\left[\mathcal{H}_i\xi_i \circ \xi_i^{-1} \circ \zeta \circ \xi_i\right] \to \mathcal{H}_*.$$

Observe that the isometries $\xi_i^{-1} \circ \zeta \circ \xi_i$ take points close to $\mathcal{H}_+$ to points close to $\mathcal{H}_*$. Since the mapping class group acts properly discontinuously on Teichmüller space, it follows that there are only finitely many elements in $M(\Sigma)$ that do this. By the box principle, one of these occurs infinitely many times in the sequence $\xi_i^{-1} \circ \zeta \circ \xi_i$. Let this isometry be $\xi^{-1} \circ \zeta \circ \xi$. Since

$$d([\mathcal{H}_*], [\mathcal{H}_*]) = D(\zeta)$$

it follows that the displacement of ζ is realized at

$$\left[\mathcal{H}_+\xi^{-1}\right].$$

q.e.d.

Definition 4.1.16. The <u>spectrum</u> of a hyperbolic structure $\mathcal{H}$ on Σ is the set

$$\Sigma(\mathcal{H}) := \{\ln(\gamma) \mid \gamma \text{ is a simple closed geodesic in } \Sigma\}.$$

Promise 4.1.17 *For any hyperbolic surface, closed geodesics of length less than $3+\sqrt{2}$ do not intersect.*

Promise 4.1.18 *Any collection of pairwise non intersecting non-homotopic loops on a surface of genus g has at most $3g-3$ elements.*

Corollary 4.1.19. *For any hyperbolic structure $\mathcal{H}$,*

$$\left|\Sigma(\mathcal{H}) \cap \left(-\infty, \ln\left(3+\sqrt{2}\right)\right]\right| \leq 3g-3.$$

q.e.d.

Promise 4.1.20 *Let γ be a simple closed curve on Σ that is not homotopically trivial. For each hyperbolic structure $\mathcal{H}$, there is a unique geodesic $\gamma_{\mathcal{H}}$ homotopic to γ. Moreover, the map*

$$\ell_\gamma : [\mathcal{H}] \mapsto \ln(\textit{lenght of } \gamma_{\mathcal{H}})$$

is well defined and satisfies the inequality

$$|\ell_\gamma([\mathcal{H}_1]) - \ell_\gamma([\mathcal{H}_2])| \leq d_{\mathcal{T}_\Sigma}([\mathcal{H}_1], [\mathcal{H}_2]).$$

Choose L greater than all $D_{\lambda_\zeta}([\mathcal{H}_i])$. Since no $\mathcal{T}_\varepsilon$ contains all $[\mathcal{H}_i]$, it follows that there is an index i for which

$$\Sigma(\mathcal{H}_i) = M \uplus N$$

with

- $M \neq \emptyset$.
- $\sup M < \ln\bigl(3+\sqrt{2}\bigr)$.
- $\sup M + L < \inf N$.

We claim that the curves from which the lengths in M arise form an invariant system. Let Δ denote the set of homotopy classes of those closed geodesics.

Observe that

$$\Sigma(\mathcal{H}) = \Sigma(\mathcal{H}\zeta) = M \uplus N.$$

Thus, we may ask whether ζ respects the decomposition into M and N. The answer is "yes" because of (4.1.20): The curves γ in Δ are those with logarithmic length relative to $\mathcal{H}_i$ in M:

$$\ell_\gamma \mathcal{H}_i \in M.$$

Since

$$|\ell_\gamma \mathcal{H}_i - \ell_\gamma \mathcal{H}_i \zeta| \leq d(\mathcal{H}_i, \mathcal{H}_i\zeta) \leq L,$$

it follows from $\sup M + L < \inf N$ that

$$\ell_\gamma \mathcal{H}_i \zeta = \ell_{\zeta\circ\gamma} \mathcal{H} \in M.$$

Thus, ζ permutes the homotopy classes in Δ. A final fact proves the ζ is reducible:

Promise 4.1.21 *If a homeomorphism ζ permutes a finte set Δ of non-parallel, pairwise disjoint simple closed curves then these homotopy classes can simultaneously realized by simple closed curves which are permuted by a homeomorphism homotopic to ζ.*

4.1.2 Second Proof: Compactifying Teichmüller Space