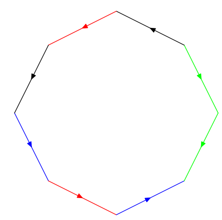


Table 6.1: A polygon diagram

6.1.2 Second Proof: Compactifying Teichmüller Space

6.2 Classification of Closed Surfaces

We saw that each surface has a triangulation. Compact surfaces have finite triangulations. In this section, we shall see that one can put these combinatorial data into a standard form.

The torus is obtained from the square by identifying opposite edges. In general, a polygon diagram is a polygon whose edges are marked with orientation arrows and colors such that each color occurs exactly twice, see figure (6.1). From a polygon diagram, we obtain a topological space by gluing edges of the same color together so that their arrows match up.

Exercise 6.2.1. Show that the space defined by a polygon diagram is a closed surface.

Proposition 6.2.2. *Every closed surface Σ can be described by a polygon diagram.*

The proof is an interpolation between two-dimensional simplicial complexes and polygon diagrams. Thus, we need a notion that generalizes both.

Definition 6.2.3. A polygon complex is a collection of polygons whose edges are colored and marked with orientation arrows.

Observation 6.2.4. *The following are obvious:*

1. *A two-dimensional simplicial complex is a polygon complex if and only if every vertex and every edge are contained in a two-simplex. In general, simplicial complexes whose maximal simplices all have the same dimension are called* <u>*chamber complexes*</u>*.*

2. *The polygon diagrams are precisely those polygonal complexes that consist of just one polygon.*

3. *Any polygon complex gives rise to a topological space by identifying edges of the same color respecting the orientation of the edges.*

4. *Any polygonal complex can be subdivided to yield a two-dimensional chamber complex.*

Proof of (6.2.2). Since Σ can be triangulated, there is a polygon complex realizing Σ. Now suppose, we had a polygon complex realizing Σ with more than one polygon. Since Σ is connected, there is a pair of equi-colored edges in two different tiles. We reduce the polygon complex by gluing these two tiles along their this pair of edges. Thereby, we form a bigger polygon. Since this process decreases the number of polygons in the complex, it will stop and we arrive at a polygon diagram for Σ. **q.e.d.**

We can improve upon this quite a bit. Recall that a polygon diagram represents a surface by identification of its edges. Thus certain points on the boundary of the polygon represent identical points in the surface. We call any two such boundary points in a polygon diagram <u>equivalent</u>. We call to edges equivalent if their mid-points are equivalent.

Proposition 6.2.5. *Any surface that is not homeomorphic to the sphere has a polygon diagram all of whose corners are equivalent.*

Definition 6.2.6. Let us call a polygon diagram a <u>one-vertex-diagram</u> if all corners are equivalent.

Figure 6.1: First case – the two edges have the green corner in common.

Proof. Color the corners of the polygon diagram according to their equivalence class. Suppose you need more than one color. In this case, a bigon represents the sphere. Thus, we assume that the polygon has at least four edges.

We will give a procedure for getting rid of any specified color. Suppose, we want to eliminate green. As green is not the only color, there will be an edge connecting a green corner to a corner of a different color, say blue. This edge has a color, say red, which specifies a partner edge. There are two cases. Either the two red edges have a corner in common or not.

Suppose the two edges have a corner in common. Then their arrow either point both toward that corner or away from that corner – this follows from the coloring of the vertices. We can than "swallow" that common corner into the interior of the polygon diagram. The case, where the green vertex is swallowed is shown. In that case, we reduce the number of green vertices by two. If the blue vertex is swallowed, the number of green vertices decreases by one.

Suppose the two red edges have no common corner. Then pick one of the red edges and move along this edge starting in its blue corner. The next corner you meet is the green corner of this edge. Continue your path along the polygon until you reach the next non-green corner. Cut of this region and glue it to the other red edge. This reduces the number of green corners by one.

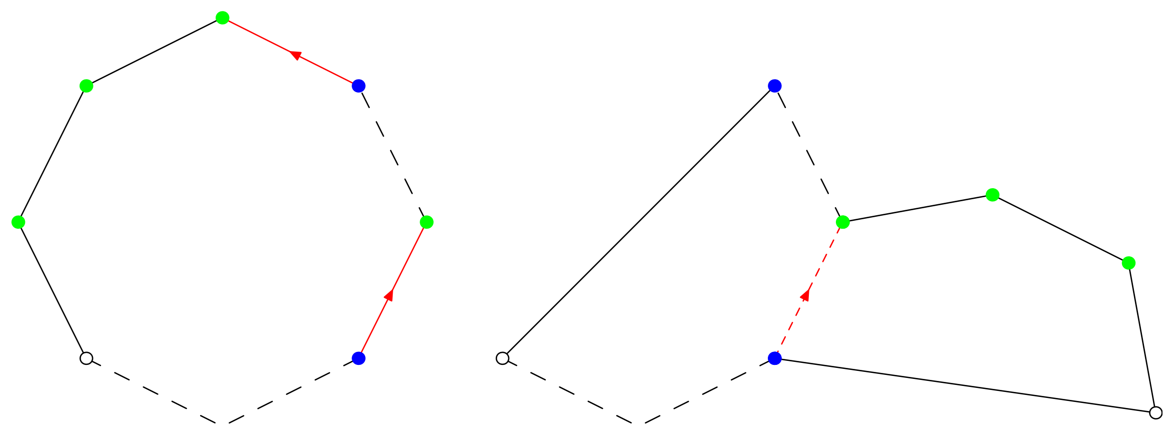


Figure 6.2: Second case – the two edges do not overlap.

Continue this process until all green corners are gone. If you need still more than one color, rid the picture of the next. **q.e.d.**

Definition 6.2.7. A simplicial complex is <u>orientable</u> if all its simplices can be given orientations compatible with the inclusion of faces as subsimplices. Note that subdivisions of simplicial complexes inherit orientations. Thus, orientability of a triangulated surface does not depend on the triangulation. We call a surface orientable if it has an orientable triangulation.

Remark 6.2.8. Let us discuss orientability of surfaces. Think of a realization of the surface as a polygon complex. Take some big sheet of paper whose two sides are colored red and yellow. Cut out the polygons of the complex. If the edge identifications allow you to glue the pieces so that crossing an edge will never get you from a red side to a green side, then you obtain an oriented surface.

For a polygon diagram, the criterion for orientability given in (6.2.8) is also necessary:

Exercise 6.2.9. Prove: A polygon diagram describes an orientable surface if and only if, for each edge-color a, the two edges of

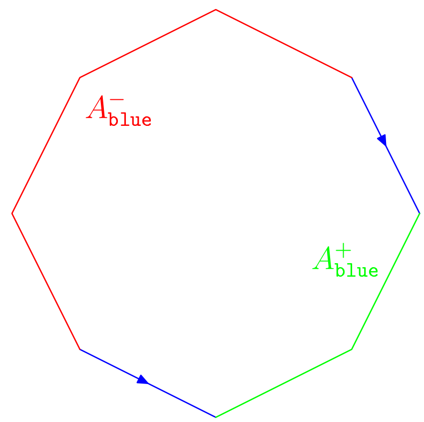


Figure 6.3: The two arcs defined by a pair of equivalent edges.

color a are oriented oppositely in the boundary circle of the polygon diagram.

Corollary 6.2.10. *In a one-vertex-diagram for an orientable surface, adjacent edges are inequivalent.*

Proof. Suppose we had a pair of equivalent adjacent edges. Since the surface is orientable, these edges are oppositely oriented. In this case, however, the corner spanned by these two edges is inequivalent to any other corner. Thus, we are not dealing with a one-vertex-diagram. **q.e.d.**

Thus, in a one-vertex-diagram for an orientable surface, any color a defines two edges with opposite orientations that cut the boundary into two non-empty arcs: The arc A^+_a, toward which the edges of color a point, and the arc A^-_a, away from which the edges point. See figure 6.3.

Observation 6.2.11. *In any one-vertex-diagram for an orientable surface and any edge-color a, there is a pair of equivalent edges such that one of them lies on A^+_a and the other one lies on A^-_a.*

Proof. Suppose not, then the corners at the edges of color a would fall into two distinct equivalence classes. **q.e.d.**

Definition 6.2.12. The genus g standard polygon diagram is the regular $4g$ polygon whose edges are colored with $2g$ colors $a_1, \ldots, a_g$ and $b_1, \ldots, b_g$ and marked so that the boundary reads the word

$$\bullet \xrightarrow{a_1} \bullet \xrightarrow{b_1} \bullet \xleftarrow{a_1} \bullet \xleftarrow{b_1} \bullet \xrightarrow{a_2} \bullet \xrightarrow{b_2} \bullet \xleftarrow{a_2} \bullet \xleftarrow{b_2} \bullet \cdots \bullet \xrightarrow{a_g} \bullet \xrightarrow{b_g} \bullet \xleftarrow{a_g} \bullet \xleftarrow{b_g}$$

The g-torus is the surface obtained from the genus g standard polygon diagram.

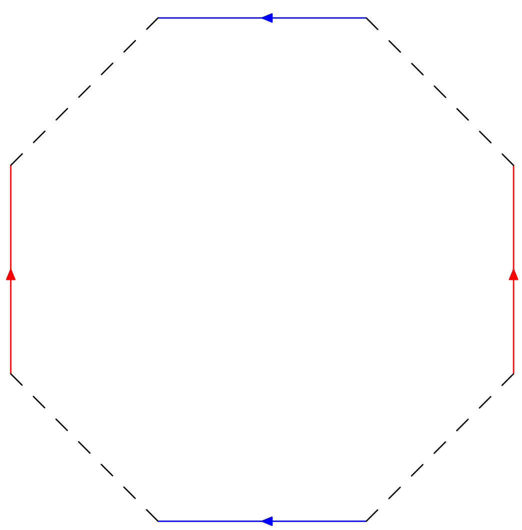
Theorem 6.2.13. *Every closed oriented surface is either a sphere or a g-torus for some $g \geq 1$.*

Proof. Let us start with a one-vertex-diagram for the surface. We will use cut and paste to transform the diagram until we obtain a genus g standard polygon diagram.

Let us call a sequence of four edges a run if it has the form

$$\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \xleftarrow{a} \bullet \xleftarrow{b} \bullet.$$

If every edge occurs in a run then we have the a standard polygon for some genus. Thus, we want to eliminate edge colors that do not occur in runs. Let a be a color whose corresponding edges do not form a run. By (6.2.11), we know that there is another color b such that the polygon diagram looks essentially like this:



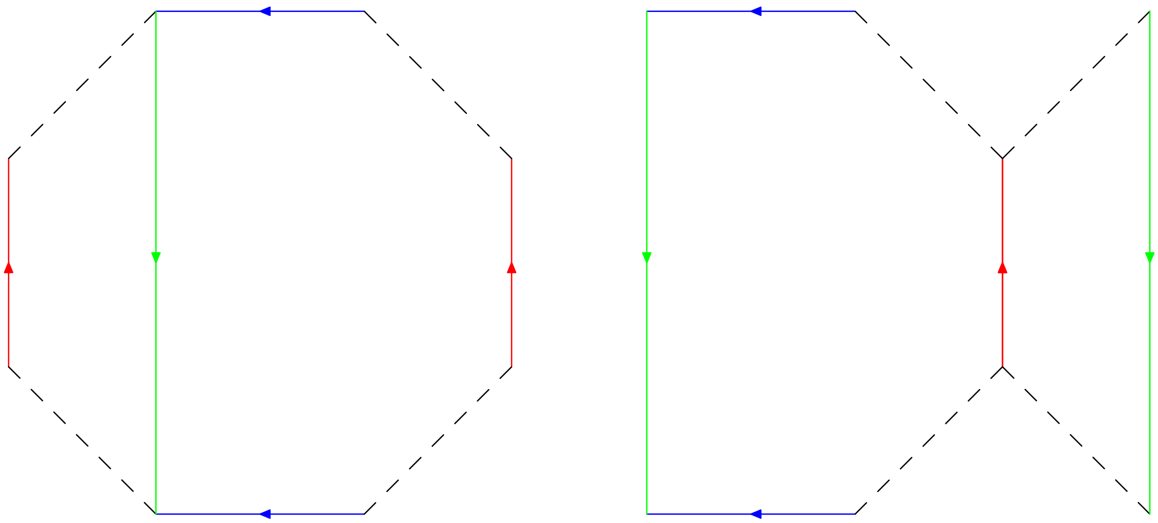
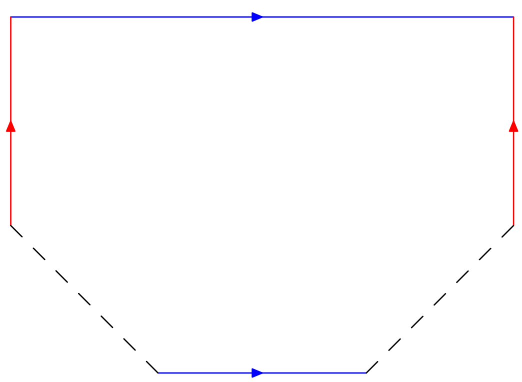


Figure 6.4: The first cut.

We cut and past as illustrated in figure 6.4. Note that the runs in the dashed arcs are not destroyed.

The first cut put us in a situation where we have three edges of a run but the forth partner of the middle edge might be somewhere:



We can create a run by cut and past ash shown in figure 6.5. Again, we do not destroy any runs previously created. **q.e.d.**

Exercise 6.2.14. Show that any non-orientable surface has a one-vertex-diagram whose boundary reads the colors

$$\bullet \xrightarrow{a_1} \bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \bullet \xrightarrow{a_2} \bullet \xrightarrow{a_3} \bullet \xrightarrow{a_3} \bullet \cdots \bullet \xrightarrow{a_g} \bullet \xrightarrow{a_g}$$

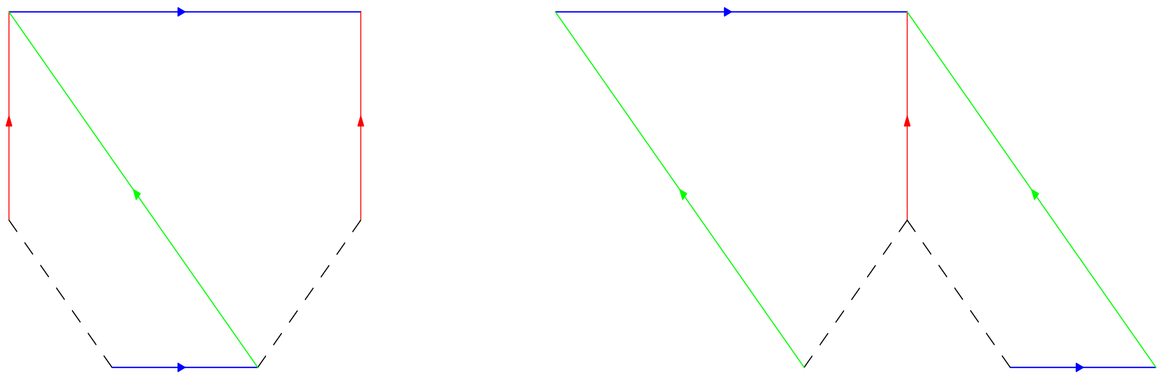


Figure 6.5: The second cut.

for some $g \geq 0$.

6.3 Poincaré's Theorem

Theorem 6.3.1 (Poincaré). *Let D be a polygon diagram drawn in the hyperbolic plane such that the lengths of its edges and the interior angles at its corners satisfy the following two conditions:*

1. *Equivalent edges have the same length.*

2. *The angles of all corners in an equivalence class sum up to 2π.*

Then there is a tiling of the hyperbolic plane by isometric copies of D such that each at edge of two copies of D meet along a pair of equivalent edges. Moreover, the coloring preserving symmetries of this tiling are a group of hyperbolic isometries of $\mathbb{H}^2$ isomorphic to the fundamental group of the surface defined by D.

Remark 6.3.2. The conditions say that the polygon diagram D can tile the hyperbolic plane locally around edges and vertices. Thus, they are clearly necessary conditions for the existence of a global tiling. The theorem says, if a tile tiles locally it tiles globally.