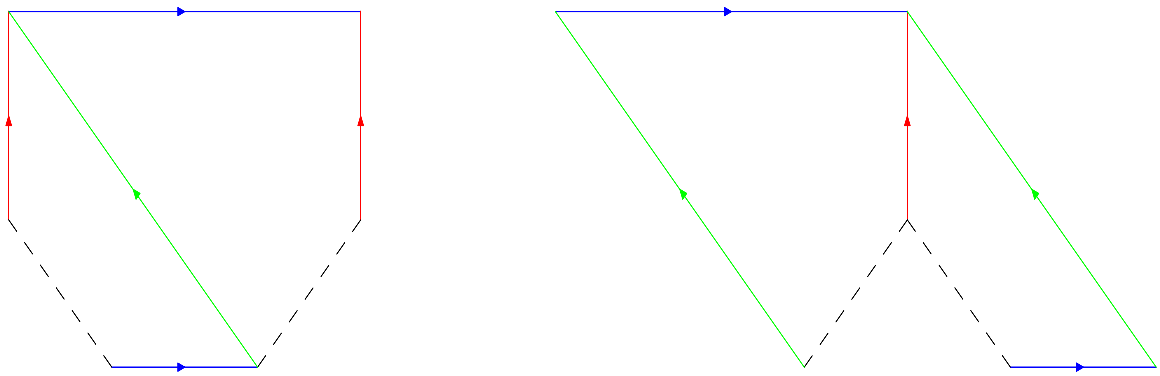


Figure 6.5: The second cut.

for some $g \geq 0$.

6.3 Poincaré's Theorem

Theorem 6.3.1 (Poincaré). *Let D be a polygon diagram drawn in the hyperbolic plane such that the lengths of its edges and the interior angles at its corners satisfy the following two conditions:*

1. *Equivalent edges have the same length.*
2. *The angles of all corners in an equivalence class sum up to 2π.*

Then there is a tiling of the hyperbolic plane by isometric copies of D such that each at edge of two copies of D meet along a pair of equivalent edges. Moreover, the coloring preserving symmetries of this tiling are a group of hyperbolic isometries of $\mathbb{H}^2$ isomorphic to the fundamental group of the surface defined by D.

Remark 6.3.2. The conditions say that the polygon diagram D can tile the hyperbolic plane locally around edges and vertices. Thus, they are clearly necessary conditions for the existence of a global tiling. The theorem says, if a tile tiles locally it tiles globally.

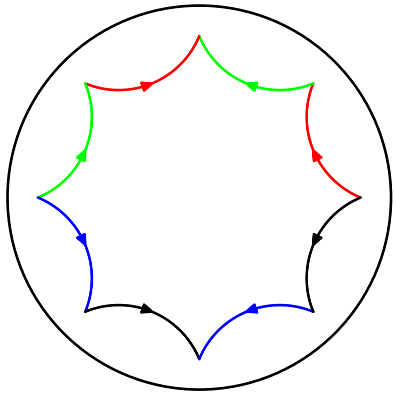


Figure 6.6: The genus 2 standard polygon diagram can be drawn in the hyperbolic plane so that all edges have equal length and all interior angles are $\frac{\pi}{4}$. This gives rise to a tiling of $\mathbb{H}^2$ by regular 8-gons. The group of coloring preserving symmetries of this tiling is the fundamental group of the 2-torus.

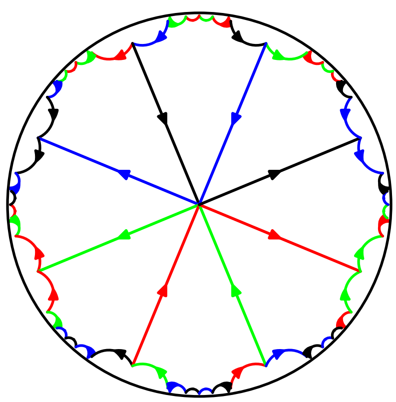


Figure 6.7: The neighborhood of a vertex in a tiling by regular genus 2 standard polygons.

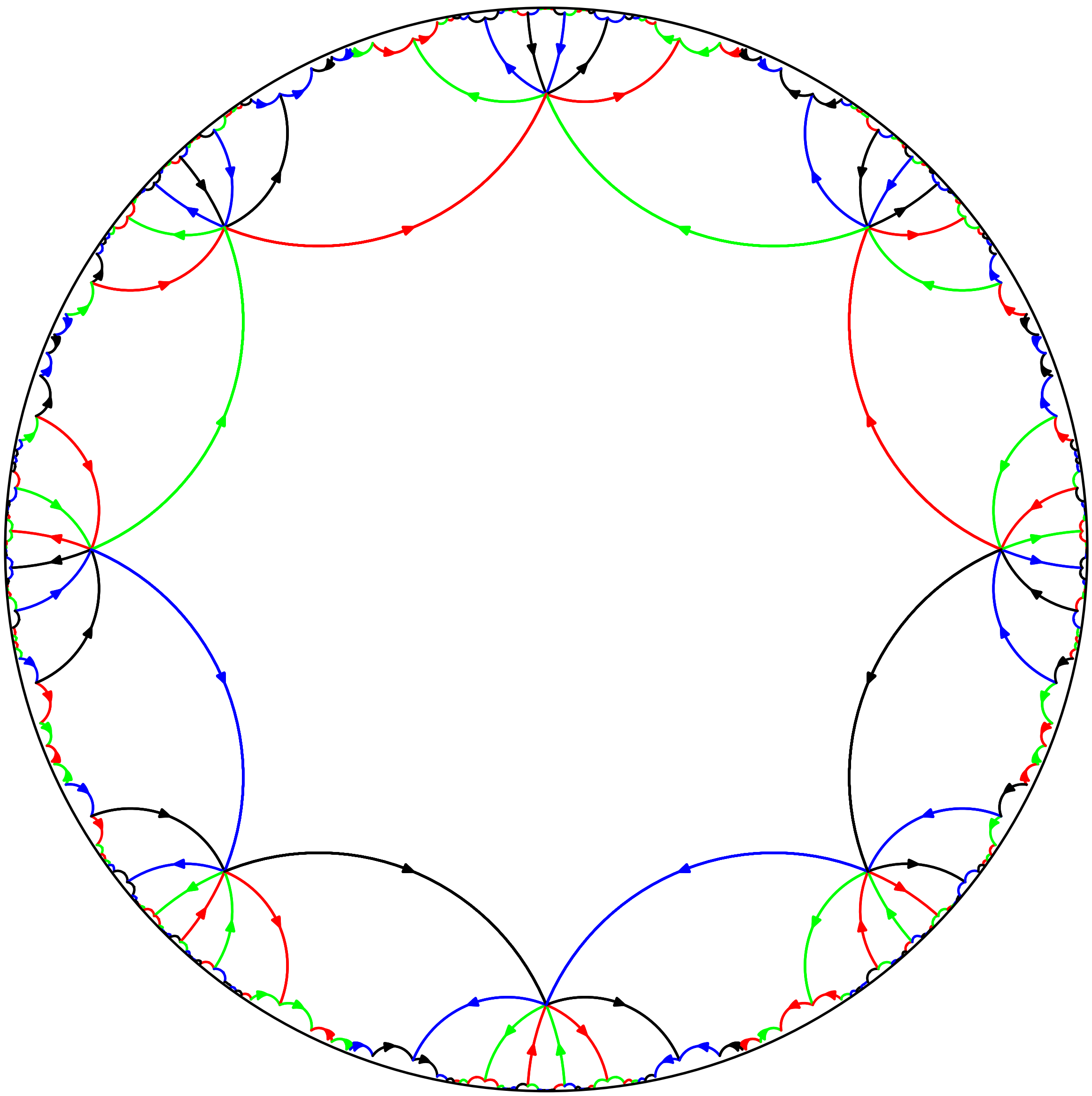
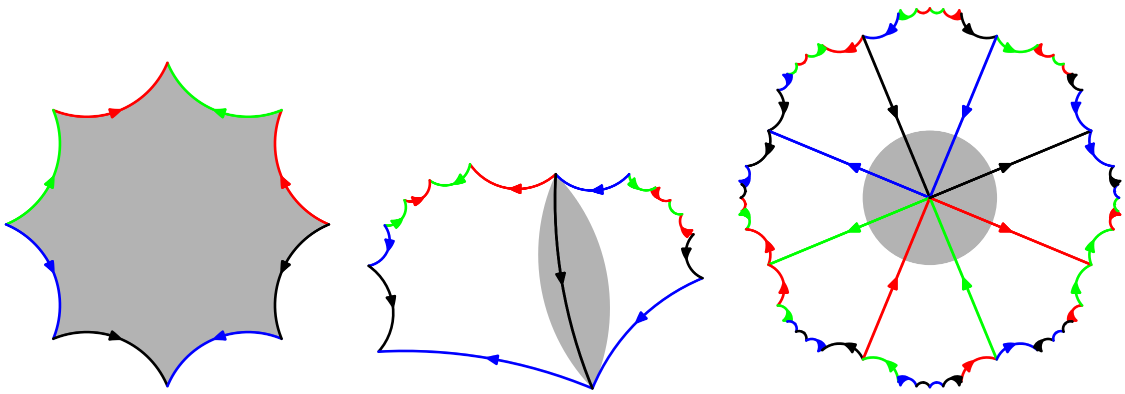


Figure 6.8: An impression of the tiling.



type I type II type III

Figure 6.9: Chart types.

Remark 6.3.3. Although the theorem is stated for polygon diagrams in the hyperbolic plane, it also holds for polygons in the Euclidean plane and even in the sphere. The proof carries over to these cases unchanged.

Exercise 6.3.4. Let Σ be a surface, let P be a polygon, and let $f : P \to \Sigma$ be a map that realizes Σ by identifying the edges of P in pairs. Prove that the universal cover $\tilde{\Sigma}$ is naturally tiled with copies of P that intersect only along their boundaries.

Proof of (6.3.1). Let Σ be the surface defined by D and let $\tilde{\Sigma}$ its universal cover. By (6.3.4), $\tilde{\Sigma}$ is tiled by topological copies of D in the way the theorem requires. Our strategy will be to put a hyperbolic structure on $\tilde{\Sigma}$ and prove that it is isometric to $\mathbb{H}^2$.

Recall that the tiles are defined as lifts $D \to \tilde{\Sigma}$ that take D homeomorphically to its image. Moreover, these lifts make the following diagram commute:

$$\begin{tikzcd} & \tilde{\Sigma} \arrow[d] \\ \mathbb{H}^2 \supset D \arrow[ur] \arrow[r] & \Sigma \end{tikzcd}$$

Now, we define a hyperbolic structure by three types of charts; see figure 6.9.

type I: In the interior $D^{(2)}$ of a tile $\tilde{D}$, we use the fact that we have a continuous inverse $\varphi_{\tilde{D}} : D^{(2)} \to D \subset \mathbb{H}^2$, which we declare to be a chart map. We say that the interior of D is the *defining piece* for the type I charts.

type II: The second type of charts will give us neighborhoods of edges. Let a be an edge-color and let e and e_- be the two open edges of color a. Fix two disjoint neighborhood U_{e_+} and U_{e_-} of e_+ and e_- in D that do not contain the end points of these edges. We form a subset $U_a \subset \mathbb{H}^2$ by gluing together two hyperbolic translates of U_{e_+} and U_{e_-} – recall that D is drawn in $\mathbb{H}^2$ and that e_+ and e_- have the same length. Note that each edge e' in $\tilde{\Sigma}$ of color a has a neighborhood $V_{e'}$ that is homeomorphically identified with U via the map $U_{e_+} \cup U_{e_-} \to U_a$. The induced homeomorphisms

$$\varphi_{e'} : V_{e'} \to U_a$$

are our second collection of coordinate charts. We say that the open sets U_{e_+} and U_{e_-} are the defining pieces for the type II charts.

type III: To define the third type of charts, fix a positive real number R such that the hyperbolic discs of radius R around all the corners of $D \subset \mathbb{H}^2$ are disjoint. Now fix an equivalence class (vertex-colors) a of corners in D. Translate the open R-neighborhood of these corners in the hyperbolic plane so that they form a local picture $U \subset \mathbb{H}^2$ for neighborhoods V_w of vertices w of color a in $\tilde{\Sigma}$. The canonical homeomophisms

$$\varphi_w : V_w \to U_a$$

will be our chart. The R-neighborhood of corners in D are the defining pieces for the type III charts.

The domains of these charts form an open cover of $\tilde{\Sigma}$. This follows since the defining pieces form an open cover of D Note that charts of type II and III are assebled by moving pieces of D via hyperbolic isometries. It follows that coordinate changes are hyperbolic isometries. Thus, we have defined a hyperbolic structure on $\tilde{\Sigma}$. Deck transformations of $\tilde{\Sigma}$ move the tiles and respect the gluing pattern. Thus, by construction of the hyperbolic structure, deck transformations become isometries with respect to this structure. Equivalently, we could say that we have, in fact, constructed a hyperbolic structure on Σ.

The hyperbolic structure on $\tilde{\Sigma}$ is complete. This follows since the cover of D by the defining pieces has a Lebesgue number. As a consequence, we infer that the simply connected cover $\tilde{\Sigma}$ is isometric to $\mathbb{H}^2$. Thus, the tiling of $\tilde{\Sigma}$ is the tiling of $\mathbb{H}^2$ that we were looking for. **q.e.d.**

6.4 The Dehn-Nielsen Theorem for Higher Genus Surfaces

Exercise 6.4.1. Prove: In a closed surface with a fixed hyperbolic structure, every closed curve is freely homotopic to a unique closed geodesic – here, a closed geodesic need not be simple.

Defintion. Let G be a group with a fixed generating system Σ. The Cayley graph $\Gamma_\Sigma(G)$ is a directed graph whose vertices are the elements of G. For each vertex g and each generator $x \in \Sigma$, there is an edge from g to gx. We ignore the orientation of these edges and define a metric on the vertex set by declaring all edges to have length 1: The metric

$$d_\Sigma : G \times G \to \mathbb{R}$$

is then given by shortest paths – note that $\Gamma(G)$ is connected since Σ generated G.

Exercise 6.4.2. Let G and H be groups generated by the *finite* generating sets Σ and Ξ, respectively. Let $\varphi : G \to H$ be a group