$$\begin{tikzcd} \tilde{\Sigma} \arrow[r, "\tilde{\zeta}"] & \mathbb{H}^2 \\ \tilde{\Sigma} \arrow[u, "\tau"] \arrow[r, "\tilde{\zeta}"'] & \mathbb{H}^2 \arrow[u, "\eta(\tau)"'] \end{tikzcd}$$

commutes for each deck transformation τ. Thus, we use $\tilde{\zeta}$ to define a hyperbolic structure on $\tilde{\Sigma}$, which visibly descends to a hyperbolic structure $\mathcal{H}$ on Σ. The homeomorphism $\tilde{\zeta}$ is a developing map for $\mathcal{H}$, and using this developing map, we see that η is the holonomy representation induced by the hyperbolic structure $\mathcal{H}$. **q.e.d.**

6.6 Short Geodesics

We will prove that short simple closed geodesics on a closed hyperbolic surface either coincide or are disjoint. Let us first give a quick an dirty reason why something like that should be true.

Proposition 6.6.1. *Let Σ be a closed hyperbolic surface and let γ_1 and γ_2 be two non-homotopic simple closed geodesics of length $< \operatorname{arcosh}(\frac{5}{4})$. Then, $\gamma_1 \cap \gamma_2 = \emptyset$.*

Proof. Suppose the two loops had an intersection point. We look at the universal cover $\mathbb{H}^2$. We lift the point of intersection and we lift the loops to intersecting geodesic lines, which are the axes of the corresponding deck transformations. The lengths of the loops are precisely the displacements of the two deck transformations.

Let us apply both deck transformations to both geodesics. The key idea is that we cannot obtain a quadrilateral since it had to be a parallelogram as isometries preserve angles. Since there are no parallelograms in the hyperbolic plane, we obtain a contradiction. Thus, the displacements have to be large enough to ensure that the shifted geodesics to not intersect. In terms of figure 6.12, this means that the left picture is forbidden. The right picture is the extreme case of what is just barely permissible.

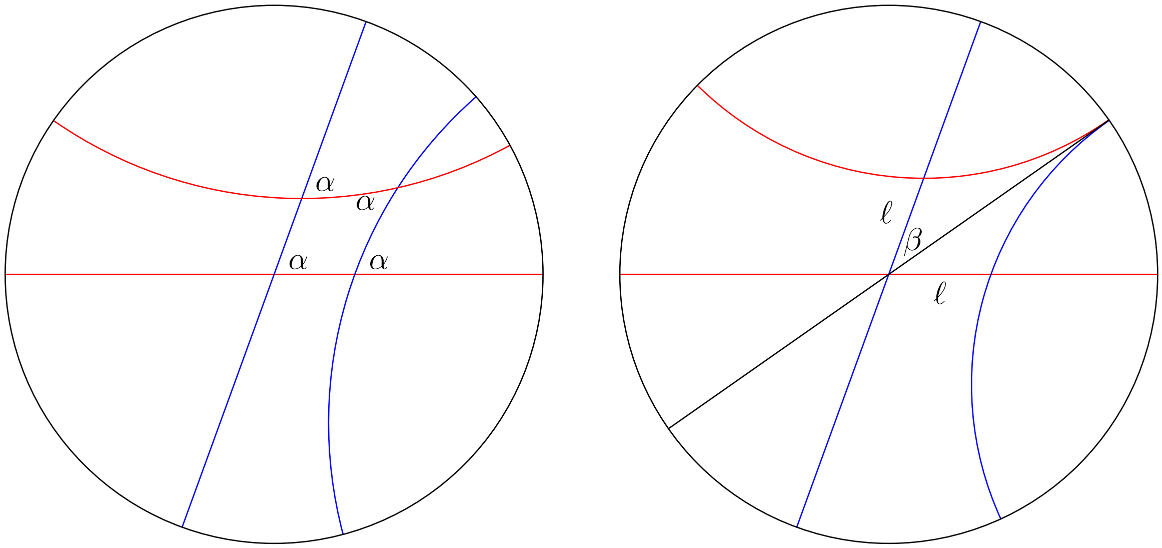
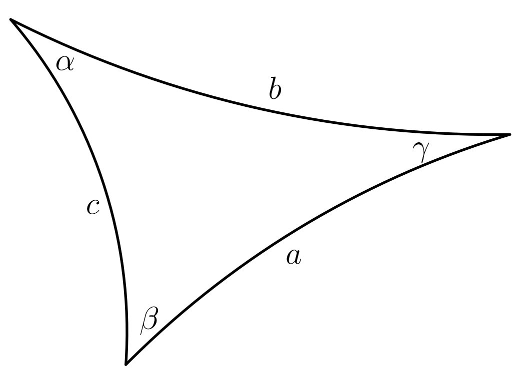


Figure 6.12: Geodesic parallelogram.

To work out the numbers, we consider the right hand of figure 6.12. We want to find a number ℓ such that the red and blue geodesics form a quadrilateral in $\mathbb{H}^2$ provided both displacements are strictly less than ℓ. First observe that shrinking one displacement even further will make intersections even more likely. Thus, we may assume that both displacements in the just permissible picture equal ℓ.

This minimum displacement ℓ for which one vertex of the quadrilateral lies on the boundary, depends on the angle β. To compute it, we need the following formula from hyperbolic trigonometry:

$$\cosh(c) = \frac{\cos(\alpha)\cos(\beta) + \cos(\gamma)}{\sin(\alpha)\sin(\beta)}$$



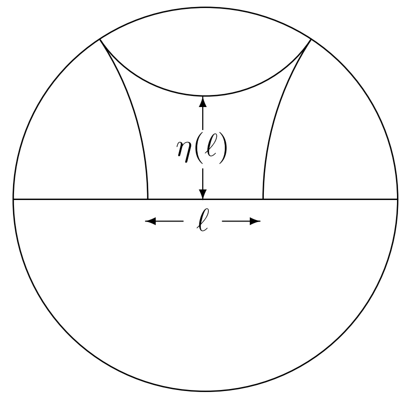


Figure 6.13: The width function.

We obtain:

$$\cosh(\ell) = \frac{\cos(\beta)\cos(\pi - 2\beta) + 1}{\sin(\beta)\sin(\pi - 2\beta)}.$$

It turns out that this function is increasing and can be extended continuously at $\beta = 0$. The value is found by L'Hospital's rule and yields the estimate

$$\cosh(\ell) \geq \frac{5}{4}.$$

Verification of this claim is most conveniently done using a computer algebra system. **q.e.d.**

Now we follow John H. Hubbard and derive the real thing.

Definition 6.6.2. The width function $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as follows. For any positive real number ℓ, draw a line segment of length ℓ on you favorite geodesic in $\mathbb{H}^2$. At its endpoints, draw the perpendiculars and extend them into one side of the hyperbolic plane until they hit the boundary. This way, you obtain two points on the boundary, one for each perpendicular. Join these boundary points by a geodesic. The distance from this geodesic to your favorite one is $\eta(\ell)$. (See figure 6.13.)

Observation 6.6.3. *The width function is monotonically decreasing.* **q.e.d.**

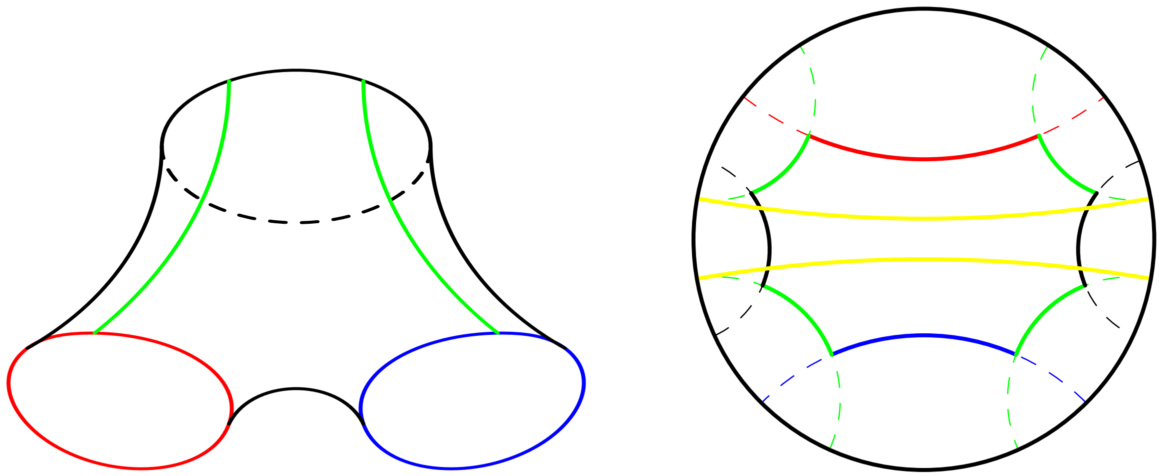


Figure 6.14: A hyperbolic pair of pants with cut lines.

Lemma 6.6.4. *In a hyperbolic pair of pants with totally geodesic boundary circles γ_1, γ_2, and γ_3 of lengths ℓ_1, ℓ_2, and ℓ_3, respectively, you can draw an annulus of width $\eta(\ell_i)$ around γ_i and all three annuli will be pairwise disjoint.*

Proof. Consider the pair of pants in figure 6.14 and let the red boundary circle be γ_1 and the blue circle be γ_2. Fix shortest lines from the third circle to γ_1 and γ_2. Note that these two green arcs will be geodesics, they will be perpendicular to the boundary, and they will be disjoint. Cut along the green geodesic arcs. We obtain a right-angled octagon as shown in the right figure. Note that the green geodesics do not intersect since they have common perpendiculars. Thus, the two yellow geodesics that determine $\eta(\ell_1)$ and $\eta(\ell_2)$ do not intersect. The claim now follows. **q.e.d.**

Theorem 6.6.5. *Let $\{\gamma_1, \gamma_2, \ldots\}$ be a set of pairwise non-homotopic simple closed geodesics in a closed hyperbolic surface with lengths $\ell_1, \ell_2, \ldots$. Then the open $\eta(\ell_i)$-neighborhoods of the loops γ_i are pairwise disjoint.*

Proof. Extend the set of curves to a complete pair of pants decomposition of the surface and apply (6.6.4). **q.e.d.**

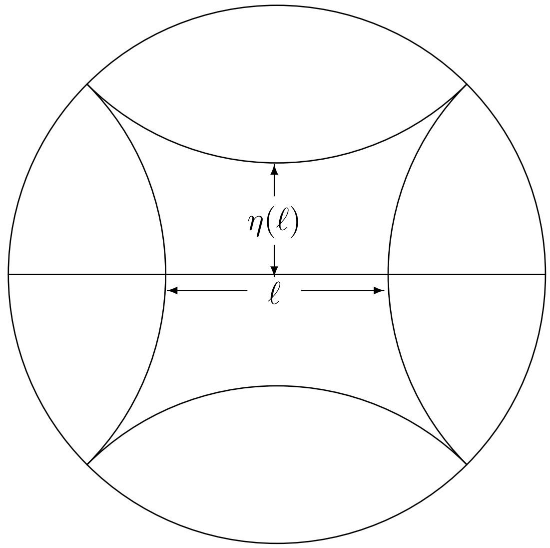


Figure 6.15: Solving $\ell = 2\eta(\ell)$.

Corollary 6.6.6. *Let γ_1 and γ_2 be two simple closed geodesics of lengths ℓ_1 and ℓ_2 on a closed hyperbolic surface. If $\ell_2 < 2\eta(\ell_1)$ then the loops are either disjoint or coincide.*

Proof. Suppose the intersection of the two loops in non-empty but the two loops do not coincide, then γ_2 intersects γ_1 transversally. In this case, it has to pass through the whole $\eta(\ell_1)$-width annulus around γ_1. Thus, $\ell_2 \geq 2\eta(\ell_1)$ as the annulus extends to both sides of γ_1. **q.e.d.**

Exercise 6.6.7. Show that the number $\ln(3+2\sqrt{2})$ is the unique solution to the equation $\ell = 2\eta(\ell)$. (Hint: look at figure 6.15.)

Corollary 6.6.8. *If two non-homotopic simple closed geodesics in a closed hyperbolic surface have both length $< \ln(3+2\sqrt{2})$, then these loops are disjoint.*

Proof. Let γ_1 and γ_2 be two simple closed geodesic curves of length ℓ_1 and ℓ_2. We suppose $\ell_1, \ell_2 < \ln(3+2\sqrt{2})$. Thus, we have

$$\ell_2 < \ln(3+2\sqrt{2}) < 2\eta(\ell_1)$$

whence the claim follows from (6.6.6). **q.e.d.**