Important Words – a Dictionary

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This is a preliminary draft. Comments, suggestions and, in particular, corrections are welcome.

Properties

Let G be a group.

Definition 1. G is finite if it has only finitely many elements.

Definition 2. G is torsion if every element has finite order.

Definition 3. G is torsion free if no element has finite order.

Definition 4. G is cyclic if it is generated by one element.

Definition 5. G is indicable if it admits an epimorphism onto the infinite cyclic group.

Definition 6. G is abelian if the commutator subgroup is trivial.

Definition 7. G is s-nilpotent if it is trivial or the commutator subgroup is $(s-1)$-nilpotent.

Definition 8. G is nilpotent if it is s-nilpotent for some s.

Definition 9. G is simple if it has at most two normal subgroups.

Definition 10. G is linear if it has a faithfull representation of finite dimension over some field (not necessaryly of characteristic 0).

Definition 11. G is of type F if there is a finite Eilenberg-Maclane space $\mathrm{K}(G,1)$. An Eilenber-Maclane space for G is a CW-complex with fundamental group G and a contractible universal cover (equivalentlty, one may require the higher fundamental groups to vanish).

Definition 12. G is finitely generated if G has a finite generating system.

Definition 13. G is finitely presented if G has a finite generating system and all relations among the generators follow from a finite set of relations.

Definition 14. G has finiteness length $\leq m$ if there is an Eilenberg-Maclane space $\mathrm{K}(G,1)$ with finite m-skeleton.

Definition 15. G has solvable word problem if is is finitely generated and for a fixed generating set there is an algorithm that decides whether a given word in the generators represents the identity.

Definition 16. G has solvable conjugacy problem if it is finitely generated and for a fixed generating set there is an algorithm that decides whether two given words represent conjugated elements.

Definition 17. G has Serre's Property FA if every action of G on a tree has a global fixed point.

Definition 18. G has Kashdan's Property (T) if if every unitary representation that has almost invariant vectors has an invariant vector. Here a unitary representation of a group G on a Hilbert space $\mathcal{H}$ is said to have almost invariant vectors, if, for any finite subset $K \subseteq G$ and any $\varepsilon > 0$, there is a unit vector $\mathbf{u} \in \mathcal{H}$ satisfying

$$|g\mathbf{u} - \mathbf{u}| < \varepsilon.$$

Definition 19. G is amenable if there is a left-invariant finitely additive probabilty measure defined on all subsets of G.

Definition 20. G has polynomial growth if it is finitely generated and the volume of a ball in the Cayley graph grows polynomially with the radius.

Definition 21. G has exponential growth if it is finitely generated and the volume of a ball in the Cayley graph grows expoentially with the radius.

Definition 22. G has intermediate growth if it is finitely generated and has neither polynomial nor exponential growth. (Note that growth is at least polynomial and at most exponential.)

Definition 23. G satisfies a quadratic isoperimetric inequality if if has a finite presentation $\mathcal{P}$ and there is a quadratic function $q : \mathbb{N} \to \mathbb{N}$ such that any loop of length l in the Cayley complex for $\mathcal{P}$ bounds a disc of area $\leq q(l)$.

Prefixes

Let "blah", "foo", and "bar" be properties, and let G be a group.

Definition 24. G is foo–by–bar if there is a short exact sequence

$$1 \to H \to G \to F \to 1$$

where H is foo and F is bar.

Remark: You might find something like foo–by–bar–by–blah, which means (foo-by–bar)–by–blah.

Definition 25. G is meta-blah if it is blah–by–blah.

Definition 26. G is virtually blah if G has a blah-subgroup of finite index.

Definition 27. To put this differently, G is residually blah, if every non-trivial element of G maps non-trivially to at least one blah-quotient of G.

Definition 28. G is locally blah if every finitely generated subgroup of G is blah.

Definition 29. G is poly-blah if there is a subnormal series

$$1 = G_1 \trianglelefteq G_2 \trianglelefteq G_3 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = G$$

with all quotients G_{i+1}/G_i being blah.

Example 30. Some well known concepts are just shorthands:

- solvable = poly-abelian
- coherent = locally finitely presented

Metaproperties

Definition 31. Let "blah" be a property. It is called subgroup closed if every subgroup of a blah group is blah.

It is quotient closed if every epimorphic image of a blah group is blah.

It is called extension closed if every meta-blah group is blah.

Remark 32. The following properties are subgroup closed:

- finite
- free
- cyclic
- abelian
- nilpotent

- solvable
- amenable

Exercise 33. *Prove: If foo and bar are subgroup closed, so is foo–by–bar.*

Remark 34. The following properties are quotient closed:

- finite
- abelian
- nilpotent
- solvable
- finitely generated
- Serre's Property FA

Remark 35. The following properties are extension closed:

- finite
- solvable
- finitely generated
- finitely presented
- finiteness length $\leq m$
- finite type
- Serre's Property FA

The Unamed Famous

There is a bunch of theorem that fit into the following schemes:

- If $\ldots$, then G has only finitely many conjugacy classes of finite subgroups.
- If $\ldots$, then all solvable subgroups of G are finitely generated and virtually abelian.
- If $\ldots$, then all elements of infinite order in G have translation length bounded away from 0.