Proof. Since F is infinite, any finite quotient is proper whence abelian. Therefore, the elements of the commutator subgroup cannot be separated from the identity in any finite quotient. The commutator subgroup, however, is non-trivial – in fact, it is infinite. **q.e.d.**

Exercise 4.42. Show that F is torsion-free.

Theorem 4.43. *Every non-abelian subgroup of F contains a copy of $\mathbb{Z} \times \mathbb{Z}$.*

Proof. Let f and g be two elements that do not commute. We have to find a copy of $\mathbb{Z} \times \mathbb{Z}$ in the subgroup $\langle f, g \rangle$.

Since f has only finitely many break points, the closed set

$$\{t \mid f(t) = t\}$$

decomposes into finitely many intervals. As the same holds for g, it follows that the open set

$$\{t \mid f(t) \neq t \text{ or } g(t) \neq t\}$$

decomposes into finitely many open intervals $I_0, \ldots, I_r$.

We consider the commutator $f_0 := [f, g]$. We will find a conjugate of f_0 inside $\langle f, g \rangle$ that commutes with f_0. The key idea is to conjugate such that the support of f_0 in I_0 is moved off itself, which is possible since this homemomorphism f_0 is the identity in small neighborhoods of the endpoints of I_0. So we only have to prove the existence of this conjugating element.

Let t be any point in I_i. Obviously, $\inf \langle f, g \rangle \cdot t$ is a global fix point for the action of $\langle f, g \rangle$ in the closure of I_i. Clearly, it has to be $\inf I_i$. Hence this points is a limit point of the orbit of t. Hence the problem in the preceding paragraph has a solution and we can conjugate f_0 as to move its support off itself inside I_0. Call this conjugate g_0.

Now, consider $f_1 := [f_0, g_0]$. By construction, this element is the identity on I_0. So we run the same argument as above in the interval I_1. This way, we can take care of all intervals. Eventually, we have two elements f_r and g_r that commute. **q.e.d.**

Corollary 4.44. *F does not contain non-abelian free groups.* **q.e.d.**

4.3 Amenability

It is not known whether F is amenable. However, if F is amenable, it is at least not obviously so.

Theorem 4.45. *Thompson's group F is not elementary amenable.*

Proof. We prove $F \notin EG_\alpha$ for all α. And, of course, this is done by transfinite induction or contradiction. We proceed by contradiction. So assume there was an ordinal α with $F \in EG_\alpha$. Then we can take α to be minimal with this property.

Since F is neither abelian nor finite, $\alpha > 0$ and so F sneaked into EG_α as an extension or as a direct union of groups in lower strata. The direct union possibility is ruled out since F is finitely generated. Thus, we have to deal with ways of representing F as an extension.

Let $N_\beta \hookrightarrow F \twoheadrightarrow Q_0$ be a short exact sequence where Q_0 is a proper quotient. By (4.34), Q_0 is abelian. Hence $N_\beta \in EG_\beta$ contains $[F, F]$.Hence $[F, F] \in EG_\beta$. However, since $[F, F]$ contains a copy of F, it follows that $F \in EG_\beta$. Thus, α was not minimal. **q.e.d.**

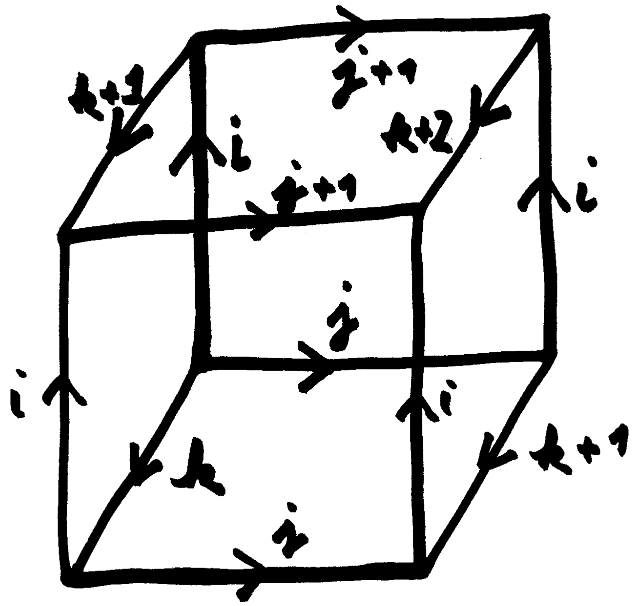
4.4 Finiteness Properties

We will construct a very nice (i.e., contractible) cube complex upon which F acts. The construction is due to K.S. Brown and R. Geoghegan [BrGe84].

This construction starts from the Cayley complex for the infinite presentation

$$\tilde{F} := \langle x_0, x_1, \ldots \mid x_m x_i = x_i x_{m+1} \text{ for } i < m \rangle .$$

Thus, we have a vertex for each element in F. From each vertex countably many edges issue, and the relations give rise to squares. This complex $\Gamma_{\tilde{F}}$ is simply connected. Our goal is to fill in cubes as to kill off higher homotopy groups. To do this, we fill in all the obvious cubes: For any square whose edges are labeled by j, k and $k+1$ – we assume $j<k$ –, and any $i<j$, we have a cube



We continue in higher dimensions. Thus for each vertex in $\Gamma_{\tilde{F}}$ and each tupel $(q_0 < q_2 < \ldots < q_r)$, we have a cube issuing from that vertex whose edges are labeled by the indices q_i. Note that each of these cubes has a unique vertex with all edges pointing away (the <u>source</u>) and a unique vertex with all edges coming in (the <u>sink</u>). Let Y denote the cube complex constructed this way, and put $X := F \backslash Y$.

Theorem 4.46 ([BrGe84]). *Y is contractible.*

Proof. Let P be the monoid of positive elements in F and let Y_+ be the subcomplex of Y spanned by $P \subset Y$. This is the complex formed by all those cubes whose sink is in P.

Claim A. Y_+ is contractible.

Proof. We will do Morse theory. There is an obvious height function on the vertices: the length of the positive word – or alternatively, the number of non-leaf nodes in the forest representing that vertex. Moving along an outgoing edge increases the height by 1. Thus, the height extends linearly to the cubes. This is a combinatorial Morse function as it is non-constant on edges. Since the minimal set is just one point (the identity vertex), contractibility of Y_+ would follow from contractibility of descending links ().

The descending link is the part of the link in a vertex v spanned by its incoming edges. These correspond to splitting off a generator on the right, i.e., deleting a terminal caret in the forest. Thus we can tell which incoming edges issue from vertices in Y_+. More importantly, we see that we can delete all terminal carets and find a source in Y_+ which gives rise to a simplex in the link of v that connects all the descending vertices. Thus, we proved that the descending link is a simplex and therefore contractible. The claim follows. □

Now we consider the cover of Y by translates fY_+.

Claim B. For any finite set of elements $f_0, \ldots, f_m \in F$, there is an element $f \in F$ such that

$$f_0 Y_+ \cap \cdots \cap f_m Y_+ = f Y_+.$$

Proof. This follows from the corresponding claim about translates fP which in turn follows by induction from the claim that for any $f \in F$, there is a $g \in P$ such that

$$P \cap fP = gP.$$

Writing f as a pair of trees, we see that in order to find a positive right multiple of f, we first have to annihilate the bottom tree. Afterward, we will end up with a right multiple of the top tree. Thus, g is the element represented by the top tree of f. $\square$

From this claim, it follows at once, that the translates fY_+ form a cover of Y by contractible subcomplexes such that any intersection of these subcomplexes is contractible. The nerve of the cover is the simplex with vertex set F. Hence, by (B.11), the space Y has the homotopy type of a big simplex. **q.e.d.**

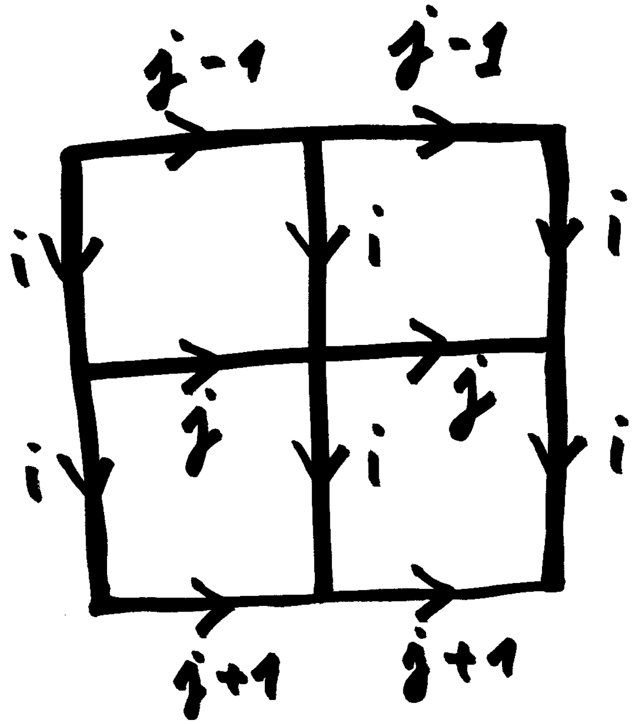
Theorem 4.47. *Y is CAT(0).*

Proof. The edges around each vertex v come in pairs: For each generator x_i, there is one incoming edge and one outgoing edge, both labeled by i. We denote the incoming edge by the pair $(i,1)$ and the outgoing edge by $(i,1)$. These pairs represent the vertices in $\mathrm{Lk}\,(v)$. In order to determine the simplices, we have to know which of these edges span a cube. Obviously, there are no cubes that involve both, $(i,0)$ and $(i,1)$. Hence we can represent cubes in the star of v by tupels

$$((q_1, \varepsilon_1), \ldots, (q_1, \varepsilon_1))$$

with $q_i < q_{i+1}$.

The following picture assumes $i < j$ and shows the edge labels of all possible squares involving these to labels.



The incoming edges only can form a square if $i < j - 1$. In general, a tupel

$$((q_1, \varepsilon_1), \ldots, (q_1, \varepsilon_1))$$

of edges spans a cube if $q_i < q_{i+1} - \varepsilon_i$.

This condition, however, is satisfied for a tupel if and only if it is satisfied for all of its subtupels of size two. This is, a set of vertices in $\mathrm{Lk}(v)$ spans a simplex if and only if any two of them are joined by an edge. Thus, $\mathrm{Lk}(v)$ is a flag complex. It follows from Gromov's lemma that Y is CAT(0) provided it is simply connected. That, in turn follows from the fact that the 2-skeleton of Y is a Cayley complex for F. **q.e.d.**

We give a description of the quotient X. This is also a cube complex. X has one vertex. The edges in the 1-skeleton are indexed by non-negative integers. We index higher dimensional cubes by the indices of the edges pointing toward the sink of the cube. Then, an ascending sequence

$$\mathbf{q} := (q_1, \ldots, q_m)$$

describes a cube if and only if $0 \leq F_1$ and $q_i + 2 \leq q_{i+1}$. The $2m$ faces of the cube $\mathbf{q}$ are given by the boundary operators

$$\begin{aligned} \partial_i^+ \mathbf{q} &:= (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots q_m) \\ \partial_i^- \mathbf{q} &:= (q_1, \ldots, q_{i-1}, q_{i+1} - 1, \ldots q_m - 1). \end{aligned}$$

Theorem 4.48. *X is homotopy equivalent to s CW-complex that has one vertex and two cells in each dimension ≥ 1.*

Proof. We distinguish three types of cells. For each dimension ≥ 1, we have two essential cells $(0, 3, \ldots, 3m-3)$ and $(1, 4, \ldots, 3m-2)$. We call an m-cube $\mathbf{q} = (q_1, \ldots, q_m)$ collapsible if there is an index i such that $q_{i+1} \neq q_i + 3$ and such that, for the last index j with that property, $q_{j+1} = q_j + 2$. In this case

$$\partial_j^+ \mathbf{q} = (q_1, \ldots, q_{j-1}, q_{j+1}, \ldots q_m)$$

is called the free face of the collapsible cube $\mathbf{q}$. Cells that are neither essential nor collapsible are considered redundant.

Observe that the free face $\partial_j^+ \mathbf{q}$ of a collapsible cube $\mathbf{q}$ is redundant since $q_{j+1} \geq q_{j-1} + 4$ and $q_{i+1} = q_i + 3$ for all $i > j+1$.

On the other hand, each redundant cell $\mathbf{p} = (p_1, \ldots, p_m)$ is the free face of a unique collapsible cell since for the last index i with $p_{i+1} \neq p_i + 3$, we have $p_{i+1} \geq p_i + 4$ whence

$$(p_1, \ldots, p_i, p_{i+1} - 2, p_{i+1}, \ldots, p_m)$$

is a collapsible cell with free face $\mathbf{p}$ – note that we have to allow $i = 0$ here because of cubes like $(2, 5, 8, \ldots)$.

Now consider the collapsible cube $\mathbf{q} = (q_1, \ldots, \ldots q_m)$ with free face $\partial_j^+ \mathbf{q}$ which is to say that we have

$$q_i = q_j + 3\,(i - j) - 1$$

for all $i > j$. The following statements are easy to see:

1. For $i < j$, the face $\partial_i^- \mathbf{q}$ is collapsible.

2. For $i > j$, the tupel $\partial_i^- \mathbf{q}$ precedes the free face $\partial_j^+ \mathbf{q}$ in the lexicographic order.

3. Also, $\partial_j^- \mathbf{q}$ precedes the free face in the lexicographic order.

We define $X_{-0} := X_{+0} := X_0$, and inductively

$$X_{-m} := X_{+m-1} \cup \text{ essential cells in dimension } m$$
$$X_{+m} := X_{-m} \cup \text{ redundant cells in dimension } m \cup \text{ collapsible cells in dimension } m$$

Obviously, we have

$$X_{+m-1} \subseteq X_{-m} \subseteq X_m \subseteq X_{+m}.$$

Moreover, we have $X_{-m} \simeq X_{+m}$ since the latter is obtained from the former by a transfinite sequence of elementary expansions which are performed according to the lexicographic order. Since the lexicographic order is a well ordering, this actually makes sense.

Now, we are ready to construct the space Z which is homotopy equivalent to X but uses at most two cells per dimension. Put $Z_0 := X_0$. Higher skeleta are constructed by induction together with the homotopy equivalence. So assume we have already a homotopy equivalence

$$\pi_{m-1} : X_{+m-1} \to Z_{m-1}.$$

Put

$$Z_m := Z \cup B_0(m) \cup B_1(m)$$

where the attaching maps are induced by the maps that embed the boundaries of the two essential m-cubes in X_{+m-1}. Thus we obtain a homotopy equivalence

$$\pi_m : X_{-m} \to Z_m$$

and $X_{-m} \simeq X_{+m}$ gives us the equivalence

$$\pi_m : X_{+m} \to Z_m$$

which completes the proof. **q.e.d.**