Example 5.22. The infinite cyclic group is arithmetic. In fact, it is the arithmetic subgroup of the $\mathbb{Q}$-group

$$\begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix}.$$

Fact 5.23 (Mal'cev). *Any torsion free finitely generated nilpotent group is arithmetic.*

Fact 5.24 (from rational homotopy theory). *The mapping class group of any simply connected finite CW-complex is arithmetic. Conversely, any torsion free arithmetic group arises this way.*

5.1 Preliminary Observations

Arithmetic groups are groups of integer matrices with determinant 1. This already implies some properties:

Observation 5.25. *Every finitely generated subgroup of an arithmetic group has a solvable word problem.*

Observation 5.26. $\mathrm{SL}_n(\mathbb{Z})$ *is residually finite.* **q.e.d.**

Corollary 5.27. *Arithmetic subgroups are residually finite.* **q.e.d.**

Exercise 5.28. Prove that $\mathrm{SL}_n(\mathbb{Z})$ is generated by elementary matrices, i.e., matrices that have 1s in the diagonal and precisely one additional 1 in an off-diagonal slot.

Exercise 5.29 (extra credit). Prove that $\mathrm{SL}_n(\mathbb{Z})$ is generated by two elements for $n \geq 5$. Remark: The statement holds for $n \geq 2$. However, a proof of the more general statement distinguishes between n even and n odd.

Exercise 5.30 (Minkowski (1887)). Show that the kernel of the map

$$\mathrm{SL}_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}_p)$$

is torsion free for any odd prime p. Hint: If M is a torsion element of $\mathrm{SL}_n(\mathbb{Z})$ then the roots of its characteristic polynomial are roots of unity. This tells you something about the way it factors over $\mathbb{Z}$.

Corollary 5.31. *Arithmetic groups are virtually torsion free.* **q.e.d.**

5.2 $\mathrm{SL}_2(\mathbb{Z})$ and the Hyperbolic Plane

5.2.1 The Symmetric Space of $\mathrm{SL}_2(\mathbb{R})$

The group $\mathrm{SL}_2(\mathbb{R})$ acts on the complex projective line $\mathbb{P}^1(\mathbb{C})$ in an obvious way. The complex projective line is the Riemann sphere, and since the coefficients of matrices in $\mathrm{SL}_2(\mathbb{R})$ are real, the equator of the Riemann sphere is invariant under this action. Moreover, the action does not swap the northern and southern hemispheres. Hence, there is an induced action on the northern hemisphere – the north pole is i. This action is given by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d}.$$

The kernel of this action is the center of $\mathrm{SL}_2(\mathbb{R})$:

$$\{\mathbb{I}_2, -\mathbb{I}_2\}.$$

The northern hemisphere is a well known model for the hyperbolic plane $\mathbb{H}^2$.

Exercise 5.32. Prove that $\mathbb{H}^2$ has constant curvature -1.

Exercise 5.33. Show that Möbius transformations are isometries of $\mathbb{H}^2$.

Exercise 5.34 (extra credit). Show that any orientation preserving isometry of $\mathbb{H}^2$ is given by a Möbius transformation.

Exercise 5.35. Show that geodesics in $\mathbb{H}^2$ are vertical lines or half circles orthogonal to the real axis.

Definition 5.36. A horizontal line or a circle tangent to the real axis in $\mathbb{H}^2$ is called a horocircle.

Exercise 5.37. Show that the action of $\mathrm{SL}_2(\mathbb{R})$ takes horocircles to horocircles.

Observation 5.38. *Elements of the form* $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ *act as translations that preserve the imaginary part and shift the real part by* b*. These elements are called translations.*

Elements of the form $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ *act act transitively on the imaginary line. They move lines in the upper half plane to parallel lines. These elements are dilations.*

Corollary 5.39. *The action of* $\mathrm{SL}_2(\mathbb{R})$ *on* $\mathbb{H}^2$ *is transitive.* **q.e.d.**

We determine the stabilizer of i: First, we have

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mathrm{i} &= \mathrm{i} \\ a\mathrm{i} + b &= -c + d\mathrm{i} \\ a = d \qquad & b = -c. \end{aligned}$$

Now, the determinant gives:

$$a^2 + b^2 = 1.$$

It follows that the stabilizer of i is precisely the group $\mathrm{SO}(2)$. This is a compact Lie group of rotations. Since these rotations act transitively on the unit tangent vectors at i, we obtain the following strengthening of (5.39)

Observation 5.40. $\mathrm{SL}_2(\mathbb{R})$ *acts transitively on the following sets:*

- *The set of embeddings* $\mathbb{R} \longrightarrow \mathbb{H}^2$.

- *The unit sphere bundle of the Riemannian manifold* $\mathbb{H}^2$.

Theorem 5.41. *Any compact subgroup of* $\mathrm{SL}_2(\mathbb{R})$ *is conjugate to a subgroup of* $\mathrm{SO}(2)\,\mathbb{R}$

Proof. A compact subgroup has bounded orbits. The hyperbolic plane is negatively curved and simply connected. Hence, compact subsets have unique centers – the center of a bounded subset is the center of a minimal covering disk. Since the group acts by isometries (5.33), the compact subgroup fixes the center of any of its orbits. Thus, any compact subgroup fixes a point of the hyperbolic plane.

By transitivity of the action (5.39), we find an element that moves the fixed point to i. This element conjugates the compact subgroup into $\mathrm{Stab}\,i$. **q.e.d.**

Corollary 5.42. $\mathbb{H}^2 = \mathrm{SL}_2(\mathbb{R})\,/\,\mathrm{SO}(2)$.

Definition 5.43. An element $M \in \mathrm{SL}_2(\mathbb{R})$ is called

- elliptic if $|\mathrm{tr}(M)| < 2$,
- parabolic if $|\mathrm{tr}(M)| = 2$, and
- hyperbolic if $|\mathrm{tr}(M)| > 2$.

Observation 5.44. *The characteristic polynomial of*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

is

$$\begin{vmatrix} 1 - xa & xb \\ xc & 1 - xd \end{vmatrix} = x^2 - (a+d)\,x + 1.$$

Hence the matrix $M \in \mathrm{SL}_2(\mathbb{R})$ *has two conjugate complex eigenvalues if it is elliptic. This is to say,* M *fixes a point in* $\mathbb{H}^2$. *In particular,* M *is conjugate to a rotation. If* M *is hyperbolic, it has two real eigenvalues whence it has two fixed points on* $\partial(\mathbb{H}^2) = \mathbb{P}^1(\mathbb{R})$. *Finally, if* M *is parabolic, it has one fixed point on the boundary* $\partial(\mathbb{H}^2)$. **q.e.d.**

5.2.2 A Fundamental Domain for $\mathrm{SL}_2(\mathbb{Z})$

Definition 5.45. Let G act on $\mathbb{H}^2$ by isometries. A strong fundamental domain for the action is a subset $D \subseteq \mathbb{H}^2$ such that every G-orbit has precisely one point in D.

Observation 5.46. *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. *Then we have:*

$$\begin{aligned} M\frac{-d \pm 1}{c} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{-d \pm 1}{c} \\ &= \frac{-ad \pm a + bc}{-cd \pm c + dc} \\ &= \frac{-1 \pm a}{\pm c} \\ &= \frac{a \mp 1}{c}. \end{aligned}$$

This computation shows that M *takes the half circle of radius* $\frac{1}{|c|}$ *centered at* $\frac{-d}{c}$ *to the half circle of the same radius centered at* $\frac{a}{c}$. *Since* M *preserves the orientation, we see that*

$$M\left\{ z \in \mathbb{H}^2 \,\middle|\, \left| z - \frac{-d}{c} \right| < \frac{1}{|c|} \right\} = \left\{ z \in \mathbb{H}^2 \,\middle|\, \left| z - \frac{a}{c} \right| > \frac{1}{|c|} \right\}$$

and

$$M\left\{ z \in \mathbb{H}^2 \,\middle|\, \left| z - \frac{-d}{c} \right| > \frac{1}{|c|} \right\} = \left\{ z \in \mathbb{H}^2 \,\middle|\, \left| z - \frac{a}{c} \right| < \frac{1}{|c|} \right\}.$$

Corollary 5.47. *Fix* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$. *The* $\langle M \rangle$*-orbit of any point intersects*

$$\left\{ z \in \mathbb{H}^2 \,\middle|\, \left| z - \frac{-d}{c} \right| \geq \frac{1}{|c|} \text{ or } \left| z - \frac{a}{c} \right| > \frac{1}{|c|} \right\}$$

in at most one point.

Put

$$\overline{D} := \left\{ z \in \mathbb{H}^2 \,\middle|\, |z| \geq 1 \text{ and } -\frac{1}{2} \leq \Re(z) \leq \frac{1}{2} \right\}$$

and

$$\Sigma := \left\{ M \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, M\overline{D} \cap \overline{D} \neq \emptyset \right\}.$$

We shall first determine Σ. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. If there are two points $x, y \in \overline{D}$ with

$$Mx = y$$

then (5.47) implies $c \in \{-1, 0, 1\}$.

<u>$c = 0$</u>: Since $\mathrm{Det}(M) = 1$, we have $a = d = \pm 1$. It follows that M either acts as the identity or as a translation. Now $b \in \{-1, 0, 1\}$ follows, and we obtain

$$M \in \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \right\}.$$

It is obvious that all these matrices belong to Σ.

<u>$c = \pm 1$</u>: There are only three half circles of radius $1 = \frac{1}{|c|}$ centered at integer points that intersect $\overline{D}$ non-trivially. Thus (5.47) implies $a, d \in \{-1, 0, 1\}$. Once we pick a and d the last entry b is determined by $\mathrm{Det}(M) = 1$. Thus, we have the following candidates:

$$\pm \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \pm \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \quad \pm \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}, \quad \pm \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix},$$

$$\pm \begin{pmatrix} \pm 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & -1 \\ 1 & \pm 1 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

By inspection, one establishes that the following matrices actually belong to Σ:

$$\pm \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \pm \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \pm \begin{pmatrix} \pm 1 & -1 \\ 1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & \pm 1 \end{pmatrix}, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence we have

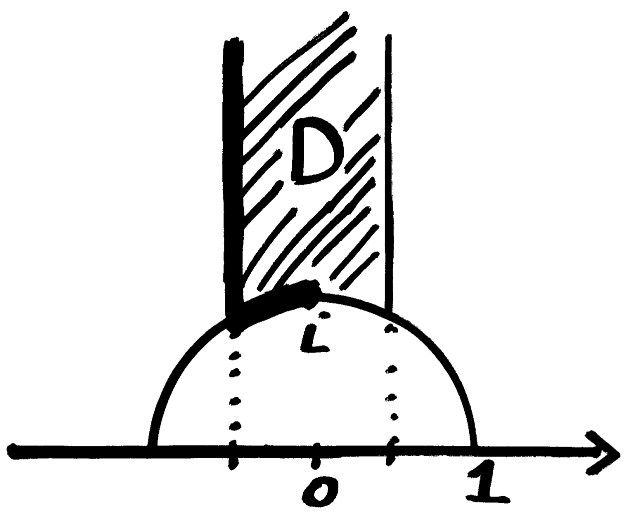
$$\Sigma = \left\{ \begin{array}{c} \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \\ \pm\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \pm\begin{pmatrix} \pm 1 & -1 \\ 1 & 0 \end{pmatrix}, \pm\begin{pmatrix} 0 & -1 \\ 1 & \pm 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{array} \right\}$$

Exercise 5.48. Show that, for any $M \in \mathrm{SL}_2(\mathbb{Z})$,

$$M\overline{D} \cap \overline{D}$$

does not contain a non-empty open subset of $\mathbb{H}^2$.

Let us define a subset D of $\overline{D}$ by excluding the right boundary and the open right half of the bottom boundary. Thus D is given by the following picture:



Lemma 5.49. $\mathbb{H}^2 = \mathrm{SL}_2(\mathbb{Z})\, D$.

Proof. We claim that the following algorithm eventually moves every $z \in \mathbb{H}^2$ into D. Put $z_0 := z$ and define two sequences of points by the following rules:

- Let

  $$z_i' := z_i + n$$

  where $n \in \mathbb{Z}$ is chosen such that

  $$-\frac{1}{2} \le \Re(z_i') < \frac{1}{2}.$$

• Put

$$z_{i+1} := \begin{cases} -\frac{1}{z_i'} & \texttt{for } |z_i'| < 1 \\ z_i' & \texttt{otherwise.} \end{cases}$$

Note that $z_s' \in \mathrm{SL}_2(\mathbb{Z})\, z_i$ and $z_{s+1} \in \mathrm{SL}_2(\mathbb{Z})\, z_i'$. Hence it suffices to prove that, eventually, $z_i \in D$.

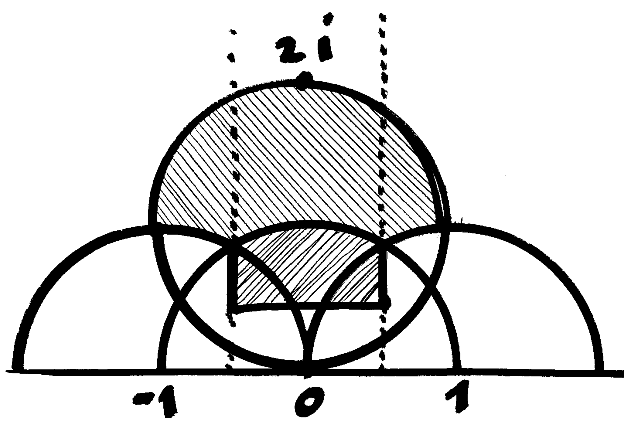
First observe that

$$\Im(z_i') \leq \frac{1}{2}$$

implies

$$|z_{i+1}| \geq 2\,|z_i'|$$

since $|\Re(z_i')| \leq \frac{1}{2}$. Thus, we have $\Im(z_i') = \Im(z_i) > \frac{1}{2}$ for i large enough. for such an i, let $z_i' = x + \mathrm{i}y$ and assume $|z_i'| < 1$. We claim $z_{i+1} \in D$. This is apparent from the following picture:



which is valid by (5.37). **q.e.d.**

Our discussion so far can be summarized as follows:

Proposition 5.50. *The collection of closed subsets $M\overline{D}$ where $M \in \mathrm{SL}_2(\mathbb{Z})$ forms an $\mathrm{SL}_2(\mathbb{Z})$-invariant tiling of $\mathbb{H}^2$ with ideal triangles.* **q.e.d.**

From this, we can derive a good deal of information about $\mathrm{SL}_2(\mathbb{Z})$.

Theorem 5.51. *The group* $\mathrm{SL}_2(\mathbb{Z})$ *has only finitely many conjugacy classes of finite subgroups.*

Proof. Let $F \leq \mathrm{SL}_2(\mathbb{Z})$ be finite. Then there is a point $x \in \mathbb{H}^2$ such that

$$Fx = x.$$

Chose $M \in \mathrm{SL}_2(\mathbb{Z})$ such that $Mx \in \overline{D}$. Then

$$MFM^{-1} \subseteq \Sigma.$$

But Σ is finite. **q.e.d.**

Exercise 5.52. Let G be a residually finite group that has only finitely many conjugacy classes of finite subgroups. Show that G is virtually torsion free.

Corollary 5.53. $\mathrm{SL}_2(\mathbb{Z})$ *is virtually torsion free.*

Theorem 5.54. *The group* $\mathrm{SL}_2(\mathbb{Z})$ *is finitely presented.*

Proof. Let U be a contractible open neighborhood of $\overline{D}$ contained the union of all tiling triangles that intersect $\overline{D}$. Then (A.10) applies. **q.e.d.**

We also note

Proposition 5.55. *Show that* D *is a strong fundamental domain for* $\mathrm{SL}_2(\mathbb{Z})$ *in* $\mathbb{H}^2$.

Proof. By (5.49), half of the claim is already proved. Thus we only have to show that no two points in D are in the same $\mathrm{SL}_2(\mathbb{Z})$-orbit. This is done by inspection of the elements in Σ. **q.e.d.**