Chapter 1

Coxeter Groups and Artin Groups

1.1 Artin Groups

Let M be a Coxeter matrix with index set S. The Artin group defined by M is given by the presentation:

$$A_M := \left\langle s \in S \;\middle|\; \underbrace{sts\cdots}_{m_{s,t}\text{ factors}} = \underbrace{tst\cdots}_{m_{s,t}\text{ factors}} \right\rangle.$$

The Coxeter matrix M defines a Coxeter group W_M at the same time. The canonical homomorphism

$$A_M \;\to\; W_M$$

is surjective. An Artin group is said to be of finite type if the associated Coxeter group is finite.

Remark 1.1.1. Sometimes a group G is called of finite type or of type F if it has a finite Eilenberg-Maclane complex. Therefore the statement

> Artin groups of finite type are of finite type.

is actually meaningfull. It happens to be true.

1.1.1 The Braid Group

!!! This whole section needs PICTURES !!!

Configuration Spaces as Hyperplane Arrangements

The labeled configuration space of n points in the plane is

$$\tilde{C}_n := \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}.$$

An element in this space is a set of n points in the plane that are labeled so that we can tell them apart. The symmetric group Perm_n on n letters acts on these configurations by permuting the labels. Hence the quotient

$$C_n := {}_{\mathrm{Perm}_n}\backslash^{\tilde{C}_n}$$

is the configuration space of n-point subsets in the plane.

Definition 1.1.2. The braid group B_n is the fundamental group of C_n. The pure braid group P_n is the fundamental group of $\tilde{C}_n$.

Observation 1.1.3. *The projection*

$$\pi : \tilde{C}_n \to C_n$$

is a covering map with Perm_n *acting as its group of deck transformations. Consequently, we have a short exact sequence*

$$P \hookrightarrow B_n \twoheadrightarrow \mathrm{Perm}_n$$

of groups. In particular, the pure braid group is a finite index normal subgroup of the braid group.

Out first goal is to prove that configuration spaces are Eilenberg-Maclane spaces for braid groups. Later, we will find smaller Eilenberg-Maclane spaces.

Theorem 1.1.4 (Fadell-Neuwirth 1962 [?, Corollary 2.2]). *The space* $\tilde{C}_n$ *is a* $\mathrm{K}(P_n, 1)$*. Consequently,* C_n *is a* $\mathrm{K}(B_n, 1)$*.*

We will follow the proof in [**?**].

For any finite set $P \subset \mathbb{C}$ of punctures, put

$$\tilde{C}_{P,n} := \{(z_1, \ldots, z_n) \mid z_i \notin P \text{ and } z_i \neq z_j \text{ for } i \neq j\}.$$

This is the configuration space of n labeled points in a plane with $m := |P|$ punctures. Note that up to homeomorphism, the position of the punctures does not matter since all m-punctures planes are homeomorphic.

Fact 1.1.5. *The map*

$$\begin{aligned} \pi : \tilde{C}_{P,n} &\to \mathbb{C} - P \\ (z_1, \ldots, z_n) &\mapsto z_1 \end{aligned}$$

is a fibre bundle whose fibre over $z \in \mathbb{C} - P$ *is* $\tilde{C}_{P\cup\{z\},n-1}$.

This fact allows us to “freeze” the points of the configuration one by one: Since fibre bundles are fibrations, we have a long exact sequence of homotopy groups

$$\cdots \to \pi_m\left(\tilde{C}_{P\cup\{z\},n-1}\right) \to \pi_m\left(\tilde{C}_{P,n}\right) \to \pi_m\left(\mathbb{C} - P\right) \to \pi_{m-1}\left(\tilde{C}_{P\cup\{z\},n-1}\right) \to \cdots$$

which proves

$$\pi_m\left(\tilde{C}_{P\cup\{z\},n-1}\right) = \pi_m\left(\tilde{C}_{P,n}\right) \text{ for } m \geq 2$$

since $\mathbb{C} - P$ has trivial homotopy groups in dimension 2 and above. Applying this observation repeatedly, we conclude that for $m \geq 2$:

$$0 = \pi_m\left(\tilde{C}_{\{z_1,\ldots,z_n\},0}\right) = \pi_m\left(\tilde{C}_{\{z_1,\ldots,z_{n-1}\},1}\right) = \cdots = \pi_m\left(\tilde{C}_{\{z_1\},n-1}\right) = \pi_m\left(\tilde{C}_{\emptyset,n}\right).$$

This proves (1.1.4) as $\tilde{C}_{\emptyset,n} = \tilde{C}_n$.

Shrinking the Eilenberg-Maclane Space

The space of all configuration deformation retract onto the subspace $\tilde{C}_n^0$ of all those configuration whose center of gravity is 0. Note that the symmetric group Perm_n acts on $V = \{(t_1, \ldots, t_n) \mid \sum_i t_i = 0\} \leq \mathbb{R}^n$ by permuting the coordinates. This is, in fact, the geometric representation of Perm_n as a finite reflection group. Decomposing the n-tuples in $\tilde{C}_n^0$ into real and imaginary parts, we obtain

$$\tilde{C}_n^0 = V \times V - \bigcup_{H \in \mathcal{H}} H \times H$$

where $\mathcal{H}$ is the set of walls defining S as a finite reflection group on V.

Let X be the Moussong comlpex associated to Perm_n. Recall that this is a convex polyhedrong in V given as the convex hull of a point chosen in a sector such that it has distance $\frac{1}{2}$ to all walls bounding its sector. Shrinking configurations if necessary by rescaling them using a real scalar yields a deformation retraction of $\tilde{C}_n^0$ onto

$$Y_n := X \times X - \bigcup_{H \in \mathcal{H}} H \times H.$$

Note that Y_n is an Eilenberg-Maclane space for the pure braid group. Let us define a poset

$$\mathcal{A}_n := \{(c, v) \mid c \text{ cell in } X, \ v \text{ vertex in } c\}$$

where the order is given by

$$(c, v) \preceq (d, w) \text{ if and only if } c \leq d \text{ and } v = \pi_c(w).$$

We will prove

Lemma 1.1.6. *There is a cover* $Y_n = \bigcup_{\alpha \in \mathcal{A}_n} U_\alpha$ *by convex open sets indexed by the element of* $\mathcal{A}_n$ *such that for any subset* $\sigma \subset \mathcal{A}_n$*,*

$$U_\sigma := \bigcap_{\alpha\in\sigma} U_\alpha \neq \emptyset \text{ if and only if } \sigma \text{ is a chain in } \mathcal{A}_n.$$

Corollary 1.1.7. *The geometric realization of* $\mathcal{A}$ *is an Eilenberg-Maclane space for the pure braid group.*

Proof. For any closed cell c in the Moussong complex, let $\mathcal{H}_c$ denote the set of walls cutting through c. Note that removing these walls chops up the Moussong comlex into convex open subsets. The set of these subsets is in 1-1-correspondence to the vertices of C: Each vertex of c pick the convex open set $C_{(c,v)}$ that contains v.

On the other hand, let D_c be the open star of the barycenter of c in the barycentric subdivision of X. Then D_c is, again, a convex open subset of X. Finally, put

$$U_{(c,v)} := D_c \times C_{(c,v)}.$$

This is a cover of X by convex open sets.

!!! finish this !!! **q.e.d.**

Corollary 1.1.8. *The geometric realization* $|\mathcal{A}_n|$ *is an Eilenberg-Maclane space for the pure braid group* B_n*.*

Remark 1.1.9. All of this is Perm_n-equivariant. Thus

$$\mathrm{Perm}_n \backslash |\mathcal{A}_n|$$

is an Eilenberg-Maclane space for the braid group.

As a consequence, we can actually work out a presentation for the braid group B_n. Let us consider the case of B_3 first. Here, the underlying Coxeter group is the symmetric group on 3 letters with standard genrating set given by two transpositions. Our Eilenberg-Maclane complex has precisely one 2-cell, which is a hexagon, two edges, and one vertex. The tricky part is to figure out, how the 2-cell is attached.

!!! ... !!!

It turns out, that we get the following presentation of the braid group B_n:

$$B_n = \left\langle s_1, \ldots, s_n \;\middle|\; \begin{array}{ll} s_i s_j s_i = s_j s_i s_j & \texttt{for} \;\; |i-j| \geq 2 \\ s_i s_j = s_j s_i & \texttt{for} \;\; |i-j| = 1 \end{array} \right\rangle .$$

Exercise 1.1.10. Show that the $\mathrm{H}_3\big({}_{\mathrm{Perm}_n}\backslash^{|\mathcal{A}_4|}\big)$ is non-trivial. Infer that B_4 does not have an Eilenberg-Maclane complex of dimension ≤ 2.

Exercise 1.1.11. Prove that $B_3 = \langle a, b, c \mid ab = bc = ca \rangle$.

Exercise 1.1.12. More generally, prove that

$$B_n = \left\langle x_{[i,j]} \;\; (i \neq j) \;\middle|\; \begin{array}{ll} x_{[i,j]} x_{[j,k]} = x_{[j,k]} x_{[k,i]} & \texttt{if} \;\; [i,j,k] \\ x_{[i,j]} x_{[k,l]} = x_{[k,l]} x_{[i,j]} & \texttt{if} \;\; [i,j,k,l] \end{array} \right\rangle$$

where we put a cyclic ordering on $\{1, 2, \ldots, n\}$ and $[a, b, \ldots]$ denotes the fact that the listed elements form a cycle in their given order. In particular, the generators are indexed by cycles of length 2.

Exercise 1.1.13. Prove that $B_3 = \langle a, b, c, s \mid ab = bc = ca = s \rangle$. Moreover, show that the Cayley 2-complex (i.e., the universal cover of the canonical 2-complex associated to this presentation) admits a $\mathtt{CAT}(0)$ metric. (This implies that the presentation 2-complex for this presentation is an Eilenberg-Maclane space for B_3.)

Exercise 1.1.14. Decide whether the presentation 2-complex for the presentation

$$B_3 = \langle a, b, c \mid ab = bc = ca \rangle$$

is an Eilenberg-Maclane complex for B_3.

1.1.2 General Artin Groups

Fact 1.1.15 (van Lek). *Let M be a Coxeter matrix over S, and let $J \subseteq S$ be a set of generators with restricted Coxeter matrx M_J. Then the canonical homorphism*

$$A_{M_J} \to A_M$$

is injective. The image is the subgroup generated by J.

Fact 1.1.16. *The space*

$$X \times X - \bigcup_H H \times H$$

is homotopy equivalent to the poset

$$\mathcal{A}_M := \{(c, v) \mid c \textit{ cell in } X, \ v \textit{ vertex in } c\}$$

where the order is given by

$$(c, v) \preceq (d, w) \textit{ if and only if } c \leq d \textit{ and } v = \pi_c(w) \, .$$

The fundamental group of these spaces is the pure Artin group.

This space is conjectured to be an Eilenberg-Maclane space.

Fact 1.1.17 (Charney-Davis). *The poset $\mathcal{A}_M$ is an Eilenberg-Maclane space for the pure Artin group P_M, provided any two Artin generators generate a finite subgroup, i.e., the Coxeter matrix M is <u>2-spherical</u>. One obtains an Eilenberg-Maclane space for the corresponding Artin group A_M by modding out the group action of $WM]$. In particular, Artin groups of finite type have a finite Eilenberg-Maclane complex.*

1.1.3 Artin Groups of Finite Type

Fact 1.1.18 (Brieskorn-Saito). *Artin groups of finite type have solvable word and conjugacy problem.*

Fact 1.1.19 (Charney). *Artin groups of finite type are biautomatic.*

Fact 1.1.20 (Bestvina). *Artin groups of finite type have the look and feel of CAT(0)-groups: Let A be an Artin group of finite type. Then the following hold:*

1. *The group A contains only finitely many conjugacy classes of finite subgroups.*
2. *Every solvable subgroup of A is finitely generated and virtually abelian.*
3. *The set of translation lengths is bounded away from 0. (Note that Artin groups of finite type have a finite Eilenberg-Maclane complex by (1.1.17) and are, therefore, torsion free.)*

Fact 1.1.21 (Squier). *An Artin group of finite type over the generating set S is a duality group of dimension $|S|$.*

Fact 1.1.22 (Krammer, Cohen-Wales). *Arting groups of finite type are linear.*

1.1.4 Right-Angled Artin Groups and the Example of M. Bestvina and N. Brady

Right-angled Artin groups are also known as graph groups since the data determining the presentation can most easyly be visualized as a graph: To any graph Γ with vertex set $\mathcal{V}$, we associate the group

$$G_\Gamma := \langle v \in \mathcal{V} \mid vw = wv \text{ if there is an edge } v — w \text{ in } \Gamma \rangle .$$

Note that there is a canonical homomorphism

$$\begin{aligned} \varphi : G_\Gamma &\rightarrow \mathbb{Z} \\ v &\mapsto 1 \end{aligned}$$

whose kernel will be denoted by K_Γ.

In this section, we also identify Γ with its associated flag complex, i.e., the simplicial complex that shares the vertices with Γ and whose simplices are cliques in Γ: A set of vertices forms a simplex if the vertices are pairwise connected by edges.

Theorem 1.1.23 (Bestvina-Brady [?]). *If Γ is a finite flag complex then the following hold:*

1. *K_Γ is of type F_m if and only if Γ is $(m-1)$-connected.*
2. *K_Γ is of type FP_m if and only if Γ is $(m-1)$-acyclic.*

This section is devoted to a proof of this result. Note that the theorem allows one to construct groups with prescribed finiteness properties. In particular, we could take Γ to be 1-acyclic but not simply connected and infer:

Corollary 1.1.24. *There is a group of type $\mathrm{FP}2$ that is not finitely presented.* **q.e.d.**

First, we construct an Eilenberg-Maclane space for G_Γ. Let $T_\mathcal{V}$ a product of a family of circles $\mathbb{S}^1_v$ indexed by the vertices in $\mathcal{V}$. We assume that all these circles have a basepoint so that we can regard them as subspaces in $T_\mathcal{V}$. For any subset σ of $\mathcal{V}$ we regard the torus $T_\sigma = \times_{v\in\sigma} \mathbb{S}^1_v$ as a subtorus of $T_\mathcal{V}$. We put

$$Q_\Gamma := \bigcup_{\sigma \text{ simplex}} T_\sigma$$

and let

$$X_\Gamma := \tilde{Q}_\Gamma$$

denote its universal cover.

Observation 1.1.25. *The complex Q_Γ has precisely one vertex P, and the link of this vertex is*

$$\mathrm{Lk}\,(P) = \mathbb{S}(\Gamma) = \bigcup_{\sigma \text{ simplex}} \mathbb{S}^\sigma \subset \mathbb{R}^\mathcal{V}$$

where $\mathbb{S}^\sigma$ denotes the unit sphere in $\mathbb{R}^\sigma \subseteq \mathbb{R}^\mathcal{V}$. The cuibical structure on Q_Γ induces the triangulation on $\mathbb{S}(L)$ given by

$$\mathbb{S}^\sigma = \mathop{\ast}_{v\in\sigma} \mathbb{S}^{\{v\}}.$$

Note that X_Γ is a piecewise Euclidean cube complex and all its vertex links are isomorphic to $\mathbb{S}(\Gamma)$.

Exercise 1.1.26. Show that $\mathbb{S}(\Gamma)$ is a flag complex.

Corollary 1.1.27. *X_Γ is CAT(0) and, therefore, contractible.* **q.e.d.**

The canonical homomorphism φ has a topological representative

$$h : Q \to \mathbb{S}^1$$

that is piecewise linear and restricts to the degree 1 map on each $\mathbb{S}^1_v$. It lifts to a piecewise linear map

$$h : X \to \mathbb{R}$$

which is affine on each cube in X.

Definition 1.1.28. A <u>combinatorial Morse function</u> on a piecewise Euclidean complex is a real valued function h that is affine on closed cells, non-constant on edges, and has a discrete set of critical values, i.e., the image of the 0-skeleton is discrete in $\mathbb{R}$.

The descending (ascending) link of a vertex v is that part of its link spanned by those cells for which v is a maximum (minimum) for h.

The s-level set is the h-preimage of the real number s. The s-sublevel set is the preimage of $(-\infty, s]$. For any closed interval I, we call its h-preimage the I-slice.

Lemma 1.1.29. *Let $r < s$ be two real numbers such that there are no critical values in $[r, s]$. Then the r-sublevel and s-sublevel sets are homotopy equivalent. Similarly, any two slices whose difference does not contain vertices are homotopy equivalent.*

Proof. Observe that the level set cut through the polyhedral cells of the complex. Thereby, the upper level set creates a free face in each cell. You can collapse the top-dimensional material in the affected cells away. This defines a deformation retraction. Now induct on lower dimensional material. **q.e.d.**

!!! Finish this !!!