Chapter 1

Coxeter Groups and Artin Groups

1.1 Euclidean Reflection Groups

Let

- $\mathbb{E}$ be a Euclidean space, and let
- $\mathcal{H}$ be a set of hyperplanes satisfying the following:

  1. $\mathcal{H}$ is locally finite, i.e., a set of hyperplanes such that any compact subset of $\mathbb{E}$ intersects only finitely many hyperplanes from $\mathcal{H}$.
  2. $\mathcal{H}$ is a W-invariant subset of $\mathbb{E}$ where W is the subgroup of $\mathrm{Isom}(\mathbb{E})$ generated by all reflections ρ_H with $H \in \mathcal{H}$.

Definition 1.1.1. Such a group W is called a Euclidean reflection group.

Exercise 1.1.2. Assume that $\mathcal{H}$ is finite. Show that $\bigcap_{H\in\mathcal{H}} H \neq \emptyset$. See (1.1.16) for a more elaborate statement.

We want to derive a presentation for W.

1.1.1 The Chamber Decomposition of $\mathbb{E}$

A chamber is a complementary component of $\mathcal{H}$, i.e., a component of $\mathbb{E} - \bigcup_{H \in \mathcal{H}} H$. Note that the closure of a chamber C is a convex polytope (possibly non-compact). The faces of this polytope span hyperplanes that belong to $\mathcal{H}$. We say that those hyperplanes from $\mathcal{H}$ are supporting C. For any chamber C, we denote by

- $|C|$ the set of hyperplanes in $\mathcal{H}$ supporting C.

  Two chambers, C and D, are called adjacent along H if $H \cap C = H \cap D$ is a CoDim-1-face. In this case, we write

$$C|_H D.$$

  They are called adjacent if they are adjacent along some H. In this case, we write

$$C|D.$$

  Note that adjacency and adjacency along H are symmetric and reflexive relations.

  A gallery is a sequence

$$C_0|C_1|\cdots|C_r$$

  of chambers such that C_i is adjacent to C_{i+1} for all $i < r$. If $C_i|_H C_{i+1}$, we say that the gallery crosses H at this step.

  The last index r gives the length of the gallery, which henceforth is the number of hyperplanes that are crossed by the gallery. The distance

- $\delta(C, D)$ of the chambers C and D is the minimum length of a gallery connecting them. Note that two chambers are adjacent if and only if their distance is at most 1.

Exercise 1.1.3. Show that C and D are H-adjacent if and only if H supports both and $\{C, \rho_H C\} = \{D, \rho_H D\}$.

Observation 1.1.4. *Any two chambers are connected by a gallery of finite length.* **q.e.d.**

Exercise 1.1.5. Prove that a gallery from C to D has minimum length if and only if it does not cross any hyperplane twice. Moreover, the set of hyperplanes that are crossed by a minimum length gallery from C to D is precisely the set of those $H \in \mathcal{H}$ that separate C from D. In particular, this set is the same for all those minimum length galleries.

Note that W acts on the set $\mathcal{C}$ of chambers by distance preserving permutations.

Observation 1.1.6. *If the hyperplane H supports the chamber C, then $\rho_H C|_H C$.*

Let us fix an arbitrary chamber

- C^*, the <u>fundamental chamber</u>. Put
- $S := \{\rho_H \mid H \in |C^*|\}$.

Lemma 1.1.7. *W acts transitively on $\mathcal{C}$ and is generated by S.*

Proof. Let

$$C^* = C_0|C_1|\cdots|C_{r-1}|C_r$$

be any gallery starting at C^*. We will show that there are elements $w_i \in \langle S \rangle$ with $C_i = w_i C^*$. This is an easy induction: Suppose w_i has been found already. Let H be the hyperplane with $C_i|_H C_{i+1}$. Then $w_i^{-1}H$ is a hyperplane in $\mathcal{H}$ that supports C^*. Thus

$$\rho_H = w_i s w_i^{-1} \text{ for some } s \in S$$

and

$$C_{i+1} = \rho_H C_i = \rho_H w_i C^* = w_i s w_i^{-1} w_i C^* = w_i s C^*.$$

This way, we constructed an element $w_{i+1} = w_i s \in \langle S \rangle$.

Since every chamber can be connected to C^* by a gallery, the subgroup $\langle S \rangle$ already acts transitively on $\mathcal{C}$.

Consider $H \in \mathcal{H}$. Let $C = wC^*$ (where $w \in \langle S \rangle$) be a chamber supported by H. As we already have observered, there is an element $s \in S$ such that

$$\rho_H = wsw^{-1} \in \langle S \rangle .$$

Thus the generating set for W is contained in $\langle S \rangle$. **q.e.d.**

Lemma 1.1.8. *Let $\underline{s} = s_1 s_2 \cdots s_r$ be a word representing $w \in W$. If this word is a minimum length representative for w, then its length r equals $\delta(C^*, wC^*)$. Otherwise, one can obtain a shorter word representing w by deleting two of the letters, i.e., there are two indices $i < j$ such that*

$$w = s_1 \cdots s_{i-1} s_{i+1} \cdots s_{j-1} s_{j+1} \cdots s_L.$$

Proof. Put

- $w_i := s_1 \cdots s_i$,
- $C_i := w_i C^*$, and let
- H_i be the hyperplane satisfying $s_i = \rho_{H_i}$.

We claim that the corresponding gallery

$$C^* = C_0|_{w_0 H_1} C_1|_{w_1 H_2} \cdots C_{r-2}|_{w_{r-2} H_{r-1}} C_{r-1}|_{w_{r-1} H_r}$$

does not cross any hyperplane twice provided that $s_1 \cdots s_r$ is a minimum word length representative for w. Then the claim follows from (1.1.5).

So let us suppose that

$$w_{i-1}H_i = w_{j-1}H_j$$

for some $i < j$. We conclude

$$w_{i-1}s_i w_{i-1}^{-1} = w_{j-1}s_j w_{j-1}^{-1}$$

whence

$$s_1 \cdots s_{i-1}s_i s_{i-1} \cdots s_1 = s_1 \cdots s_{j-1}s_j s_{j-1} \cdots s_1.$$

Thus,

$$1 = s_i \cdots s_{j-1}s_j s_{j-1} \cdots s_{i+1}.$$

Multiplying from the right, we obtain

$$s_{i+1} \cdots s_{j-1} = s_i \cdots s_j$$

which implies that we have a shorter word for w:

$$w = s_1 \cdots s_{i-1}s_{i+1} \cdots s_{j-1}s_{j+1} \cdots s_L$$

This is a contradiction. **q.e.d.**

Corollary 1.1.9. *The action of W on $\mathcal{C}$ is simply transitive.* **q.e.d.**

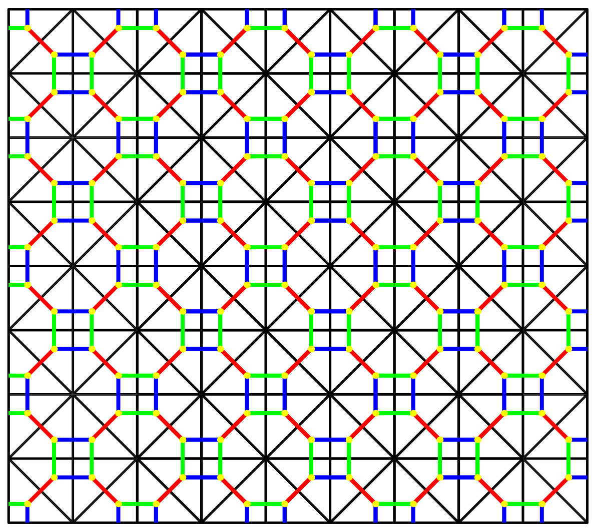
This corollary allows us to draw the Cayley graph of W with respect to S. Since all generators have order 2, we simplify matters by ommiting all the bi-gons that would arise that way. Thus, we define the <u>reduced Caley graph</u>

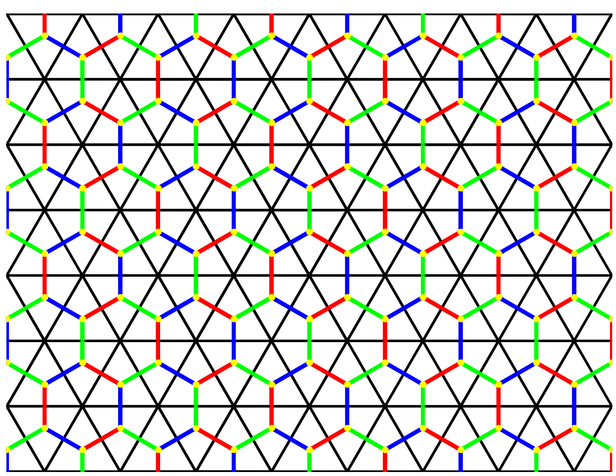
$$\Gamma := \Gamma_S(W)$$

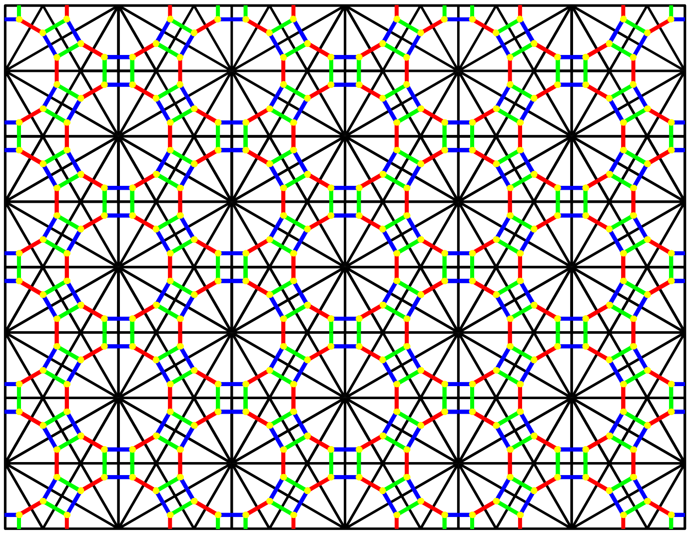
of W to have a vertex for each group element and an edge (labelled by s) for each unordered pair $\{w, ws\}$. Note that W acts from the left.

Observation 1.1.10. *Pick a point inside the fundamental chamber. The W-orbit of this point can be identified with the vertex set of Γ. The edges of Γ correspond to* CoDim*-1-faces in the chamber decomposition of $\mathbb{E}$. In fact, we can connect the vertices by edges perpendicular to those faces. This way, the Cayley graph is W-equivariantly embedded in $\mathbb{E}$.*

Example 1.1.11. Here are the planar reflection groups whose fundamental chambers are triangles:







1.1.2 The Coxeter Matrix

The Coxeter Matrix of the pair (W,S) is the $S\times S$-matrix

$$M := (m_{s,t} := \mathrm{ord}_W(st))_{s,t\in S}\,.$$

The entries are taken from $\{1,2,3,\ldots,\infty\}$. Note that M is symmetric and satisfies:

$$m_{s,t}=1 \quad \text{if and only if} \quad s=t. \tag{1.1}$$

Theorem 1.1.12. *The group W has the presentation*

$$W=\langle s\in S \mid (st)^{m_{s,t}}=1 \ \textit{for}\ m_{s,t}<\infty\rangle\,.$$

Proof. The given relations obviously hold. To deduce any given other relation, realize the relation as a closed loop in the Cayley graph. This graph lies in the ambient Euclidean space. Find a bounding disk that intersects the CoDim-2-skeleton of the chamber decomposition transversally. Now see the van Kampen diagram. **q.e.d.**

For each s let $\mathbf{u}_s$ be the unit vector perpendicular to the hyperplane inducing the reflection s. (There is a choice here: we use the vector that points away from the fundamental chamber.)

Exercise 1.1.13. Show that for any $s, t \in S$,

$$\langle \mathbf{u}_s, \mathbf{u}_t \rangle = \begin{cases} -\cos\left(\frac{\pi}{m_{s,t}}\right) & \text{for } m_{s,t} \text{ finite} \\ -1 & \text{for } m_{s,t} \text{ infinite}. \end{cases}$$

Now, we can settle the question, whether S is finite.

Proposition 1.1.14. *The fundamental chamber has finite support.*

Proof. Suppose otherwise. Then the set of unit vectors $\mathbf{u}_s$ had an accumulation point by compactness of the unit sphere. However, their pair-wise scalar products are negative. **q.e.d.**

Corollary 1.1.15. *The set $\mathcal{H}$ decomposes into fintely many parallelity classes.*

Proof. Suppose otherwise, then, by compactness, there would be hyperplanes that span arbitrary small angles. Take a point very close to their intersection that lies in a chamber. Since the angles around faces of chambers are bounded away from 0, we have a contradiction. **q.e.d.**

Exercise 1.1.16. Show that the following are equivalent:

1. $\mathcal{H}$ is finite.

2. W is finite.

3. W is torsion.

4. $\bigcap_{H \in \mathcal{H}} H \neq \emptyset$.

Corollary 1.1.17. *A Euclidean reflection group W is virtually free abelian.*

Proof. Consider the action of W upon the sphere at infinity. By (1.1.15), this sphere is decomposed into finitely many regions, upon which W acts by spherical isometries. The image of W in $\mathrm{Isom}(\mathbb{S})$ is a finite Euclidean reflecion group by (1.1.16). The kernel of the homomorphism consists of translations. **q.e.d.**

1.1.3 The Cocompact Case

In this section, we assume that the fundamental chamber has compact closure. All the result are valid in the general case, though. In deed, we will prove them for arbitrary Coxeter groups later.

Observation 1.1.18. *Every point of $\mathbb{E}$ is either contained in a chamber or belongs to the closures of at least two adjacent chambers. In the latter case, it has a translate in the closure of C^*. Thus, the closure of the fundamental chamber is a fundamental domain for the action of W, i.e., the translates of the closure cover $\mathbb{E}$ while the translates of C^* stay disjoint.*

Theorem 1.1.19. *W has only finitely many finite subgroups up to conjugacy.*

Proof. A finite subgroup fixes a point. This point is a translate of some point in $\overline{C^*}$. Thus any finite subgroup is conjugate to a subgroup of a stabilizer of a point in $\overline{C^*}$. There are only finitely many of those since C^* has only finitely many faces. **q.e.d.**

Theorem 1.1.20. *The conjugacy problem in W is solvable.*

Proof. !!! Do the CAT(0) proof !!! **q.e.d.**

1.2 Coxeter Groups

Definition 1.2.1. Let S be a set. A Coxeter matrix over S is a symmetric matrix $M = (m_{s,t})_{s,t \in S}$ with entries $m_{s,t}$ in $\{1, 2, 3, \dots, \infty\}$ such that

$$m_{s,t} = 1 \text{ if and only if } s = t.$$

The Coxeter group defined by M is the group given by the presentation

$$W = \langle s \in S \mid (st)^{m_{s,t}} = 1 \text{ if } m_{s,t} \text{ finite} \rangle .$$

The pair (W, S) is called a Coxeter system.

Example 1.2.2. Every Euclidean reflection group is a Coxeter group.

Coxeter groups are defined by generators and relations. In general, it is hard to tell wheter a group given in this manner is trivial or not. So our first problem will be to see that Coxeter groups are not trivial.

Observation 1.2.3. *Every defining relation of W has even length. Thus, there is a well defined surjective homomorphism*

$$W \to C_2$$

sending each generator in S to the generator of C_2. In particular, none of the generators is trivial in W. **q.e.d.**

Thus, every generator generates a subgroup of order 2 inside W.

1.2.1 The Geometric Representation

To show that the generators have order 2, we used a representation of W. Now, we shall extend this method to show that the products st also have the orders that we would expect from the presentation.

Definition 1.2.4. Let (W, S) be a Coxeter system with Coxeter matrix M. Let $V := \bigoplus_{s \in S} \mathbb{R} e_s$ be the real vector space generated by S: To avoid confusion, we denote the basis vector corresponding to s by e_s.

Define a bilinear form on V by

$$\langle e_s, e_t \rangle_M := \begin{cases} -\cos\left(\frac{\pi}{m_{s,t}}\right) & \text{if } m_{s,t} < \infty \\ -1 & \text{if } m_{s,t} = \infty \end{cases}$$

and define an action of W on V where the generator s acts as the linear automorphism

$$\rho_s : e_t \mapsto e_t - 2 \langle e_s, e_t \rangle_M e_s.$$

This action defines the geometric representation

$$\rho : W \to \mathrm{Aut}(V).$$

Exercise 1.2.5. Check that the geometric representation does exist, i.e., check that the automorphisms ρ_s satisfy the defining relations of W.

Lemma 1.2.6. *The order of st in M is given by the entry $m_{s,t}$ of the Coxeter matrix.*

Proof. Note that the action of the subgroup $\langle s, t \rangle$ leaves the subspace $V_{s,t} := \langle e_s, e_t \rangle$ invariant.

$m_{s,t} = \infty$: The action hits e_t as follows:

$$e_t \xrightarrow{\rho_s} e_t + 2e_s \xrightarrow{\rho_t} 3e_t + 2e_s \xrightarrow{\rho_s} 3e_t + 4e_s \xrightarrow{\rho_t} 5e_t + 4e_s \xrightarrow{\rho_s} \cdots$$

Thus, the product $\rho_t \rho_s$ has infinite order.

$m_{s,t} < \infty$: In this case, the bilinear form $\langle -, - \rangle_M$ restricts to a positive definite bilinear form on $V_{s,t}$, and a direct computation shows that the product $\rho_t \rho_s$ is a rotation about of order $m_{s,t}$. **q.e.d.**

Corollary 1.2.7. *Thus, the generators s and t span a copy of the dihedral group $D_{m_{s,t}}$ inside W.* **q.e.d.**

Exercise 1.2.8. Show that W is finite if the bilinear form $\langle -,- \rangle_M$ is positive definite.

Exercise 1.2.9. Show that if W is finite, then there is a unique bilinear form $\langle -,- \rangle$ on V characterized by the following properites

1. $\langle -,- \rangle$ is positive definite.

2. All basis vectors e_s have unit length.

3. The action of W preserves $\langle -,- \rangle$.

Moreover, this bilinear form is $\langle -,- \rangle_M$.

Corollary 1.2.10. *Finite Euclidean reflection groups and finite Coxeter groups are the very same thing.*

Remark 1.2.11. The classification of finite Coxeter groups is done by classifying all Coxeter matrices that are positive definite.

Exercise 1.2.12. A Coxeter system is called irreducible if there is no generator that commutes simultaneously with all the others. Classify all irreducible Coxeter systems over three generators whose Coxeter groups are finite. (Hint: You should recover descriptions of the Platonic solids along the way; in fact, the existence of the Platonic solids can be derived from this classification.)

1.2.2 The Geometry of a Coxeter System

We studied Euclidean reflection groups by means of the assiciated Chamber system upon which the group acts. To study general Coxeter groups, we will construct the geometry from the group. So, we will construct a chamber system from the (reduced) Cayley complex $\Gamma_S(W)$ for the Coxeter presentation. The vertices of the Cayley complex are the chambers, and two chambers are s-adjacent if they are joined by an edge with labes s. Of course an edge path in the Cayley complex is a gallery in the chamber system. We will see that this chamber system allows reflections and half spaces.

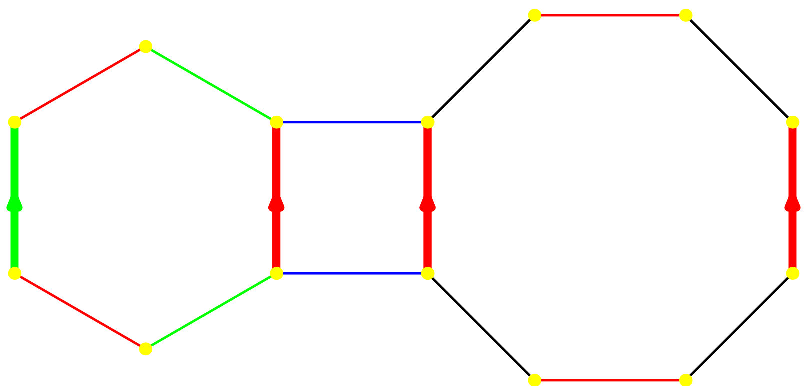
Definition 1.2.13. Two edges e and e' in $\Gamma_S(W)$ are opposite if they are contained in a relator disc and have maximal distance in this circle. We write $e \longleftrightarrow e'$. The edges e and e' are parallel if $e = e'$ or if there is a finite sequence

$$e = e_0 \longleftrightarrow e_1 \longleftrightarrow e_2 \longleftrightarrow \cdots \longleftrightarrow e_r = e'.$$

We write $e \parallel e'$.

Parallelity is an equivalence relation. Its equivalence classes are called walls.

It is useful to extend the notion of parallelism to oriented edges. Let us consider opposite edges first. Inside a relator disc, an oriented edge induces an orientation of the boundary circle of the disc. We call two oriented edges of a relator disc opposite, if they induce opposite orientations of the boundary circle and their underlying geometric edges are opposite. As above, parallelism is defined as the transitive closure of opposition. Equivalence classes of oriented edges under parallelism are called oriented walls



Observation 1.2.14. *Let $\overrightarrow{e}_0$ and $\overrightarrow{e}_1$ are opposite oriented edges in a relator cell. Then removing these two edges cuts the boundary circle of the relator disc into two arcs; and the arc from $\iota(\overrightarrow{e}_0)$ to $\iota(\overrightarrow{e}_1)$ reads the same word as the arc from $\tau(\overrightarrow{e}_0)$ to $\tau(\overrightarrow{e}_1)$. (Here, we use that relator discs are not "crushed".)*

By induction, it follows that if $\overrightarrow{e}$ and $\overrightarrow{e}'$ are parallel, then there is a group element $w \in W$ such that

$$\iota(\overrightarrow{e}) = \iota(\overrightarrow{e}')\, w \text{ and } \tau(\overrightarrow{e}) = \tau(\overrightarrow{e}')\, w.$$

Note that we are multiplying from the right, which in general will tear edges apart.

Observation 1.2.15. *Let us observe a "local converse": Each relator disc is in its own right the Cayley graph of a finite dihedral group. Let $\overrightarrow{e}$ and $\overrightarrow{e}'$ be two oriented edges in this cell. If there is a group element w in the dihedral group such that*

$$\iota(\overrightarrow{e}) = \iota(\overrightarrow{e}')\, w \text{ and } \tau(\overrightarrow{e}) = \tau(\overrightarrow{e}')\, w$$

then $\overrightarrow{e}$ and $\overrightarrow{e}'$ are either opposite or identical.

Corollary 1.2.16. *An (oriented) wall either avoids a relator cell or intersects it in a pair of opposite (oriented) edges.* **q.e.d.**

Proof. Let $\overrightarrow{e}$ be an oriented edge in a relator cell. We have to show that the only parallel edge in this cell is the opposite one. So let $\overrightarrow{e}'$ be any other parallel edge in this relator cell. We know that there is an element $w \in W$ such that

$$\iota(\overrightarrow{e}) = \iota(\overrightarrow{e}')\,w \text{ and } \tau(\overrightarrow{e}) = \tau(\overrightarrow{e}')\,w.$$

Since both edges belong to the relator cell, the element w actually belongs to the dihedral subgroup generated by the two labels around the relator cell. Now, it follows from the local converse that $\overrightarrow{e}$ and $\overrightarrow{e}'$ are opposite or identical. **q.e.d.**

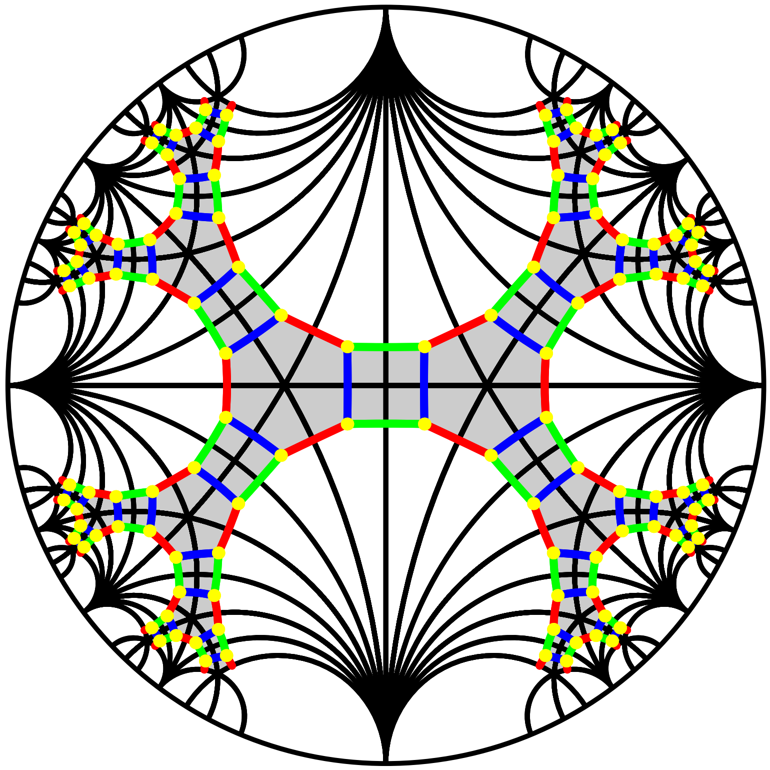
The same reasoning actually yields:

Corollary 1.2.17. *Let e be an edge and let $\overrightarrow{e}$ and $\overleftarrow{e}$ denote the two corresponding oriented edges. Then $\overrightarrow{e}$ and $\overleftarrow{e}$ are not parallel.*

In particular, every wall can be oriented in precisely two ways. **q.e.d.**

Let h be a wall. The <u>boundary</u> $\partial(h)$ of h is the set of vertices (chambers) that are incident with at least one edge in h – recall that a wall is an equivalence class of edges.

Example 1.2.18. Here is the Cayley graph for the group $\left\langle \mathbf{b},\mathbf{g},\mathbf{r} \,\middle|\, \mathbf{b}^2=\mathbf{g}^2=\mathbf{r}^2=(\mathbf{br})^3=(\mathbf{bg})^2=1 \right\rangle$. drawn in the hyperbolic plane. Oberserve how the axes for the reflections intersect groups of two or four edges perpendicularly. These are precisely the walls. The shaded regions are the relator discs.



Let h be a wall and let g be an edge path (a gallery).

Let

- $\mathrm{hits}\,(h, \mathbf{g})$ denote the number of times that the edge path g passes through the wall h.

Definition 1.2.19. An elementary homotopy of an edge path in the Cayley graph of a Coxeter group is one of the following two types of moves:

1. Replacing a subpath reading part of a relator disc by the complementary part of the relation.

2. Adding or removing a backtracking edge.

Two paths in the Cayley graph are called homotopic if one can be obtained from the other by a finite sequence of elementary homotopy.

Observation 1.2.20. *Since elementary homotopies correspond to substitutions in words waranted by the defining relations, two galleries are homotopic if and only if they connect the same end points.*

Observation 1.2.21. *Given a wall and a path, the number of crossings between the wall and the gallery changes by an even number during any elementary homotopy of the gallery. This follows from (1.2.16) Thus for a given wall h and two galleries $\mathbf{g}_0$ and $\mathbf{g}_1$, we have*

$$\mathrm{hits}\,(h, \mathbf{g}_0) \equiv \mathrm{hits}\,(h, \mathbf{g}_1) \mod 2.$$

In particular, the endpoints of an edge inside h cannot be connected by a gallery that does not cross h.

Corollary 1.2.22. *Every wall separates Γ.*

Observation 1.2.23. *If $\overrightarrow{e}$ and $\overrightarrow{e}'$ are parallel their terminal vertices can be joined by a path that does not intersect the wall they belong to. This follows by induction from the corresponding statement about opposite oriented edges, which is obvious.*

Corollary 1.2.24. *Each wall separates the Cayley graph into precisely two half spaces.*

Proof. We already know that walls separate. That there are not more than two components follows from (1.2.23). **q.e.d.**

Lemma 1.2.25. *Associated to each wall, there is a unique element in W that acts like a reflection along the wall.*

Proof. Let e be an edge in the wall. Then there is a unique element in W that interchanged its endpoints. (This is, indeed, true for any edge: There is a unique group element taking the initial point to the terminal point. But then, it has to swap the two points, because the action of W preserves the labelling of edges by generators.)

Now, just check that this swap condition extends to edges that are opposite in a relator disc. **q.e.d.**

Corollary 1.2.26. *Half spaces are convex, i.e., if two chambers lie in a given half space, then so does every minimal chamber between them.* **q.e.d.**

Corollary 1.2.27. *The gallery distance of two chambers is the number of walls seperating them.* **q.e.d.**

Definition 1.2.28. A morphism of graphs is a distance non-increasing map from the vertices of graph to the vertices of another graph. A folding of a graph is an idempotent graph endomorphism $f : \Gamma \rightarrow \Gamma$ such that the preimage of each vertex v is either empty or contains precisely two vertices (one of which is v). The image α_f of a folding is called a half space or a root. Two foldings f and f' are opposite if their images are disjoint and the following hold:

$$\begin{aligned} f &= f \circ f' \\ f' &= f' \circ f. \end{aligned}$$

Any two opposite foldings f and f' induce a reflection

$$\begin{aligned} \rho : \Gamma &\rightarrow \Gamma \\ v &\mapsto \begin{cases} f(v) & \text{if } v \in \alpha_{f'} \\ f(v) & \text{if } v \in \alpha_f \end{cases} \end{aligned}$$

Exercise 1.2.29. Show that a (locally finite) graph is the Cayley graph of a (finitely generated) Coxeter group if and only if the following conditions holds:

1. For each oriented edge $\overrightarrow{e}$ there is a unique folding $f_{\overrightarrow{e}}$ of Γ satisfying $f_{\overrightarrow{e}}(\iota(\overrightarrow{e})) = \tau(\overrightarrow{e})$.
2. If $\overrightarrow{e}$ and $\overleftarrow{e}$ are opposite orientations of the same underlying geometric edge, then $f_{\overrightarrow{e}}$ and $f_{\overleftarrow{e}}$ are opposite foldings.

Above, we introduced the geometric representation of W on the vector space V spanned by $\{e_s \mid s \in S\}$. Let V^* be the dual of V. It turns out that the induced action of W on V^*,

$$\begin{aligned} \tau : W &\rightarrow \mathrm{Aut}(V^*) \\ w : \lambda &\mapsto \lambda \circ \rho_w, \end{aligned}$$

gives another description of the chamber system: For any s, define the posite and negative halfspace in V^* by

$$\begin{aligned} U_s^+ &:= \{\lambda \in V^* \mid \lambda(e_s) > 0\} \\ U_s^- &:= \{\lambda \in V^* \mid \lambda(e_s) < 0\}. \end{aligned}$$

The <u>Tits cone</u>

$$C := \{\lambda \in V^* \mid \lambda(e_s) > 0 \text{ for all } s \in S\}$$

is the intersection of the positive cones.

Exercise 1.2.30. Show that for every $w \in W$,

$$\tau_w(C) \subseteq U_s^+ \text{ if and only if } |sw| = |w| + 1$$

and

$$\tau_w(C) \subseteq U_s^- \text{ if and only if } |sw| = |w| - 1.$$

Exercise 1.2.31. Infer from (1.2.30) that the geometric representation is faithful.

Corollary 1.2.32. *Finitely generated Coxeter groups are linear.* **q.e.d.**

1.2.3 The Deletion Condition

In (1.1.8), we have seen, that the pair (W, S) for a Euclidean reflection group satisfies the Deletion Condition:

Definition 1.2.33 (Deletion Condition). Let (W, S) be a pair where W is a group and S is a generating set for W consisting entirely of elements of order 2. We say that this pair satisfies the Deletion Contition if:

> For any non-reduced word $s_1 \cdots s_r$ over S there are two indices i and j such that
>
> $$s_1 \cdots s_r =_W s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r.$$
>
> The carets indicate ommision.

This is, one can delete two letters from any non-minimum-length word to obtain a shorter representative for the same element of W.

In this section, we will recognize (W, S) as a Coxeter system using the Deletion Condition.

Lemma and Definition 1.2.34 (Exchange Condition). *The pair (W, S) satifies the Exchange Condition, i.e.:*

> *Let $s_1 \cdots s_r$ and and $t_1 \cdots t_r$ be two reduced words over S representing the same element $w \in W$. If $s_1 \neq t_1$, then there is an index $i \in \{2, \ldots, r\}$ such that*
>
> $$w =_W s_1 t_1 \cdots \hat{t}_i \cdots t_r.$$

Proof. This is a formal consequence of the Deletion Condition: From

$$s_1 \cdots s_r =_W t_1 \cdots t_r,$$

we obtain

$$s_2 \cdots s_r =_W s_1 t_1 \cdots t_r$$

where the right hand is longer than the left hand whence there must be a pair of letters that can be dropped without changing the value of the product. However, one of the two letters must be the leading s_1: Otherwise, we had

$$s_2 \cdots s_r =_W s_1 t_1 \cdots \hat{t}_i \cdots \hat{t}_i \cdots t_r$$

whence

$$s_1 \cdots s_r =_W t_1 \cdots \hat{t}_i \cdots \hat{t}_i \cdots t_r$$

contradicting the minimality of the initial words.

Thus, we have

$$s_2 \cdots s_r =_W t_1 \cdots \hat{t}_i \cdots \cdots t_r$$

whence

$$s_1 \cdots s_r =_W s_1 t_1 \cdots \hat{t}_i \cdots \cdots t_r.$$ **q.e.d.**

The Coxeter Matrix of the pair (W, S) is the $S \times S$-matrix

$$M := \left(m_{s,t} := \operatorname{ord}_W(st)\right)_{s,t \in S}.$$

The entries are taken from $\{1, 2, 3, \ldots, \infty\}$. Note that M is symmetric and satisfies:

$$m_{s,t} = 1 \quad \text{if and only if} \quad s = t. \qquad (1.2)$$

Any symmetric matrix satisfying (1.2) is called a Coxeter matrix.

An elementary M-reduction is one of the following moves:

1. Delete a subword ss.

2. Replace a subword $\underbrace{sts\cdots}_{m_{s,t} \text{ letters}}$ by $\underbrace{tst\cdots}_{m_{s,t} \text{ letters}}$.

Theorem 1.2.35 (Tits). *Let* $\underline{s} = s_1 \cdots s_{|\underline{s}|}$ *be a reduced word over* S*. Then* $\underline{s}$ *can be obtained from any word* $\underline{t} = t_1 \cdots t_{|\underline{t}|}$ *by a sequence of elementary* M*-reductions.*

Proof. This is also a purely formal consequence of the Deletion Condition. Let us first prove the theorem under the additional hypothesis that $\underline{t}$ is reduced, as well. In this case, $|\underline{s}| = |\underline{t}|$ and only moves of type (2) are possible. We induct on the length of the words.

Assume first that $s_1 = t_1$. Then $s_2 \cdots s_{|\underline{s}|}$ and $t_2 \cdots t_{|\underline{s}|}$ are two reduced words representing the same group element. By induction, we can pass from one to the other by elementary M-reductions.

So assume $s_1 \neq t_1$. So we can apply the exchange condition both ways and obtain

$$\begin{aligned} s_1 \cdots s_{|\underline{s}|} &=_W s_1 t_1 \cdots \hat{y}_i \cdots t_{|\underline{s}|} \\ t_1 \cdots t_{|\underline{s}|} &=_W t_1 s_1 \cdots \hat{x}_i \cdots s_{|\underline{s}|} \end{aligned}$$

Note that both equations actually can be realized by M-reduction since the words start with identical letters. Thus, we only have to realize an M-reduction to pass from $s_1 t_1 \cdots \hat{y}_i \cdots t_{|\underline{s}|}$ to $t_1 s_1 \cdots \hat{x}_i \cdots s_{|\underline{s}|}$. If $m_{s,t} = 1$, we are done. Otherwise we apply the exchange condition again:

!!! ... !!! (Finish this)

Now let us drop the assumption that $\underline{t}$ is reduced. It suffices to prove that $\underline{t}$ can be shortened by M-reductions. We induct on the length of $\underline{t}$. If $\underline{t}_2 \cdots \underline{t}_{|\underline{t}|}$ is not reduced, we apply the induction hypothesis to this subword.

So we assume that $\underline{t}_2 \cdots \underline{t}_{|\underline{t}|}$ is reduced. Then we find

$$\underline{t}_1 \cdots \underline{t}_{|\underline{t}|} =_W \underline{t}_2 \cdots \hat{t}_i \cdots \underline{t}_{|\underline{t}|}$$

whence $\underline{t}_2 \cdots \underline{t}_{|\underline{t}|}$ can be transformed into $t_1\underline{t}_2 \cdots \hat{t}_i \underline{t}_{|\underline{t}|}$ by M-reductions. (Both of these words are reduced, so we are in the case that we have discussed already.) Now, we can shorten:

$$\underline{t}_1 \cdots \underline{t}_{|\underline{t}|} \xrightarrow{M} \underline{t}_1 t_1 \underline{t}_2 \cdots \hat{t}_i \underline{t}_{|\underline{t}|} \xrightarrow{M} \underline{t}_2 \cdots \hat{t}_i \underline{t}_{|\underline{t}|}.$$

The final step is an operation of type (1). **q.e.d.**

Corollary 1.2.36. *The pair (W, S) is a Coxeter system.*

Proof. A relation is a word that evaluates to 1 in W. Therefore, any relation can be transformed into the empty word by M-reductions. However, these correspond to the relations of the Coxeter presentation. **q.e.d.**

1.2.4 The Moussong Complex

The goal in this section is to describe a piecewise Euclidean CAT(0) complex upon which the Coxeter group W acts cocompactly, properly, and discontinuously. We can find such a complex, provided the generating set S of reflections is finite. The existence of such a complex settles a lot of questions at once:

Corollary 1.2.37. *For every finitely generated Coxeter group the following hold:*

1. *W has solvable conjugacy problem.*
2. *W has only fintely many conjugacy classes of finite subgroups.*

The construction start with the Cayley graph. For any subset $J \subseteq S$, we define J-residues to be the components of the Cayley graph after all edges whose labels are not in J have been removed. So we restrict ourselves to edges with labels in J and look at the connected components of the resulting graph. By (1.2.10), every finite Coxeter group W_J is a Euclidean reflection group acting on some Euclidean space $\mathbb{E}$. The reflections are induced by finitely many hyperplanes that all pass through a common point. The hyperplanes chop up $\mathbb{E}$ into chambers. In one of these chambers find a point that has distance $\frac{1}{2}$ from all the walls. The orbit of this points spans a convex polyhedron P_J all of whose edges have length 1. Indeed, the 1-skeleton of this convex polyhedron is a Cayley graph for the finite Coxeter group W_J. The following exercise justifies all these claims.

Exercise 1.2.38. For any subset $J \subseteq S$, let M_J be the submatrix of M whose rows and columns have indices in J, and let (W_J, J) be the Coxeter system defined by M_J. Prove:

1. The inclusion $J \hookrightarrow S$ induces an injective group homomorphism $W_J \to W$ that identifies the group W_J with the subgroup of W generated by $J \subseteq S$.

2. in view of the preceeding result, we regard W_J as a subgroup of W. These subgroups are called special parabolic subgroups. Prove that

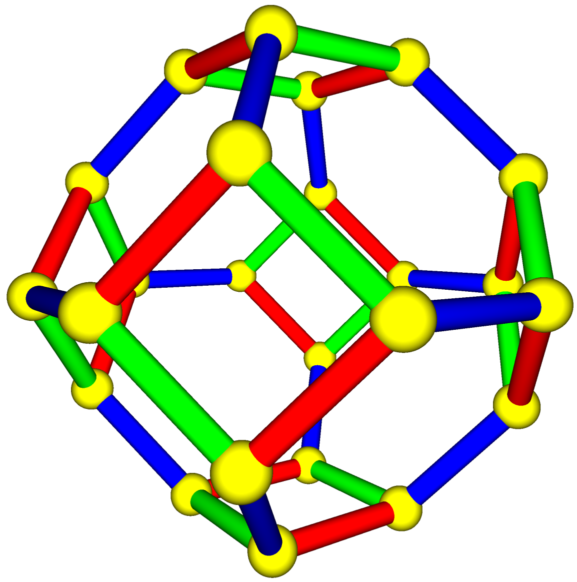
$$W_{J \cap I} = W_J \cap W_I$$

for any two subsets $J, I \subseteq S$.

3. The J-residues in Γ_W are in bijective correspondence to the left cosets of W_J.

4. Every J-residue is isomorphic to the Cayley graph of W_J with respect to the generating set J.

Example 1.2.39. Here is the polyhedron for $\left\langle \mathbf{b}, \mathbf{g}, \mathbf{r} \,\middle|\, \mathbf{b}^2 = \mathbf{g}^2 = \mathbf{r}^2 = (\mathbf{bg})^3 = (\mathbf{br})^3 = (\mathbf{gr})^2 = 1 \right\rangle$, which is the symmetric group on four letters:



Note how the faces correspond to cosets of special parabolic subgroups.

The <u>Moussong complex</u> X for the Coxeter system (W, S) is defined by the following procedure:

- Start with with the Cayley graph, and declare all edges to be of length 1. Observe that the edges with label s correspond precisely to the $\{s\}$-residues.

- Construct the 2-skeleton by glueing in polygons $P_{\{s,t\}}$ for any pair $\{s,t\}$ of generators that generate a finite subgroup. More precisely, if the $\{s,t\}$-residues are finite then $P_{\{s,t\}}$ is a polygon whose boundary is isomorphic to these residues. The isomorphism induces attaching maps that we use to glue in one copy of $P_{\{s,t\}}$ for each residue.

- The 3-skeleton is defined similarly. For every $J \subseteq S$ of size three, we glue in copies of P_J if the J-residues are finite. Note that the boundary sphere of P_J consists of polygons that are isomorphic to the cells P_I for strict subsets $I \subset J$. Thus, we find the boundary spheres of our 3-cells in the 2-skeleton that has been constructed already.

- Proceed on higher skeleta until every finite residue is geometrically realized.

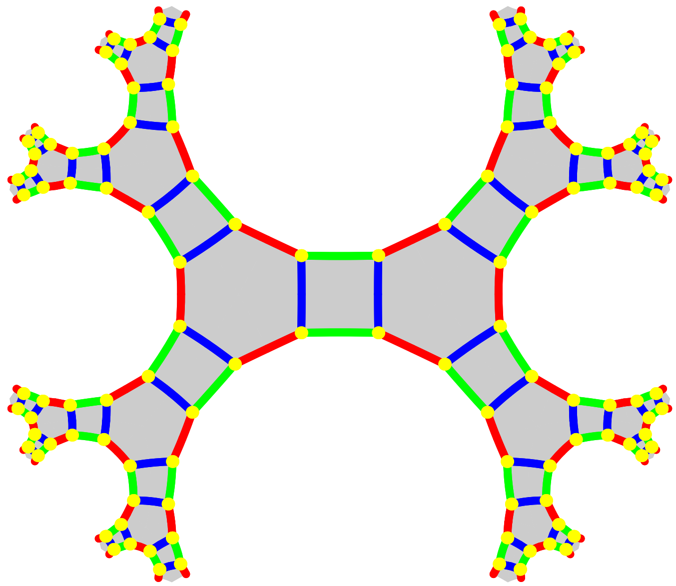
Observation 1.2.40. *The Moussong complex carries a natural piecewise Euclidean structure: all its cells are convex polyhedra in Euclidean space, and all attaching maps identify lower dimensional cells isometrically with faces of higher dimensional cells.*

Observation 1.2.41. *The 1-skeleton of X is the Cayley graph. The 2-skeleton is the Cayley complex of the Coxeter presentation for W: The 2-cells in X are precisely the cells whose boundaries read valid relations in W. It follows that X is simply connected.*

Corollary 1.2.42. *To prove X to be CAT(0) it suffices to show that vertex links in X are CAT(1) since X is piecewise Euclidean and simply connected.*

Example 1.2.43. Here is a (distorted picture of) the Moussong complex for the group

$$\left\langle \mathbf{b}, \mathbf{g}, \mathbf{r} \,\middle|\, \mathbf{b}^2 = \mathbf{g}^2 = \mathbf{r}^2 = (\mathbf{br})^3 = (\mathbf{bg})^2 = 1 \right\rangle .$$



The grey shaded area consists of hexagons and squares that are glued in. All these polygons are regular Euclidean polygons.

Observation 1.2.44. *Since W acts transitively on the set of vertices, all vertex links are isometric.*

Theorem 1.2.45. *The vertex link L of X is CAT(1).*

Proof. We give an explicit description of the link as a piecewise spherical complex: The vertex set of L is S. For every subset $J \subseteq S$ that generates a finite subgroup in W, we glue in a spherical simplex whose edge lengths are

$$d(s,t) = \pi - \frac{\pi}{m_{s,t}}.$$

We have to show that the resulting complex is metrically flag, i.e., if we find a subset $J \subseteq S$ such that all elements are joined by an edge (equivalently, $m_{s,t}$ is finite), then this subset should generate a finite subgroup if the edge lengths can be realized by a spherical simplex.

So suppose $\{\mathbf{u}_s \mid s \in J\}$ is a collection of unit vectors whose distances realize the edge lengths. Then

$$\langle \mathbf{u}_s, \mathbf{u}_t \rangle = \cos\left(\pi - \frac{\pi}{m_{s,t}}\right) = -\cos\left(\frac{\pi}{m_{s,t}}\right)$$

which is precisely the coefficient in the bilinear form $\langle -, - \rangle_{M_J}$, which therefore is positive definite. By (1.2.8), this implies that W_J is finite as required. **q.e.d.**