Chapter 1

Geometry: Nuts and Bolts

1.1 Metric Spaces

Definition 1.1.1. A metric space is proper if all closed balls are compact.

The length pseudo metric of a metric space X is given by

$$(x, y) \mapsto \inf_{p:x \longrightarrow y} |p|.$$

If the metric and the induced length pseudo metric coincide, the space X is called a length space.

A geodoesic (segment) in a metric space X is a distance preserving map

$$\gamma : [0, |\gamma|] \to X$$

whose domain is an interval. The length of the domain is the length of the geodesic. A bi-infinite geodesic or a geodesic line is a distance preserving map

$$\gamma : \mathbb{R} \to X,$$

and a geodesic ray is a distance preserving map

$$\gamma : \mathbb{R}^+ \to X.$$

A geodesic space is a a metric space wherein any two points are joined by a geodesic. A Hadamard space is a complete geodesic space.

Exercise 1.1.2. Let X be a complete metric space. Show that X is geodesic if "it has midpoints", i.e., for every pair $\{x,y\}$ there is a point z such that $d(x,z)=d(y,z)=\frac{1}{2}d(x,y)$.

Exercise 1.1.3. Let X be a complete metric space. Show that X is a length space if "it has approximate midpoints", i.e., for every pair $\{x,y\}$ and every $\varepsilon>0$ there is a point z such that $d(x,z)\,,d(y,z)\leq\varepsilon+\frac{1}{2}d(x,y)$.

Definition 1.1.4. A sequence $f_i:X\to Y$ of maps between metric spaces is equicontinuous if for any $\varepsilon>0$ there is a $\delta>0$ such that, for every i: if $d(x,y)\leq\delta$, then $d(f_i(x)\,,f_i(y))\leq\varepsilon$.

Fact 1.1.5 (Arzelà-Ascoli). *Let X be a compact metric space and Y be a separable metric space, then every sequence of equicontinuous functions $f_i:X\to Y$ has a subsequence that converges uniformly on compact subsets to a continuous map $f:X\to Y$.*

Corollary 1.1.6. *In a complete geodesic space with unique geodesics, those geodesics vary continuously with their endpoints.*

Fact 1.1.7 (Hopf-Rinow). *Let X be a complete, locally compact, length space. Then X is a proper geodesic space.*

Let M^m_κ be the simply connected Riemannian manifold of constant curvature κ of dimension m. (There is only one up to isometry.)

Definition 1.1.8. A geodesic triangle Δ inside a metric space X is called κ-admissible if there is a triangle in M^2_κ that has the same side length. Such a triangle is called a κ-comparison triangle.

Note that every point of Δ has a corresponding point in the comparison triangle. The triangle Δ is called κ-thin if distances of points in Δ are bounded from above by the distances of their corresponding points in a κ-comparision triangle.

A complete geodesic space X is CAT(κ) if any κ-admissible triangle in X is κ-thin. The space X is said to have curvature $\leq \kappa$, if it is locally CAT(κ), i.e., every point has an open ball around it so that this ball is a CAT(κ) space.

Exercise 1.1.9. Show that any two points in a CAT(0) space are connected by a unique geodesic segment.

Fact 1.1.10 (Cartan-Hadamard). *Let X be a connected complete metric space of curvature $\leq \kappa \leq 0$. Then the universal cover of X with the induced length metric is CAT(κ). Moreover, geodesic segments in the universal cover are unique and vary continuously with their endpoints.*

Definition 1.1.11. Let X be a CAT(0) space. A flat strip in X is a convex subspace that is isometric to a strip in the Euclidean plane bounded by two parallel lines.

Exercise 1.1.12 (Flat Strip Theorem). Let X be CAT(0) and let $\gamma : \mathbb{R} \to X$ and $\gamma' : \mathbb{R} \to X$ be two geodesic lines. Show that the convex hull of these two lines is a flat strip provided that the geodesic lines are asymptotic, i.e., the function $d(\gamma(t), \gamma'(t))$ is bounded.

Exercise 1.1.13. Let X be a connected complete metric space of curvature $\leq \kappa \leq 0$. Show that every free homotopy class has a representative that is a closed geodesic. Moreover, any two such representatives bound a "flat annulus", i.e., they lift to bi-infinite geodesics in the universal cover that bound a flat strip.

Definition 1.1.14. A subset $X' \subseteq X$ is convex if with any pair of points in X it contains all geodesic segments joining them.

Fact 1.1.15. *Let X' be a convex complete subspace of the proper CAT(0) space X. Then there is a nearest-point projection*

$$\pi : X \to X'$$

that takes every point $x \in X$ to the point in X' that is nearest to x. This image point exists by properness and is unique by CAT(0)-ness. For any point x outside X', the geodesic segment $[x, \pi(x)]$ is perpendicular to X'. Moreover π is a distance non-increasing map.

1.2 Piecewise Geometric Complexes

Recall that M_κ^m is the simply connected manifold of constant curvature κ of dimension m.

Definition 1.2.1. A piecewise M_κ-complex is a CW-complex whose cells have the structure of convex polyhedra in some M_κ and whose attching maps are isometric identifications with faces.

Fact 1.2.2 (Bridson). *If a connected piecewise M_κ complex has only finitely many shapes, then it is a complete geodesic space.*

Fact 1.2.3. *Let X be a piecewise M_κ complex. For $\kappa \leq 0$, the following are equivalent:*

1. X *is CAT*(κ).

2. X *has unique geodesics.*

3. *All links in* X *are CAT*(1) *and* X *does not contain a closed geodesic.*

4. *All links in* X *are CAT*(1) *and* X *is simply connected.*

For $\kappa > 0$, *the following are equivalent:*

1. X *is CAT*(κ).

2. *For any two points* $x, y \in X$ *of distance* $d(x,y) < \frac{\pi}{\sqrt{\kappa}}$, *there is a unique geodesic segment joining* x *and* y.

3. *All links in* X *are CAT*(1) *and* X *does not contain a closed geodesic of length* $< \frac{2\Pi}{\sqrt{\kappa}}$.

Fact 1.2.4 (Gromov's Lemma). *A piecewise spherical simplicial complex all of whose edges have length* $\frac{\pi}{2}$ *is CAT*(1) *if and only if it is a* <u>*flag complex*</u>, *i.e., any collection of vertices that are pairwise joined by edges forms a simplex. (Whenever you see a possible* 1*-skeleton of a simplex, the simplex is actually there.)*

A piecewise spherical simplicial complex is called <u>metrically flag</u> if every collection of vertices that are pairwise joined by edges forms a simplex provided there is a non-degenerate spherical simplex with those edge lengths.

Fact 1.2.5 (Moussong's Lema). *Let* X *be a piecewise spherical simplicial complex all of whose edges have length* $\geq \frac{\pi}{2}$. *Then* X *is CAT*(1) *if and only if* X *is metrically flag.*

Proof that metrically flag implies CAT(1)**.** We follow the argument of D. Krammer [**?**]. !!! ... !!! **q.e.d.**

Exercise 1.2.6. Prove that a CAT(1) piecewise spherical simplicial complex is metrically flag.

1.3 Group Actions

Definition 1.3.1. Let X be a metric space, and let $\lambda : X \to X$ be an isometry. The displacement function of λ is the map

$$D_\lambda : x \mapsto d(x, \lambda(x))\,.$$

The displacement of λ is

$$D(\lambda) := \inf_{x \in X} D(x)\,.$$

The min-set of λ is the set

$$\mathrm{Min}(\lambda) := \{x \in X \mid D(x) = D(\lambda)\}\,.$$

The isometry is parabolic if its min-set is empty. It is semi-simple otherwise. A semi-simple isometry is called elliptic if its displacement is 0 and hyperbolic otherwise.

Observation 1.3.2. *Let X be CAT(0). Then the min-set of any semi-simple isometry is a closed, convex, and complete subspace.*

Observation 1.3.3. *Let X be CAT(0), let λ be a semi-simple isometry of X, and let X' be a closed, convex, complete, and λ-invariant subspace. Then*

$$D_X(\lambda) = D_{X'}(\lambda)$$

since the nearest-point projection to X' is λ-equivariant and distance non-increasing.