Exercise 1.2.6. Prove that a CAT(1) piecewise spherical simplicial complex is metrically flag.

1.3 Group Actions

Definition 1.3.1. Let X be a metric space, and let $\lambda : X \to X$ be an isometry. The displacement function of λ is the map

$$D_\lambda : x \mapsto d(x, \lambda(x)) .$$

The displacement of λ is

$$D(\lambda) := \inf_{x \in X} D(x) .$$

The min-set of λ is the set

$$\mathrm{Min}(\lambda) := \{x \in X \mid D(x) = D(\lambda)\} .$$

The isometry is parabolic if its min-set is empty. It is semi-simple otherwise. A semi-simple isometry is called elliptic if its displacement is 0 and hyperbolic otherwise.

Observation 1.3.2. *Let X be CAT(0). Then the min-set of any semi-simple isometry is a closed, convex, and complete subspace.*

Observation 1.3.3. *Let X be CAT(0), let λ be a semi-simple isometry of X, and let X' be a closed, convex, complete, and λ-invariant subspace. Then*

$$D_X(\lambda) = D_{X'}(\lambda)$$

since the nearest-point projection to X' is λ-equivariant and distance non-increasing.

Proposition 1.3.4. *Let X be a complete CAT(0) metric space and let λ be a hyperbolic isometry. Then the min-set of λ is a disjoint union of bi-infinite geodesics each of which is fixed by λ set-wise. Indeed, λ acts on each of these* <u>*axes*</u> *as a translation of amplitude $D(\lambda)$.*

Proof. Let x be in the min-set of λ. The axis through x is the union of the geodesic segments $\left[\lambda^{k-1}x, \lambda^k x\right]$. To see that this is a geodesic, assume that at one of the break points the angle was not π. The midpoints of the adjacent edges would have distance strictly less than $D(\lambda)$. **q.e.d.**

Corollary 1.3.5. *For any semi-simple isometry λ, we have $D\left(\lambda^k\right) = kD(\lambda)$.* **q.e.d.**

1.3.1 Proper Actions

Definition 1.3.6. Let G act by isometries on the metric space X. For any subset $Y \subset X$, we define the <u>big stabilizer</u> to be

$$\overline{\mathrm{Stab}}_G(S) := \{g \in G \mid gY \cap Y \neq \emptyset\}.$$

The action is said to be <u>proper</u> if every compact subset $C \subseteq X$ has a finite big stabilizer.

The action is <u>properly proper</u> if for every point $x \in X$ there is an $r > 0$ such that the closed ball $\bar{B}_r(x)$ has a finite big stabilzer.

Exercise 1.3.7. Show that every properly proper action is proper, and that every proper action on a proper metric space is properly proper.

Theorem 1.3.8. *Let G act properly properly by isometries on the metric space X. Then the following hold:*

1. *For every point $x \in X$ there is an $\varepsilon > 0$ such that*

$$\overline{\mathrm{Stab}}_G\big(\bar{B}_\varepsilon(x)\big) = \mathrm{Stab}_G(x)\,.$$

2. *The action is discontinuous, i.e., the distance between orbits induce a metric (and not just a pseudo-metric) on the quotient space.*

3. *If G acts freely, then the projection $X \to G\backslash X$ is a covering projection and a local isometry.*

Proof. The first clain is easy and the other two follow. **q.e.d.**

1.3.2 Proper Cocompact Actions

Exercise 1.3.9. Let G act properly and cocompactly on the length space X. Show that X is complete and locally compact.

By the Hopf-Rinow theorem (1.1.7), we infer:

Corollary 1.3.10. *A length space that admits a cocompact proper action is a proper metric space.* **q.e.d.**

Proposition 1.3.11. *Let G act properly and cocompactly by isometries on the proper metric space X. Then the following hold:*

1. *There are only finitely many conjugacy classes of point stabilizers.*

2. *Every element $g \in G$ acts by a semi-simple isometry.*

Proof. For both arguments, we fix a compact subset $C \subseteq X$ whose G-translates cover X.

(1) Choose a finite cover of C by finitely many balls $\bar{B}_1, \ldots, \bar{B}_r$ whose big stabilizers are all finite. Then for every $x \in X$ there is an element $g \in G$ such that $gx \in \bar{B}_i$ for some i. Thus

$$g^{-1} \operatorname{Stab}(x)\, g \subseteq \overline{\operatorname{Stab}}\big(\bar{B}_i\big) \subseteq \bigcup_i \overline{\operatorname{Stab}}\big(\bar{B}_i\big)\,.$$

However the right hand is finite. Note that we only used strict properness of the action and could do away with properness of the space.

(2) Let (x_i) be a sequence of points in X such that $D_g(x_i) \to D(g)$ as $i \to \infty$. Choose elements g_i such that $y_i := g_i x_i \in C$.

Observe that

$$D_{g_i g g^{-1}} y_i = d\big(g_i^{-1} y_i, g g_i^{-1} y\big) = d(x_i, g x_i) \to D(g)\,.$$

Thus there is an ε such that the expression

$$d\big(x, g_i g g_i^{-1} x\big)$$

is bounded for all i and all $x \in C$ by $2\operatorname{diam}(C) + D(g) + \varepsilon$. Thus there is a closed ball such that all $g_i g g_i^{-1}$ are in its big stabilizer. Therefore, the sequence $g_i g g_i^{-1}$ traverses only finitely many different group elements. Passing through several subsequences, we may assume that $g_i g g_i^{-1}$ is constant and that y_i converges to some point $y \in C$. But then $g_i g g_i^{-1}$ is semi-hyperbolic with y in its min-set. Thus g is semi-hyperbolic with $g_i^{-1} y$ in its min-set. **q.e.d.**

Definition 1.3.12. The <u>translation distance</u> of a group element $g \in G$ with respect to a given action on the metric space X is the limit

$$\tau(g) := \lim_{k \to \infty} \frac{d\big(x, g^k x\big)}{k}.$$

Exercise 1.3.13. Show that translation distances exits and is independend of the choice of $x \in X$.

Exercise 1.3.14. Let λ by a semi-simple isometry of a CAT(0) space X. Show that $\tau(\lambda) = D(\lambda)$.

Definition 1.3.15. Let $G = \langle \Sigma \rangle$ be a finitely generated group. For every element $g \in G$, the translation length (with respect to Σ) is defined as

$$\underline{\tau}(g) := \lim_{k \to \infty} \frac{|g^k|}{k}.$$

Exercise 1.3.16. Show that the translation length of a group element is well defined, i.e. the limit exists and is independent of the generating set Σ.

Corollary 1.3.17. *If G acts isometrically, cocompactly, and properly on a proper CAT(0) space, then the following hold:*

1. *The finite subgroups of G are precisely the subgroups of G that have a global fixed point in X. In particular:*
   - *(a) The group G has only finitely many conjugacy classes of finite subgroups (1.3.11(1)).*
   - *(b) An element of G has finite order if and only if it acts by an elliptic isometry.*
2. *The group G does not contain a Baumslag-Solitar group*

   $$\left\langle a, b \,\middle|\, ba^q b^{-1} = a^p \right\rangle$$

   where $q \neq p$.
3. *The set of translation distances is discrete.*
4. *There is a strictly positive lower bound $\varepsilon > 0$ on the translation length of non-torsion elements of G.*

Proof.

(1) Since the action is proper, every point stabilizer is finite. It remains to prove that every finite subgroup has a global fixed point.

For any compact subset $C \subseteq X$ let $\bar{B}_C$ denote the smallest closed ball that contains C. (It follows from X being CAT(0) that this ball exists and is unique.) Let x_C be the center of $\bar{B}_C$. Note that since $\bar{B}_C$ is defined entirely in metric terms, we have

$$\bar{B}_{gC} = g\bar{B}_C$$

and

$$x_{gC} = gx_C$$

for any group element $g \in G$.

Let $F \leq G$ be a finite subgroup. Let C be the orbit of some point $y \in X$. Note that C is F-invariant. By the preceeding considerations, the point x_C is F-invariant, too.

(2) As a matter of fact, Baumslag-Solitar groups are torsion free. Thus, if G contained a copy, the elements inside the subgroup would be hyperbolic. Since a^q and a^p are conjugate, they have the same displacement. On the other hand, their displacements have the ratio $\frac{q}{p}$. It follows that the displacement of a is 0 which is a contradiction.

(3) Suppose we had a sequence g_i of group elements whit different translation distances that converge to a limit L. Passing to a sequence of conjugates (which have the same translation lengths), we find a sequence of points $x_i \in C$ such that x_i is in the min-set of g_i. The contradiction is assumed at any accumulation point $x \in C$ as hit by the different g_i: The ball of radius $L + 3\varepsilon$ is not moved off itself by any g_i where $\tau(g_i) = D(g_i)$ is ε-close to L and x_i is ε-close to x.

(4) Observe that for any point $x \in X$, we have

$$\begin{aligned} d(x, g_1 \cdots g_r x) &\leq d(x, g_1 \cdots g_{r-1} x) + d(g_1 \cdots g_{r-1} x, g_1 \cdots g_r x) \\ &= d(x, g_1 \cdots g_{r-1} x) + d(x, g_r x) \\ &\leq \sum_{i=1}^{r} d(x, g_i x) \\ &\leq r \max_{i \in \{1,\dots,r\}} d(x, g_i x) \,. \end{aligned}$$

Thus, fixing a generating set, we can find a constant C such that for any element $g \in G$,

$$\frac{d\big(x, g^k x\big)}{k} \leq C \frac{\left|g^k\right|}{k}.$$

Passing to the limit, we find

$$\underline{\tau}(g) \geq \frac{\tau(g)}{C} \geq \varepsilon$$

for some $\varepsilon > 0$ that exists by (3). **q.e.d.**

1.3.3 Abelian and Solvable Subgroups

Exercise 1.3.18. Let G be virtually $\mathbb{Z}^n$, and let H be a subgroup of G that is isomorphic to $\mathbb{Z}^n$. Show that H has finite index in G. (Hint: Consider a finite index $\mathbb{Z}^n$ inside G and the action of H on $G/\mathbb{Z}^n$.)

Exercise 1.3.19. Let G be finitely generated. Show that G is virtually abelian provided the commutator subgroup $[G, G]$ is finite. (Hint: Let H be the centralizer of $[G, G]$ in G and show that the center of H has finite index in H and that H has finite index in G.)