Math 758 – *Your* Favorite Groups (homework 1, due Feb 5)

Exercise 1.1. Prove that a gallery from C to D has minimum length if and only if it does not cross any hyperplane twice. Moreover, the set of hyperplanes that are crossed by a minimum length gallery from C to D is precisely the set of those $H \in \mathcal{H}$ that separate C from D. In particular, this set is the same for all those minimum length galleries.

Exercise 1.2. Show that for any $s, t \in S$,

$$\langle \mathbf{u}_s, \mathbf{u}_t \rangle = \begin{cases} -\cos\left(\frac{\pi}{m_{s,t}}\right) & \text{for } m_{s,t} \text{ finite} \\ -1 & \text{for } m_{s,t} \text{ infinite.} \end{cases}$$

Exercise 1.3. Show that the following are equivalent:

1. $\mathcal{H}$ is finite.
2. W is finite.
3. W is torsion.
4. $\bigcap_{H \in \mathcal{H}} H \neq \emptyset$.

Definition. Let $G = \langle x_1, \ldots, x_r \rangle$ be a group generated by a given finite generating set $\Sigma = \{x_1, \ldots, x_r\}$.

The word problem for the pair (G, Σ) is the following:

> Is there an algorithm that takes as input any word w over $\Sigma \cup \Sigma^{-1}$ and prints "yes" if the word w evaluates in G to 1 and prints "no" otherwise?

If there is such an algorithm, the pair (G, Σ) is said to have solvable word problem.

The conjugacy problem for the pair (G, Σ) is the following:

> Is there an algorithm that takes as input any pair (w, u) of words over $\Sigma \cup \Sigma^{-1}$ and prints "yes" if the two word evaluate to conjugate elements of G and prints "no" otherwise.

If there is such an algorithm, the pair (G, Σ) is said to have solvable conjugacy problem.

Exercise 1.4. Let Σ and Ξ be two finite generating sets for G. Show that the pair (G, Σ) has a solvable word problem (conjugacy problem) if and only if the pair (G, Σ) has a solvable word problem (conjugacy problem). As the parentheses indicate, there is a proof that works for both problems.

Exercise 1.5. Let $H \hookrightarrow G \twoheadrightarrow F$ be a short exact sequence of groups where F is finite and H is finitely generated and has solvable word problem. Show that G is finitely generated and has solvable word problem.

Exercise 1.6. Let H be a subgroup of finite index in the finitely generated G. Then H is finitely generated, too. Show that H has solvable word problem if and only if G has solvable word problem.

Each problem is worth 5 points, but you can earn at most 20 points with this assignment.

Late homework will not be accepted.