Math 758 – *Your* Favorite Groups (homework 3, due Feb 19)

A morphism of graphs is a distance non-increasing map from the vertices of graph to the vertices of another graph. A folding of a graph is an idempotent graph endomorphism $f : \Gamma \to \Gamma$ such that the preimage of each vertex$v$ is either empty or contains precisely two vertices (one of which is v). The image α_f of a folding is called a half space or a root. Two foldings f and f' are opposite if their images are disjoint and the following hold:

$$\begin{aligned} f &= f \circ f' \\ f' &= f' \circ f. \end{aligned}$$

Exercise 3.1. Show that a (locally finite) graph is the Cayley graph of a (finitely generated) Coxeter group if and only if the following conditions holds:

1. For each oriented edge $\overrightarrow{e}$ there is a unique folding $f_{\overrightarrow{e}}$ of Γ satisfying $f_{\overrightarrow{e}}(\iota(\overrightarrow{e})) = \tau(\overrightarrow{e})$.
2. If $\overrightarrow{e}$ and $\overleftarrow{e}$ are opposite orientations of the same underlying geometric edge, then $f_{\overrightarrow{e}}$ and $f_{\overleftarrow{e}}$ are opposite foldings.

Exercise 3.2. Let X be a complete metric space. Show that X is geodesic if "it has midpoints", i.e., for every pair $\{x, y\}$ there is a point z such that $d(x, z) = d(y, z) = \frac{1}{2}d(x, y)$.

Exercise 3.3. Let X be a complete metric space. Show that X is a length space if "it has approximate midpoints", i.e., for every pair $\{x, y\}$ and every $\varepsilon > 0$ there is a point z such that $d(x, z)\,, d(y, z) \leq \varepsilon + \frac{1}{2}d(x, y)$.

Exercise 3.4. Show that any two points in a CAT(0) space are connected by a unique geodesic segment.

Let X be a CAT(0) space. A flat strip in X is a convex subspace that is isometric to a strip in the Euclidean plane bounded by two parallel lines.

Exercise 3.5 (Flat Strip Theorem). Let $\gamma : \mathbb{R} \to X$ and $\gamma' : \mathbb{R} \to X$ be two geodesic lines. Show that the convex hull of these two lines is a flat strip provided that the geodesic lines are asymptotic, i.e., the function $d(\gamma(t)\,, \gamma'(t))$ is bounded.

Exercise 3.6. Let X be a connected complete metric space of curvature $\leq \kappa \leq 0$. Show that every free homotopy class has a representative that is a closed geodesic. Moreover, any two such representatives bound a "flat annulus", i.e., they lift to bi-infinite geodesics in the universal cover that bound a flat strip.

Each problem is worth 5 points, but you can earn at most 20 points with this assignment.

Late homework will not be accepted.