Math 758 – *Your* Favorite Groups (homework 5, due Mar 7)

For $i \geq 0$, let

$$\nu : \mathbb{B}^{i-1} \hookrightarrow \mathbb{B}^i$$

be the embedding of $\mathbb{B}^{i-1}$ in $\mathbb{B}^i$ as the northern hemisphere and let

$$\sigma : \mathbb{B}^{i-1} \hookrightarrow \mathbb{B}^i$$

be the embedding as the southern hemisphere. A map $\pi : E \to X$ has the <u>homotopy lifting property in dimension m</u> if for every commutative diagram

$$\begin{tikzcd} \mathbb{B}^{m-1} \arrow[r, "\bar{f}"] \arrow[d, "\nu"'] & E \arrow[d, "\pi"] \\ \mathbb{B}^m \arrow[r, "f"'] & X \end{tikzcd}$$

there exists a lift $\tilde{f} : \mathbb{B}^m \to E$ such that

$$\begin{tikzcd} \mathbb{B}^{m-1} \arrow[r, "\bar{f}"] \arrow[d, "\nu"'] & E \arrow[d, "\pi"] \\ \mathbb{B}^m \arrow[r, "f"'] \arrow[ur, "\tilde{f}"] & X \end{tikzcd}$$

commutes.

A map that has the homotopy lifting property in all dimensions is called a <u>fibration</u>. A <u>fibre bundle over X with fibre F</u> is a map $f : E \to X$ such that:

1. There is an intersection closed open cover $\mathcal{U}$ of X and a familiy of <u>local trivializations</u> $(\phi_U : U \times F \to E)_{U \in \mathcal{U}}$ that are homeomorphism onto their image and preserve fibres, i.e., the following diagram

$$\begin{tikzcd} U \times F \arrow[r, "\phi_U"] \arrow[d] & E \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

commutes.

2. For every inclusion $U \hookrightarrow V$, there is a fibre preserving <u>change of coordinates</u> $\phi : U \times F \to V \times F$ such that

$$\begin{tikzcd} & U \times F \arrow[dl] \arrow[dd, "\phi"] \arrow[dr, "\phi_U"] & \\ X & & E \\ & V \times F \arrow[ul] \arrow[ur, "\phi_V"'] & \end{tikzcd}$$

commutes

Exercise 5.1. Prove that fibre bundles are fibrations.

Suppose $\pi : E \to X$ has the homotopy lifting property in dimensions $\leq m+1$. Let F be the preimage of the base point in X. Recall that every element in $\pi_m(X)$ can be represented as a map $f : \mathbb{B}^m \to X$ that takes the whole boundary sphere $\partial(\mathbb{B}^m)$ to the base point of X. Let $\tilde{f}$ be the lift in

$$\begin{tikzcd} \mathbb{B}^{m-1} \arrow[r, "\bar{f}"] \arrow[d, "\nu"'] & E \arrow[d, "\pi"] \\ \mathbb{B}^m \arrow[r, "f"'] \arrow[ur, "\tilde{f}"] & X \end{tikzcd}$$

where $\mathbb{B}^{m-1}$ entirely sent to the basepoint of E. Then the composition

$$\mathbb{B}^{m-1} \xrightarrow{\sigma} \mathbb{B}^m \xrightarrow{\tilde{f}} E$$

maps $\mathbb{B}^{m-1}$ to F and sends the boundary sphere $\partial(\mathbb{B}^{m-1})$ to the base point. Thus, it defines an element of $\pi_{m-1}(F)$.

Exercise 5.2. Prove that this map is a well defined group homomorphism $\pi_m(X) \to \pi_{m-1}(F)$.

Exercise 5.3. For $i \leq m$, show that the sequence

$$\pi_i(E) \to \pi_i(X) \to \pi_{i-1}(F)$$

is exact in the middle.

Exercise 5.4. For $i \leq m$, show that the sequence

$$\pi_i\left(X\right) \to \pi_{i-1}\left(F\right) \to \pi_{i-1}\left(E\right)$$

is exact in the middle.

Exercise 5.5. The aim of this exercise is to show that Eilenberg-Maclane complexes are unique up to homotopy equivalence. Let G be a group, and let X and Y be two cell complexes. Assume that G acts on both complexes freely and cellularly. Construct two G-equivariant maps $f : X \to Y$ and $h : Y \to X$ such that the composition $h \circ f$ is G-equivariantly homotopic to the identity on X and $f \circ h$ is G-equivariantly homotopic to the identity on Y. Conclude that

$$G\backslash X \backsimeq G\backslash Y.$$

(Hint: Use induction on skeleta for the construction of the maps as well as for the construction of the homotopies.)

Exercise 5.6. Let C_n be a non-trivial finite cyclic group. Find a "nice" Eilenberg-Maclance complex for this group and compute its homology. In particular, demonstrate that there are non-vanishin homology groups in arbitrary high dimensions.

Conclude that no Eilenberg-Maclance space for C_n has finite dimension.

It follows that the braid group B_n is torsion free.

Each problem is worth 5 points, but you can earn at most 20 points with this assignment.

Late homework will not be accepted.