Math 758 – *Your* Favorite Groups (homework 6, due Mar 17)

Exercise 6.1. Show that the $\mathrm{H}_3\big({}_{\mathrm{Perm}_n}\backslash^{|\mathcal{A}_4|}\big)$ is non-trivial. Infer that B_4 does not have an Eilenberg-Maclane complex of dimension ≤ 2.

Exercise 6.2. Prove that $B_3 = \langle a, b, c \mid ab = bc = ca \rangle$.

Exercise 6.3. More generally, prove that

$$B_n = \left\langle x_{[i,j]} \ \ (i \neq j) \ \middle| \ \begin{array}{ll} x_{[i,j]}x_{[j,k]} = x_{[j,k]}x_{[k,i]} & \text{if } [i,j,k] \\ x_{[i,j]}x_{[k,l]} = x_{[k,l]}x_{[i,j]} & \text{if } [i,j,k,l] \end{array} \right\rangle$$

where we put a cyclic ordering on $\{1, 2, \ldots, n\}$ and $[a, b, \ldots]$ denotes the fact that the listed elements form a cycle in their given order. In particular, the generators are indexed by cycles of length 2.

Exercise 6.4. Prove that $B_3 = \langle a, b, c, s \mid ab = bc = ca = s \rangle$. Moreover, show that the Cayley 2-complex (i.e., the universal cover of the canonical 2-complex associated to this presentation) admits a CAT(0) metric. (This implies that the presentation 2-complex for this presentation is an Eilenberg-Maclane space for B_3.)

Exercise 6.5. Decide whether the presentation 2-complex for the presentation

$$B_3 = \langle a, b, c \mid ab = bc = ca \rangle$$

is an Eilenberg-Maclane complex for B_3.

Exercise 6.6. Let L be a flag complex. Define

$$\mathbb{S}(L) := \bigcup_{\sigma \text{ simplex}} \mathbb{S}^{\sigma} \subset \mathbb{R}^{\mathcal{V}}$$

where $\mathcal{V}$ is the vertex set of L and $\mathbb{S}^{\sigma}$ is the unit sphere in the subspace of $\mathbb{R}^{\mathcal{V}}$ spanned by $\sigma \subseteq \mathcal{V}$. Define a triangulation of $\mathbb{S}(L)$ by

$$\mathbb{S}^{\sigma} = \mathop{\ast}_{v \in \sigma} \mathbb{S}^{\{v\}}.$$

Prove that $\mathbb{S}(L)$ is a flag complex that contains L as a retract. (It is in general not a deformation retract!)

Each problem is worth 5 points, but you can earn at most 20 points with this assignment.

Late homework will not be accepted.