Chapter 1

$\mathrm{Out}(F_n)$ and $\mathrm{Aut}(F_n)$

1.1 Topological Representatives for Automorphisms

Definition 1.1.1. Let X be a topological space with base point P. A self-homotopy equivalence is a base point preserving map $f : X \to X$ such that there is a base point preserving homotopy inverse, i.e., a base point preserving map $h : X \to X$ such that

$$f \circ h \sim \mathrm{id}_X \text{ rel } P$$

and

$$h \circ f \sim \mathrm{id}_X \text{ rel } P.$$

The mapping class group of (X, P) is the group

$$M(X, P) := \{[f]_P \mid f : X \to X \text{ is a homotopy equivalence rel. } P\}.$$

This is the group of self-homotopy equivalences modulo homotopy relative to the base point.

Observation 1.1.2. *The mappging class group is a group.* **q.e.d.**

Observation 1.1.3. *The map*

$$\begin{array}{rcl}\nu : M(X,P) & \to & \mathrm{Aut}(\pi_1(X,P)) \\ [f] & \mapsto & \alpha_{[f]} : [\gamma] \mapsto [f \circ \gamma]\end{array}$$

is a group homomorphism.

If X is the rose R_n on n pedals, this homomorphism has an inverse given as follows: An automorphism of $\pi_1(R_n, P)$ assigns to each pedal a loop in R_n. We can think of this loop as a map from its pedal to R_n. Since the base point is preserved, we can paste the maps on that we obtained for the individual pedals together. This way, we defined a map $R_n \to R_n$. **q.e.d.**

Corollary 1.1.4. $\mathrm{Aut}(F_n) = M(R_n, P)$. **q.e.d.**

1.1.1 Stallings Folds

Let Γ and Δ be two graphs. A <u>fold</u> is a map

$$f_{\overrightarrow{e},\overrightarrow{e}'} : \Gamma \to \Delta$$

that identifies two oriented edges $\overrightarrow{e}$ and $\overrightarrow{e}'$ that have the same initial vertex. A fold is called <u>singular</u> if the two edges also share their terminal vertices, it is called <u>non-singular</u> otherwise.

Observation 1.1.5. *A non-singular fold is a homotopy equivalence. A singular fold induces a non-injective map in homotopy. Its kernel is the normal subgroup generated by the loop $\overrightarrow{e}\,\mathrm{op}(\overrightarrow{e}')$.* **q.e.d.**

Observation 1.1.6. *Let $\varphi : \Gamma \to \Delta$ be a graph morphism. If φ is not locally injective, then there is a vertex v from which two edges $\overrightarrow{e}$ and $\overrightarrow{e}'$ issue that are identified by the map φ. Thus, φ factors through the fold $f_{\overrightarrow{e},\overrightarrow{e}'}$:*

$$\varphi = \varphi' f_{\overrightarrow{e},\overrightarrow{e}'}.$$ **q.e.d.**

Proposition 1.1.7 (Stallings). *Let Γ be a finite graph and let $\varphi : \Gamma \to \Delta$ be a graph morphism. Then there is a finite sequence of folds*

$$\Gamma = \Gamma_0 \xrightarrow{f_1} \Gamma_1 \xrightarrow{f_2} \Gamma_2 \xrightarrow{f_3} \cdots \xrightarrow{f_r} \Gamma_r$$

and a graph morphism

$$\psi : \Gamma_r \to \Delta$$

such that

$$\varphi = \psi \circ f_r \circ f_{r-1} \circ \cdots \circ f_1$$

and such that ψ is locally injective.

Proof. Since every fold decreases the number of edges, every sequence of folds must terminate. So you try to write $\varphi = \varphi_1 f_1$. If this succeds, you try the same on φ_1. Continue until you do not find a way of factoring through a fold. We have observed in (1.1.6) that in this case the map is locally injective. **q.e.d.**

Let $\alpha : F_n \to F_n$ be an automorphism. We realize α as a graph morphism: Subdivide R_n such that pedal l_i has as many segments as needed to write the word representing $\alpha(x_i)$. Call the subdivided rose $\overline{R}_n$. These words then define a map

$$\varphi : \overline{R}_n \to R_n.$$

Let us factor out a maximal sequence of folds:

$$\varphi = \psi \circ f_r \circ f_{r-1} \circ \cdots \circ f_1$$

such that ψ is locally injective. By (1.1.5), all the folds are non-singular, for otherwise we would not induce an isomorphism of fundamental groups.

Observation 1.1.8. *A locally injective map takes non-backtraking paths to non-backtracking paths. Thus, since φ is onto in π_1, the map $\psi : \Gamma_r \to R_n$ is an isomorphism of graphs: Look at the vertex in R_n and consider which paths in Γ_r give rise to the simple loops based at the vertex. It is immediate that for each such simple loop in R_n there has to be a corresponding loop in Γ_r based at its base point. Then, however, local injectivity rules out the existence of any other edges.* **q.e.d.**

Corollary 1.1.9. *Any assignment $x_i \mapsto w_i$ of words to generators determines a homomorphism $F_n \to F_n$. This homomorphism is an automorphism if and only if the topological representative, realized as a graph morphism $\varphi : \overline{R}_n \to R_n$, decomposes as*

$$\varphi = \psi \circ f_r \circ f_{r-1} \circ \cdots \circ f_1$$

such that all folds are non-singular and $\psi : \Gamma_n \to R_n$ is an isomorphism of graphs. This criterion can be checked algorithmically. **q.e.d.**

Corollary 1.1.10. *Every generating set of F_n that consists of precisely n elements is a free generating set.*

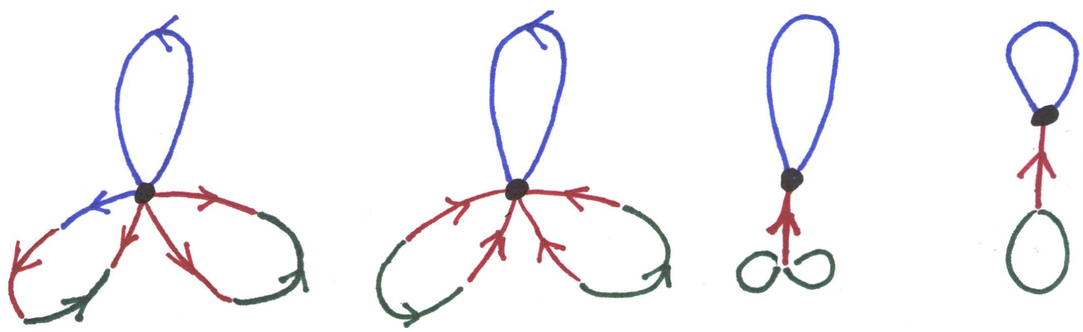
Proof. We only needed surjectivity to argue that ψ is an isomorphism of graphs. Since a non-singular fold will ensure non-surjectivity (check this in homology, if you consider it non-obvious), we infer from surjectivity alone that all folds are non-singular and the final locally injective graphmorphism is an isomorphism. Thus, every surjection $F_n \twoheadrightarrow F_n$ is an isomorphism. Compare also Grushko's Theorem (**??**), which is also proved using Stallings folds. **q.e.d.**

Example 1.1.11. Let us consider $F_3 = \langle \mathbf{b}, \mathbf{g}, \mathbf{r} \rangle$.

1. The assignment

$$\begin{aligned} \mathbf{b} &\mapsto \mathbf{b} \\ \mathbf{g} &\mapsto \mathbf{brgr}^{-1} \\ \mathbf{r} &\mapsto \mathbf{rgr}^{-1} \end{aligned}$$

has the following topological representative and crucial stages in the folding sequence:

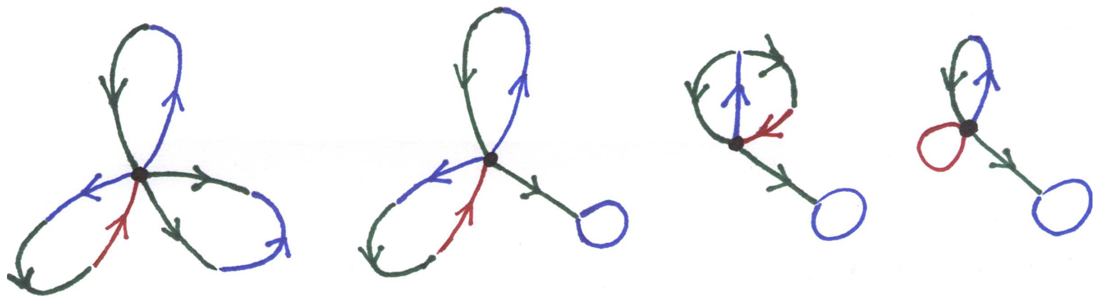


The non-singular fold at the end detects a failure of injectivity. The final picture also indicates a failure of surjectivity. Indeed, we can read off that the image of the homomorphism is the subgroup generated by $\mathbf{b}$ and $\mathbf{rgr}^{-1}$.

2. The assignment

$$\begin{aligned} \mathbf{b} &\mapsto \mathbf{gb} \\ \mathbf{g} &\mapsto \mathbf{rgb} \\ \mathbf{r} &\mapsto \mathbf{g}^{-1}\mathbf{bg} \end{aligned}$$

yields:

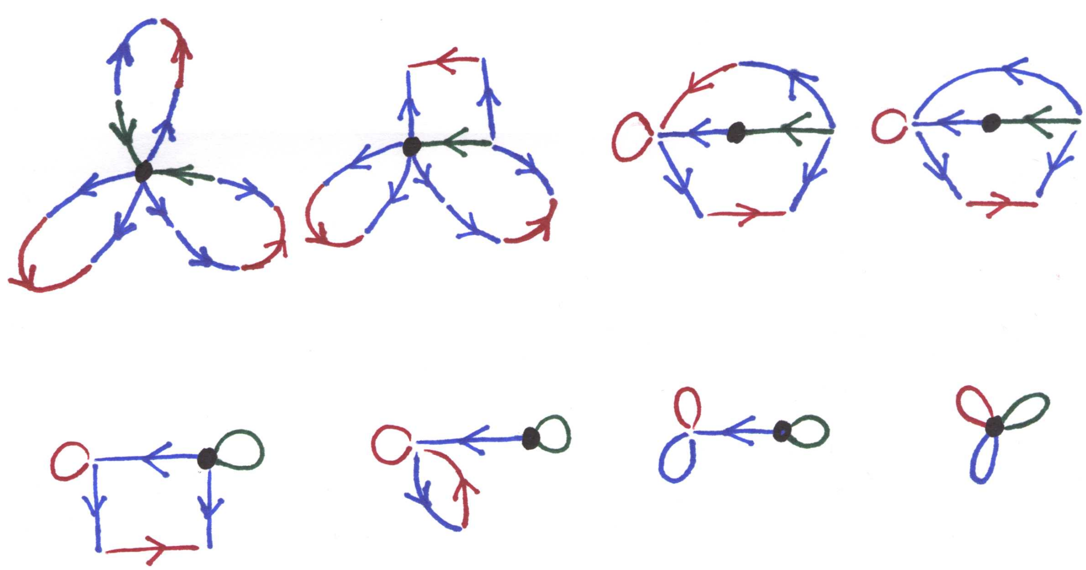


and we see that surjectivity fails.

3. The assignment

$$\begin{aligned} \mathbf{b} &\mapsto \mathbf{bbrb^{-1}g} \\ \mathbf{g} &\mapsto \mathbf{brb^{-1}g} \\ \mathbf{r} &\mapsto \mathbf{brb^{-1}} \end{aligned}$$

has the following topological representative and crucial stages in the folding sequence:



In this case, we actually have an automorphism.

1.2 A Generating Set for $\mathrm{Aut}(F_n)$

Theorem 1.2.1 (Nielsen). *The following automorphisms of F_n generate $\mathrm{Aut}(F_n)$:*

1. *Transposition of two free generators.*
2. *Inversion of a free generators.*
3. *The autmorphisms $\alpha_{i,j}$ defined as follows:*

$$x_k \mapsto \begin{cases} x_k & \text{if } k \neq i \\ x_i x_j & \text{if } k = i. \end{cases}$$

1.2.1 Proof of Nielsen's Theorem

The idea is to use Stallings folds. So let $\varphi : \overline{R}_n \to R_n$ be a graph morphism representing the automorphism $\alpha : F_n \to F_n$. We decompose

$$\varphi = \psi \circ f_r \circ \cdots \circ f_1$$

with ψ locally injective and $f : \Gamma_{i-1} \to \Gamma_i$ a fold. We know that ψ is actually an isomorphism of graphs. Thus this gives a permutation of the genertators, some of which are possibly inverted. We will want to recognize the folds as being related to the generators $\alpha_{i,j}$. However, this is not straight forward since there is no canonical identification of F_n with the fundamental groups of the intermediate graphs Γ_i.

The way to fix this, is to consider spanning trees in these graphs. Since permuting and inverting generators is covered by the generating set, we will need a data structure that just keeps track of an unodered set of free generators up to inversion (we could call a two element subset $\{g, g^{-1}\} \subset F_n$ an unsigned element).

Definition 1.2.2. So let Γ together with

1. a labeling, i.e., a graph morphism $\rho : \Gamma \to R_n$ (one should think of this as an assignment of free generators or their inverses to the oriented edges such that swapping the orientation of an edge corresponds to inverting the generator),
2. a base vertex v_0,
3. and a spanning tree T.

We will associate to this the following set of unsigned elements in F_n: For each edge e not in T, there is a reduced cyclic edge path, unique up to orientation, starting at v_0 traveling along a geodesic in T going through e and heading back to v_0 within T. Collect all these elements. Let $S(\Gamma, \rho, v_0, T)$ denote this set.

We have to study how this set changes with respect to the following transformations:

1. Change of the spanning tree.
2. Folding the graph.

Let us do change of spanning trees first. To simplify matters, we will want to change spanning trees only a little bit, say replacing one edge at a time.

Definition 1.2.3. Let Γ be a graph. The complex of forests is the simplicial complex $\mathcal{F}(\Gamma)$ whose vertices are the non-loop edges in Γ and whose simplices are the sub-forests in Γ. Note that all maximal simplices in $\mathcal{F}(\Gamma)$ have the same dimension. Such complexes are called chamber complexes—the chambers are the maximal simplices. In the case of $\mathcal{F}(\Gamma)$ the chambers are precisely the spanning trees.

Two chambers are called adjacent if they share a codimension-1 face. A gallery is a sequence of chambers such that neighboring terms in the sequence are adjacent chambers.

Lemma 1.2.4. *$\mathcal{F}(\Gamma)$ is gallery connected, i.e., any two chambers are joined by a gallery.*

Proof. Let T and T' two spanning trees, and let e be an edge of T' that does not occur in T. Adding this edge to T will create a circle. So we are to remove an edge from this circle. Note that any edge will do. At least one of the edges alon this circle does not belong to T' since this tree does not contain circles. So we can exchange an edge that is in T but not in T' by the edge e that is in T' but not in T and form a new spanning tree. However, this tree differs from T' in fewer places. So, we keep doing this until we have removed all differences (which are only finite in number). **q.e.d.** **q.e.d.**

Proposition 1.2.5. *Let T and T' be two adjacent spanning trees in the labelled and basepointed graph Γ. Then $S_T := S(\Gamma, \rho, v_0, T)$ and $S_{T'} := S(\Gamma, \rho, v_0, T')$ are related as follows:*

> *There is an element $g \in S_T \cap S_{T'}$, and for every other element $h \in S_{T'}$ there is an element $h' \in S_T$ such that one of the following holds:*

$$\begin{aligned} h &= h' \\ h &= h'g \\ h &= h'g^{-1} \\ h &= gh' \\ h &= g^{-1}h' \end{aligned}$$

Note that we can realize a transition of this type as a product of Nielsen generators.

Proof. !!! PICTURE !!! **q.e.d.**

The very same picture also yields our first result about how folds change the generating set:

Proposition 1.2.6. *Let $f:\Gamma\to\Delta$ be a fold compatible with the labeling ρ that identifies an edge in the spanning tree T with a loop. Then Δ has a spanning tree T' induced by T and a labeling τ induced by the labeling ρ. and $S_\Gamma := S(\Gamma,\rho,v_0,T)$ is related to $S_\Delta := S(\Delta,\tau,v_0,T')$ in the same way as described in (1.2.5).*

Proof. !!! PICTURE !!! **q.e.d.**

Observation 1.2.7. *A fold of two edges in the spanning tree does not affect $S(\Gamma,\rho,v_0,T)$.* **q.e.d.**

We put everything together. If a fold identifies two edges none of which is a loop, then we can change the spanning tree to contain both of these edges. The change of the spanning tree is taken care of by (1.2.5). If one of the edges is a loop, we can at least put the other edge in the spanning tree (it cannot be a loop itself, since we do not have singular folds). Afterwards, we are done by (1.2.6). Therefore, along our chain of folds, we can use Nielsen generators to realize each fold.

1.2.2 The Homotopytype of the Complex of Forests

Theorem 1.2.8. *Let Γ be a finite graph with $m+2$ vertices. Then the complex $\mathcal{F}(\Gamma)$ is homotopy equivalent to a wedge of m-spheres.*

Proof. Induct on the number of edges. Starting point is the case of a bridge edge which serves as a cone point. The Induction step is that removing a non-bride gives you a complex of the same type with fewer edges which is m-spherical by induction. Now, the relative link of the vertex corresponding to the removed edge is a the forest complex for the graph obtained by collapsing this edge. This is $(m-1)$-spherical. **q.e.d.**

Exercise 1.2.9. Find a recursive way to compute the number of spanning trees in a graph Γ.

1.3 Outer Space