Chapter 1

Coxeter Groups and Artin Groups

1.1 Euclidean Reflection Groups

Let

- $\mathbb{E}$ be a Euclidean space, and let
- $\mathcal{H}$ be a set of hyperplanes satisfying the following:
  1. $\mathcal{H}$ is locally finite, i.e., a set of hyperplanes such that any compact subset of $\mathbb{E}$ intersects only finitely many hyperplanes from $\mathcal{H}$.
  2. $\mathcal{H}$ is a W-invariant subset of $\mathbb{E}$ where W is the subgroup of $\mathrm{Isom}(\mathbb{E})$ generated by all reflections ρ_H with $H \in \mathcal{H}$.

Definition 1.1.1. Such a group W is called a Euclidean reflection group.

Exercise 1.1.2. Assume that $\mathcal{H}$ is finite. Show that $\bigcap_{H\in\mathcal{H}} H \neq \emptyset$. See (1.1.16) for a more elaborate statement.

We want to derive a presentation for W.

1.1.1 The Chamber Decomposition of $\mathbb{E}$

A chamber is a complementary component of $\mathcal{H}$, i.e., a component of $\mathbb{E} - \bigcup_{H\in\mathcal{H}} H$. Note that the closure of a chamber C is a convex polytope (possibly non-compact). The faces of this polytope span hyperplanes that belong to $\mathcal{H}$. We say that those hyperplanes from $\mathcal{H}$ are supporting C. For any chamber C, we denote by

- $|C|$ the set of hyperplanes in $\mathcal{H}$ supporting C.

  Two chambers, C and D, are called adjacient along H if $H \cap C = H \cap D$ is a CoDim-1-face. In this case, we write

$$C|_H D.$$

  They are called adjacient if they are adjacient along some H. In this case, we write

$$C|D.$$

  Note that adjaciency and adjaciency along H are symmetric and reflexive relations.

  A gallery is a sequence

$$C_0|C_1|\cdots|C_r$$

  of chambers such that C_i is adjacient to C_{i+1} for all $i < r$. If $C_i|_H C_{i+1}$, we say that the gallery crosses H at this step.

  The last index r gives the length of the gallery, which henceforth is the number of hyperplanes that are crossed by the gallery. The distance

- $\delta(C, D)$ of the chambers C and D is the minimum length of a gallery connecting them. Note that two chambers are adjacient if and only if their distance is at most 1.

Exercise 1.1.3. Show that C and D are H-adjacient if and only if H supports both and $\{C, \rho_H C\} = \{D, \rho_H D\}$.

Observation 1.1.4. *Any two chambers are connected by a gallery of finite length.* **q.e.d.**

Exercise 1.1.5. Prove that a gallery from C to D has minimum length if and only if it does not cross any hyperplane twice. Moreover, the set of hyperplanes that are crossed by a minimum length gallery from C to D is precisely the set of those $H \in \mathcal{H}$ that separate C from D. In particular, this set is the same for all those minimum length galleries.

Note that W acts on the set $\mathcal{C}$ of chambers by distance preserving permutations.

Observation 1.1.6. *If the hyperplane H supports the chamber C, then $\rho_H C|_H C$.*

Let us fix an arbitrary chamber

- C^*, the <u>fundamental chamber</u>. Put
- $S := \{\rho_H \mid H \in |C^*|\}$.

Lemma 1.1.7. *W acts transitively on $\mathcal{C}$ and is generated by S.*

Proof. Let

$$C^* = C_0|C_1| \cdots |C_{r-1}|C_r$$

be any gallery starting at C^*. We will show that there are elements $w_i \in \langle S \rangle$ with $C_i = w_i C^*$. This is an easy induction: Suppose w_i has been found already. Let H be the hyperplane with $C_i|_H C_{i+1}$. Then $w_i^{-1} H$ is a hyperplane in $\mathcal{H}$ that supports C^*. Thus

$$\rho_H = w_i s w_i^{-1} \text{ for some } s \in S$$

and

$$C_{i+1} = \rho_H C_i = \rho_H w_i C^* = w_i s w_i^{-1} w_i C^* = w_i s C^*.$$

This way, we constructed an element $w_{i+1} = w_i s \in \langle S \rangle$.

Since every chamber can be connected to C^* by a gallery, the subgroup $\langle S \rangle$ already acts transitively on $\mathcal{C}$.

Consider $H \in \mathcal{H}$. Let $C = wC^*$ (where $w \in \langle S \rangle$) be a chamber supported by H. As we already have observered, there is an element $s \in S$ such that

$$\rho_H = wsw^{-1} \in \langle S \rangle.$$

Thus the generating set for W is contained in $\langle S \rangle$. **q.e.d.**

Lemma 1.1.8. *Let $\underline{s} = s_1 s_2 \cdots s_r$ be a word representing $w \in W$. If this word is a minimum length representative for w, then its length r equals $\delta(C^*, wC^*)$. Otherwise, one can obtain a shorter word representing w by deleting two of the letters, i.e., there are two indices $i < j$ such that*

$$w = s_1 \cdots s_{i-1} s_{i+1} \cdots s_{j-1} s_{j+1} \cdots s_L.$$

Proof. Put

- $w_i := s_1 \cdots s_i$,
- $C_i := w_i C^*$, and let
- H_i be the hyperplane satisfying $s_i = \rho_{H_i}$.

We claim that the corresponding gallery

$$C^* = C_0|_{w_0 H_1} C_1|_{w_1 H_2} \cdots C_{r-2}|_{w_{r-2} H_{r-1}} C_{r-1}|_{w_{r-1} H_r}$$

does not cross any hyperplane twice provided that $s_1 \cdots s_r$ is a minimum word length representative for w. Then the claim follows from (1.1.5).

So let us suppose that

$$w_{i-1}H_i = w_{j-1}H_j$$

for some $i < j$. We conclude

$$w_{i-1}s_iw_{i-1}^{-1} = w_{j-1}s_jw_{j-1}^{-1}$$

whence

$$s_1 \cdots s_{i-1}s_is_{i-1} \cdots s_1 = s_1 \cdots s_{j-1}s_js_{j-1} \cdots s_1.$$

Thus,

$$1 = s_i \cdots s_{j-1}s_js_{j-1} \cdots s_{i+1}.$$

Multiplying from the right, we obtain

$$s_{i+1} \cdots s_{j-1} = s_i \cdots s_j$$

which implies that we have a shorter word for w:

$$w = s_1 \cdots s_{i-1}s_{i+1} \cdots s_{j-1}s_{j+1} \cdots s_L$$

This is a contradiction. **q.e.d.**

Corollary 1.1.9. *The action of W on $\mathcal{C}$ is simply transitive.* **q.e.d.**

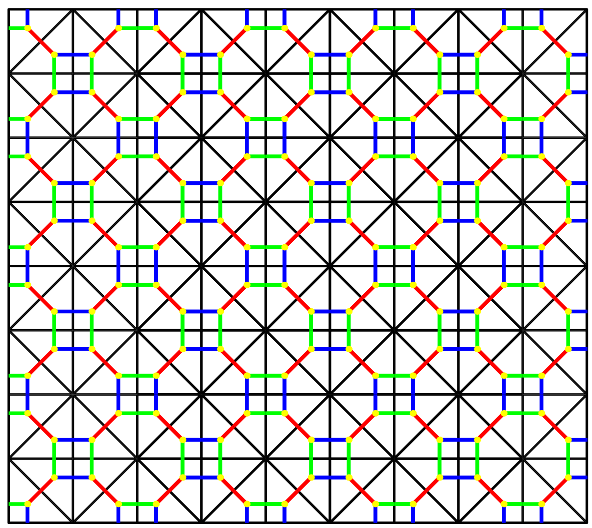
This corollary allows us to draw the Cayley graph of W with respect to S. Since all generators have order 2, we simplify matters by ommiting all the bi-gons that would arise that way. Thus, we define the reduced Caley graph

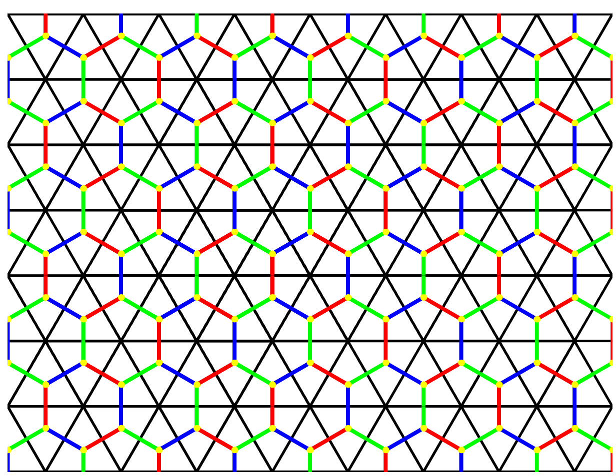
$$\Gamma := \Gamma_S(W)$$

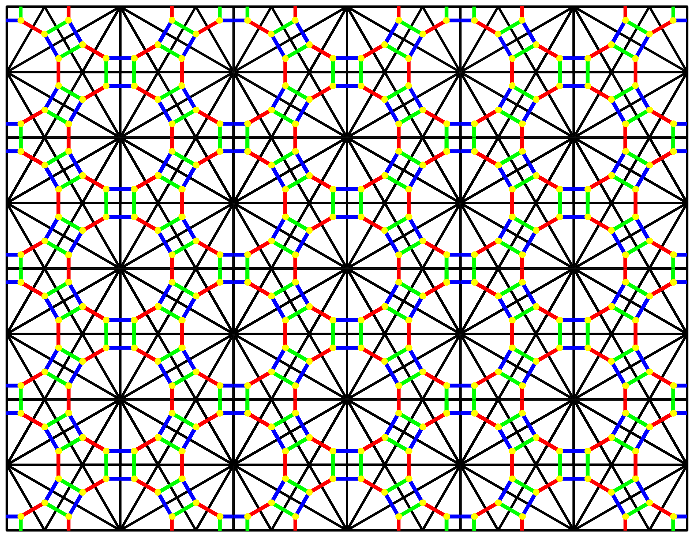
of W to have a vertex for each group element and an edge (labelled by s) for each unordered pair $\{w, ws\}$. Note that W acts from the left.

Observation 1.1.10. *Pick a point inside the fundamental chamber. The W-orbit of this point can be identified with the vertex set of Γ. The edges of Γ correspond to CoDim-1-faces in the chamber decomposition of $\mathbb{E}$. In fact, we can connect the vertices by edges perpendicular to those faces. This way, the Cayley graph is W-equivariantly embedded in $\mathbb{E}$.*

Example 1.1.11. Here are the planar reflection groups whose fundamental chambers are triangles:







1.1.2 The Coxeter Matrix

The Coxeter Matrix of the pair (W,S) is the $S\times S$-matrix

$$M := \left(m_{s,t} := \mathrm{ord}_W(st)\right)_{s,t\in S}.$$

The entries are taken from $\{1,2,3,\dots,\infty\}$. Note that M is symmetric and satisfies:

$$m_{s,t}=1 \quad \text{if and only if} \quad s=t. \tag{1.1}$$

Theorem 1.1.12. *The group W has the presentation*

$$W = \langle s\in S \mid (st)^{m_{s,t}}=1 \ \textit{for}\ m_{s,t}<\infty\rangle.$$

Proof. The given relations obviously hold. To deduce any given other relation, realize the relation as a closed loop in the Cayley graph. This graph lies in the ambient Euclidean space. Find a bounding disk that intersects the CoDim-2-skeleton of the chamber decomposition transversally. Now see the van Kampen diagram. **q.e.d.**

For each s let $\mathbf{u}_s$ be the unit vector perpendicular to the hyperplane inducing the reflection s. (There is a choice here: we use the vector that points away from the fundamental chamber.)

Exercise 1.1.13. Show that for any $s, t \in S$,

$$\langle \mathbf{u}_s, \mathbf{u}_t \rangle = \begin{cases} -\cos\left(\frac{\pi}{m_{s,t}}\right) & \text{for } m_{s,t} \text{ finite} \\ -1 & \text{for } m_{s,t} \text{ infinite.} \end{cases}$$

Now, we can settle the question, whether S is finite.

Proposition 1.1.14. *The fundamental chamber has finite support.*

Proof. Suppose otherwise. Then the set of unit vectors $\mathbf{u}_s$ had an accumulation point by compactness of the unit sphere. However, their pair-wise scalar products are negative. **q.e.d.**

Corollary 1.1.15. *The set $\mathcal{H}$ decomposes into fintely many parallelity classes.*

Proof. Suppose otherwise, then, by compactness, there would be hyperplanes that span arbitrary small angles. Take a point very close to their intersection that lies in a chamber. Since the angles around faces of chambers are bounded away from 0, we have a contradiction. **q.e.d.**

Exercise 1.1.16. Show that the following are equivalent:

1. $\mathcal{H}$ is finite.
2. W is finite.
3. W is torsion.
4. $\bigcap_{H \in \mathcal{H}} H \neq \emptyset$.

Corollary 1.1.17. *A Euclidean reflection group W is virtually free abelian.*

Proof. Consider the action of W upon the sphere at infinity. By (1.1.15), this sphere is decomposed into finitely many regions, upon which W acts by spherical isometries. The image of W in $\mathrm{Isom}(\mathbb{S})$ is a finite Euclidean reflecion group by (1.1.16). The kernel of the homomorphism consists of translations. **q.e.d.**

1.1.3 The Cocompact Case

In this section, we assume that the fundamental chamber has compact closure. All the result are valid in the general case, though. In deed, we will prove them for arbitrary Coxeter groups later.

Observation 1.1.18. *Every point of $\mathbb{E}$ is either contained in a chamber or belongs to the closures of at least two adjacent chambers. In the latter case, it has a translate in the closure of C^*. Thus, the closure of the fundamental chamber is a fundamental domain for the action of W, i.e., the translates of the closure cover $\mathbb{E}$ while the translates of C^* stay disjoint.*

Theorem 1.1.19. *W has only finitely many finite subgroups up to conjugacy.*

Proof. A finite subgroup fixes a point. This point is a translate of some point in $\overline{C^*}$. Thus any finite subgroup is conjugate to a subgroup of a stabilizer of a point in $\overline{C^*}$. There are only finitely many of those since C^* has only finitely many faces. **q.e.d.**

Theorem 1.1.20. *The conjugacy problem in W is solvable.*

Proof. !!! Do the CAT(0) proof !!! **q.e.d.**