1.2 Coxeter Groups

Definition 1.2.1. Let S be a set. A <u>Coxeter matrix</u> over S is a symmetric matrix $M = (m_{s,t})_{s,t \in S}$ with entries $m_{s,t}$ in $\{1, 2, 3, \dots, \infty\}$ such that

$$m_{s,t} = 1 \text{ if and only if } s = t.$$

The <u>Coxeter group</u> defined by M is the group given by the presentation

$$W = \langle s \in S \mid (st)^{m_{s,t}} = 1 \text{ if } m_{s,t} \text{ finite} \rangle.$$

The pair (W, S) is called a <u>Coxeter system</u>.

Example 1.2.2. Every Euclidean reflection group is a Coxeter group.

Coxeter groups are defined by generators and relations. In general, it is hard to tell wheter a group given in this manner is trivial or not. So our first problem will be to see that Coxeter groups are not trivial.

Observation 1.2.3. *Every defining relation of W has even length. Thus, there is a well defined surjective homomorphism*

$$W \to C_2$$

sending each generator in S to the generator of C_2. In particular, none of the generators is trivial in W. **q.e.d.**

Thus, every generator generates a subgroup of order 2 inside W.

1.2.1 The Geometric Representation

To show that the generators have order 2, we used a representation of W. Now, we shall extend this method to show that the products st also have the orders that we would expect from the presentation.

Definition 1.2.4. Let (W, S) be a Coxeter system with Coxeter matrix M. Let $V := \bigoplus_{s \in S} \mathbb{R} e_s$ be the real vector space generated by S: To avoid confusion, we denote the basis vector corresponding to s by e_s.

Define a bilinear form on V by

$$\langle e_s, e_t \rangle_M := \begin{cases} -\cos\left(\frac{\pi}{m_{s,t}}\right) & \text{if } m_{s,t} < \infty \\ -1 & \text{if } m_{s,t} = \infty \end{cases}$$

and define an action of W on V where the generator s acts as the linear automorphism

$$\rho_s : e_t \mapsto e_t - 2 \langle e_s, e_t \rangle_M e_s.$$

This action defines the <u>geometric representation</u>

$$\rho : W \to \mathrm{Aut}(V).$$

Exercise 1.2.5. Check that the the geometric representation does exist, i.e., check that the automorphisms ρ_s satisfy the defining relations of W.

Lemma 1.2.6. *The order of st in M is given by the entry $m_{s,t}$ of the Coxeter matrix.*

Proof. Note that the action of the subgroup $\langle s, t \rangle$ leaves the subspace $V_{s,t} := \langle e_s, e_t \rangle$ invariant.

$\underline{m_{s,t} = \infty}$: The action hits e_t as follows:

$$e_t \xrightarrow{\rho_s} e_t + 2e_s \xrightarrow{\rho_t} 3e_t + 2e_s \xrightarrow{\rho_s} 3e_t + 4e_s \xrightarrow{\rho_t} 5e_t + 4e_s \xrightarrow{\rho_s} \cdots$$

Thus, the product $\rho_t \rho_s$ has infinite order.

$\underline{m_{s,t} < \infty}$: In this case, the bilinear form $\langle -, - \rangle_M$ restricts to a positive definite bilinear form on $V_{s,t}$, and a direct computation shows that the product $\rho_t \rho_s$ is a rotation about of order $m_{s,t}$. **q.e.d.**

11

Corollary 1.2.7. *Thus, the generators s and t span a copy of the dihedral group $D_{m_{s,t}}$ inside W.*   **q.e.d.**

Exercise 1.2.8. Show that W is finite if the bilinear form $\langle -, - \rangle_M$ is positive definite.

Exercise 1.2.9. Show that if W is finite, then there is a unique bilinear form $\langle -, - \rangle$ on V characterized by the following properites

1. $\langle -, - \rangle$ is positive definite.

2. All besis vectors e_s have unit length.

3. The action of W preserves $\langle -, - \rangle$.

Moreover, this bilinear form is $\langle -, - \rangle_M$.

Corollary 1.2.10. *Finite Euclidean reflection groups and finite Coxeter groups are the very same thing.*

Remark 1.2.11. The classification of finite Coxeter groups is done by classifying all Coxeter matrices that are positive definite.

Exercise 1.2.12. A Coxeter system is called <u>irreducible</u>, if there is no generator that commutes simultaneously with all the others. Classify all irreducible Coxeter systems over three generators whose Coxeter groups are finite. (Hint: You should recover descriptions of the Platonic solids along the way; in fact, the existence of the Platonic solids can be derived from this classification.)

1.2.2 The Geometry of a Coxeter System

We studied Euclidean reflection groups by means of the assiciated Chamber system upon which the group acts. To study general Coxeter

12

groups, we will construct the geometry from the group. So, we will construct a chamber system from the (reduced) Cayley complex $\Gamma_S(W)$ for the Coxeter presentation. The vertices of the Cayley complex are the chambers, and two chambers are s-adjacent if they are joined by an edge with labes s. Of course an edge path in the Cayley complex is a gallery in the chamber system. We will see that this chamber system allows reflections and half spaces.

Definition 1.2.13. Two edges e and e' in $\Gamma_S(W)$ are opposite if they are contained in a relator disc and have maximal distance in this circle. We write $e \longleftrightarrow e'$. The edges e and e' are parallel if $e = e'$ or if there is a finite sequence

$$e = e_0 \longleftrightarrow e_1 \longleftrightarrow e_2 \longleftrightarrow \cdots \longleftrightarrow e_r = e'.$$

We write $e \parallel e'$.

Parallelity is an equivalence relation. Its equivalence classes are called walls.

Lemma 1.2.14. *If a wall intersects a relator disc, then the intersection consists of precisely one pair of opposite edges.*

Proof. Let e and e' be edges inside the relator disc $\mathbb{B}$ that are parallel. Then there is a chain of relator discs proving them parallel.

!!! PICTURE !!!

q.e.d.

Definition 1.2.15. An elementary homotopy of an edge path in the Cayley graph of a Coxeter group is the replacement of a subpath reading part of a relator disc by the complementary part of the

13

relation. Two paths in the Cayley graph are called <u>homotopic</u> if one can be obtained from the other by a finite sequence of elementary homotopy.

Observation 1.2.16. *Two paths are homotopic if and only if they connect the same end points.*

Observation 1.2.17. *Given a wall and a gallery, the number of crossings between the wall and the gallery changes by an even number during any elementary homotopy of the gallery. Thus, for any two chambers and a wall, we have a well defined notion of separation.*

Lemma 1.2.18. *Each wall separates the Cayley graph into two half spaces.*

Lemma 1.2.19. *Associated to each wall, there is a unique element in W that acts like a reflection along the wall.*

Lemma 1.2.20. *Half spaces are convex, i.e., if two chambers lie in a given half space, then so does every minimal chamber between them.*

Lemma 1.2.21. *The gallery distance of two chambers is the number of walls seperating them.*

Above, we introduced the geometric representation of W on the vector space V spanned by $\{e_s \mid s \in S\}$. Let V^* be the dual of V. It turns out that the induced action of W on V^*,

$$\tau : W \quad \to \quad \mathrm{Aut}(V^*)$$
$$w : \lambda \quad \mapsto \quad \lambda \circ \rho_w,$$

gives another description of the chamber system: For any s, define the posite and negative halfspace in V^* by

$$U_s^+ \quad := \quad \{\lambda \in V^* \mid \lambda(e_s) > 0\}$$
$$U_s^- \quad := \quad \{\lambda \in V^* \mid \lambda(e_s) < 0\} .$$

The Tits cone

$$C := \{\lambda \in V^* \mid \lambda(e_s) > 0 \text{ for all } s \in S\}$$

is the intersection of the positive cones.

Exercise 1.2.22. Show that for every $w \in W$,

$$\tau_w(C) \subseteq U_s^+ \text{ if and only if } |sw| = |w| + 1$$

and

$$\tau_w(C) \subseteq U_s^- \text{ if and only if } |sw| = |w| - 1.$$

Exercise 1.2.23. Infer from (1.2.22) that the geometric representation is faithful.

Corollary 1.2.24. *Finitely generated Coxeter groups are linear.*   **q.e.d.**

1.2.3   The Deletion Condition

In (1.1.8), we have seen, that the pair (W, S) for a Euclidean reflection group satisfies the Deletion Condition:

Definition 1.2.25 (Deletion Condition). Let (W, S) be a pair where W is a group and S is a generating set for W consisting entirely of elements of order 2. We say that this pair satisfies the Deletion Contition if:

> For any non-reduced word $s_1 \cdots s_r$ over S there are two indices i and j such that
>
> $$s_1 \cdots s_r =_W s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r.$$
>
> The carets indicate ommision.

This is, one can delete two letters from any non-minimum-length word to obtain a shorter representative for the same element of W.

In this section, we will recognize (W, S) as a Coxeter system using the Deletion Condition.

Lemma and Definition 1.2.26 (Exchange Condition). *The pair (W, S) satifies the Exchange Condition, i.e.:*

> *Let $s_1 \cdots s_r$ and and $t_1 \cdots t_r$ be two reduced words over S representing the same element $w \in W$. If $s_1 \neq t_1$, then there is an index $i \in \{2, \ldots, r\}$ such that*
>
> $$w =_W s_1 t_1 \cdots \hat{t}_i \cdots t_r.$$

Proof. This is a formal consequence of the Deletion Condition: From

$$s_1 \cdots s_r =_W t_1 \cdots t_r,$$

we obtain

$$s_2 \cdots s_r =_W s_1 t_1 \cdots t_r$$

where the right hand is longer than the left hand whence there must be a pair of letters that can be dropped without changing the value of the product. However, one of the two letters must be the leading s_1: Otherwise, we had

$$s_2 \cdots s_r =_W s_1 t_1 \cdots \hat{t}_i \cdots \hat{t}_i \cdots t_r$$

whence

$$s_1 \cdots s_r =_W t_1 \cdots \hat{t}_i \cdots \hat{t}_i \cdots t_r$$

contradicting the minimality of the initial words.

Thus, we have

$$s_2 \cdots s_r =_W t_1 \cdots \hat{t}_i \cdots \cdots \cdots t_r$$

16

whence

$$s_1 \cdots s_r =_W s_1 t_1 \cdots \hat{t}_i \cdots\cdots t_r. \qquad \textbf{q.e.d.}$$

The <u>Coxeter Matrix of the pair (W, S)</u> is the $S \times S$-matrix

$$M := (m_{s,t} := \mathrm{ord}_W(st))_{s,t \in S}.$$

The entries are taken from $\{1, 2, 3, \ldots, \infty\}$. Note that M is symmetric and satisfies:

$$m_{s,t} = 1 \quad \text{if and only if} \quad s = t. \tag{1.2}$$

Any symmetric matrix satisfying (1.2) is called a <u>Coxeter matrix</u>.

An <u>elementary M-reduction</u> is one of the following moves:

1. Delete a subword ss.

2. Replace a subword $\underbrace{sts\cdots}_{m_{s,t} \text{ letters}}$ by $\underbrace{tst\cdots}_{m_{s,t} \text{ letters}}$.

Theorem 1.2.27 (Tits). *Let $\underline{s} = s_1 \cdots s_{|\underline{s}|}$ be a reduced word over S. Then $\underline{s}$ can be obtained from any word $\underline{t} = t_1 \cdots t_{|\underline{t}|}$ by a sequence of elementary M-reductions.*

Proof. This is also a purely formal consequence of the Deletion Condition. Let us first prove the theorem under the additional hypothesis that $\underline{t}$ is reduced, as well. In this case, $|\underline{s}| = |\underline{t}|$ and only moves of type (2) are possible. We induct on the length of the words.

Assume first that $s_1 = t_1$. Then $s_2 \cdots s_{|\underline{s}|}$ and $t_2 \cdots t_{|\underline{s}|}$ are two reduced words representing the same group element. By induction, we can pass from one to the other by elementary M-reductions.

So assume $s_1 \neq t_1$. So we can apply the exchange condition both ways and obtain

$$s_1 \cdots s_{|\underline{s}|} =_W s_1 t_1 \cdots \hat{y}_i \cdots t_{|\underline{s}|}$$
$$t_1 \cdots t_{|\underline{s}|} =_W t_1 s_1 \cdots \hat{x}_i \cdots s_{|\underline{s}|}$$

17

Note that both equations actually can be realized by M-reduction since the words start with identical letters. Thus, we only have to realize an M-reduction to pass from $s_1 t_1 \cdots \hat{y}_i \cdots t_{|\underline{\mathbf{s}}|}$ to $t_1 s_1 \cdots \hat{x}_i \cdots s_{|\underline{\mathbf{s}}|}$. If $m_{s,t} = 1$, we are done. Otherwise we apply the exchange condition again: ...

Now let us drop the assumption that $\underline{t}$ is reduced. It suffices to prove that $\underline{t}$ can be shortened by M-reductions. We induct on the length of $\underline{t}$. If $\underline{t}_2 \cdots \underline{t}_{|\underline{t}|}$ is not reduced, we apply the induction hypothesis to this subword.

So we assume that $\underline{t}_2 \cdots \underline{t}_{|\underline{t}|}$ is reduced. Then we find

$$\underline{t}_1 \cdots \underline{t}_{|\underline{t}|} =_W \underline{t}_2 \cdots \hat{t}_i \cdots \underline{t}_{|\underline{t}|}$$

whence $\underline{t}_2 \cdots \underline{t}_{|\underline{t}|}$ can be transformed into $t_1 \underline{t}_2 \cdots \hat{t}_i \underline{t}_{|\underline{t}|}$ by M-reductions. (Both of these words are reduced, so we are in the case that we have discussed already.) Now, we can shorten:

$$\underline{t}_1 \cdots \underline{t}_{|\underline{t}|} \xrightarrow{M} \underline{t}_1 t_1 \underline{t}_2 \cdots \hat{t}_i \underline{t}_{|\underline{t}|} \xrightarrow{M} \underline{t}_2 \cdots \hat{t}_i \underline{t}_{|\underline{t}|}.$$

The final step is an operation of type (1). **q.e.d.**

Corollary 1.2.28. *The pair (W, S) is a Coxeter system.*

Proof. A relation is a word that evaluates to 1 in W. Therefore, any relation can be transformed into the empty word by M-reductions. However, these correspond to the relations of the Coxeter presentation. **q.e.d.**