

The Pigeonhole Principle

The Pigeonhole Principle (also sometimes called the Box Principle or the Dirichlet Box Principle) simply states that if one wants to put pigeons in holes, and there are more pigeons than there are holes, then one of the holes has to contain more than one pigeon. There is also a stronger form of the principle: if the number of pigeons is more than k times the number of holes, then some hole has at least $k + 1$ pigeons. While these facts are rather obvious, clever applications of them can be used to solve some Putnam problems. The Pigeonhole Principle can be applied, for example,

- to prove the existence of geometric objects (see problems 3 and 5),
- to solve combinatorial problems (see problems 1 and 6),
- to solve number-theoretic problems involving divisibility (see problems 2 and 4).

Below are two simple examples.

Example 1: Given five points inside an equilateral triangle of side length 2, show that there are two points whose distance from each other is at most 1.

Solution: Draw the equilateral triangle whose three vertices are the midpoints of the sides of the original triangle. This splits the original triangle into four equilateral triangles of side length 1. By the Pigeonhole Principle, one of these four triangles must contain two of the points, and the distance between those two points is at most 1.

Example 2: Suppose a soccer team scores at least one goal in 20 consecutive games. If it scores a total of 30 goals in those 20 games, prove that in some sequence of consecutive games it scores exactly 9 goals.

Solution: Let g_i be the total number of goals scored in games 1 through i . Consider the set of numbers $\{g_1, \dots, g_{20}, g_1 + 9, \dots, g_{20} + 9\}$, all of which are positive integers between 1 and 39. Since there are 40 numbers in the set, by the Pigeonhole Principle one number appears at least twice. Since the team scores at least once in each game, we have $g_i \neq g_j$ for all $i \neq j$, and therefore $g_i + 9 \neq g_j + 9$ for all $i \neq j$. It follows that there exist i and j such that $g_i + 9 = g_j$. The number of goals scored in games $i + 1, i + 2, \dots, j$ equals $g_j - g_i = 9$.

Some Practice Problems

Below are some practice problems, all of which can be solved with a clever application of the Pigeonhole Principle. For even more problems on this topic, see the web site http://www.cut-the-knot.org/do_you_know/pigeon.shtml, which is maintained by Alexander Bogomolny. Some of the problems below were modeled after problems on this page. Problems 5 and 6 were on previous Putnam exams.

Problem 1: Suppose $n \geq 2$ baseball teams play in a tournament. If no two teams play each other more than once, prove that two teams have to play the same number of games.

Problem 2: Prove that there exist two powers of 7 whose difference is divisible by 2003.

Problem 3: Fifty-one points are placed in a square of side length 1. Prove that there is a circle of radius $1/7$ that contains three of the points.

Problem 4: Prove that any collection of 31 distinct integers between 1 and 60 has the property that one divides another.

Problem 5 (2000 B6): Let B be a set of more than $2^{n+1}/n$ distinct points with coordinates of the form $(\pm 1, \pm 1, \dots, \pm 1)$ in n -dimensional space with $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

Problem 6 (1993 A4): Let x_1, \dots, x_{19} be positive integers each of which is less than or equal to 93. Let y_1, \dots, y_{93} be positive integers each of which is less than or equal to 19. Prove there exists a (nonempty) sum of some x_i 's equal to a sum of some y_j 's.