A NOTE ON THE ALMOST ONE HALF HOLOMORPHIC PINCHING
UNE NOTE SUR LE PINCEMENT HOLOMORPHE PRESQUE UN DEMI

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Abstract. Motivated by a previous work of Zheng and the second named author, we study pinching constants of compact Kähler manifolds with positive holomorphic sectional curvature. In particular we prove a gap theorem following the work of Petersen and Tao on Riemannian manifolds with almost quarter-pinched sectional curvatures.

Résumé.. Motivé par un travail précédent de Zheng et le deuxième auteur, nous étudions les constantes de pincement des variétés kähleriennes compactes avec courbure sectionnelle holomorphe positive. En particulier, nous prouvons un théorème de l'écart suivant le travail de Petersen et Tao sur variétés riemanniennes avec des courbures sectionnelles presque $\frac{1}{4}$-pincée.

1. The theorem

Let (M, J, g) be a complex manifold with a Kähler metric g, one can define the *holomorphic sectional curvature* (H) of any J-invariant real 2-plane $\pi = \text{Span}\{X, JX\}$ by

$$H(\pi) = \frac{R(X, JX, JX, X)}{||X||^4}.$$

It is the Riemannian sectional curvature restricted on any J-invariant real 2-plane (p165 [16]). In terms of complex coordinates, it is equivalent to write

$$H(\pi) = \frac{R(V, \overline{V}, V, \overline{V})}{||V||^4}$$

where $V = X - \sqrt{-1}JX \in T^{1,0}(M)$.

In this note we study pinching constants of compact Kähler manifolds with positive holomorphic sectional curvature $(H > 0)$. The goal is to prove the following rigidity result on a Kähler manifold with the almost one half pinching.

Theorem 1.1. *For any integer $n \geq 2$, there exists a positive constant $\epsilon(n)$ such that any compact Kähler manifold with $\frac{1}{2} - \epsilon(n) \leq H \leq 1$ of dimension n is biholomorphic to any of the following*

(i) $\mathbb{CP}^n$,
(ii) $\mathbb{CP}^k \times \mathbb{CP}^{n-k}$,
(iii) *An irreducible rank 2 compact Hermitian symmetric space of dimension n.*

Before we discuss the proof, let us review some background on compact Kähler manifolds with $H > 0$. The condition $H > 0$ is less understood and seems mysterious. For example, $H > 0$ does not imply positive Ricci curvature, though it leads to positive scalar curvature. Essentially one has to work on a fourth order tensor from the viewpoint of linear algebra, while usually the stronger notion of holomorphic bisectional curvature leads to bilinear forms.

Naturally one may wonder if there is a characterization of such an interesting class of Kähler manifolds. In particular, Yau ([25] and [26]) asked if the positivity of holomorphic sectional curvature can be used to characterize the rationality of algebraic manifolds. For example, is such a manifold a rational variety? There is much progress on Kähler surfaces with $H > 0$. In

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1975 Hitchin [15] proved that any compact Kähler surface with $H > 0$ must be a rational surface and conversely he constructed examples of such metrics on any Hirzebruch surface $M_{2,k} = \mathbb{P}(H^k \oplus 1_{\mathbb{CP}^1})$. It remains an interesting question to find out if Kähler metrics of $H > 0$ exist on other rational surfaces.

In higher dimensions, much less is known on $H > 0$ except recent important works of Heier-Wong (see [14] for example). One of their results states that any projective manifold which admits a Kähler metric with $H > 0$ must be rationally connected. It could be possible that any Kähler manifold with $H > 0$ is in fact projective, again it is an open question. We also remark that some generalization of Hitchin's construction of Kähler metrics of $H > 0$ in higher dimensions has been obtained in [2].

If Yau's conjecture is true, then how do we study the complexities of rational varieties which admit Kähler metrics with $H > 0$? A naive thought is that the global and local holomorphic pinching constants of H should give a stratification among all such rational varieties. Here the local holomorphic pinching constant of a Kähler manifold (M, J, g) of $H > 0$ is the maximum of all $\lambda \in (0, 1]$ such that $0 < \lambda H(\pi') \leq H(\pi)$ for any $J-$invariant real $2-$planes $\pi, \pi' \subset T_p(M)$ at any $p \in M$, while the global holomorphic pinching constant is the maximum of all $\lambda \in (0, 1]$ such that there exists a positive constant C so that $\lambda C \leq H(p, \pi) \leq C$ holds for any $p \in M$ and any J-invariant real 2-plane $\pi \subset T_p(M)$. Obviously the global holomorphic pinching constant is no larger than the local one, and there are examples of Kähler metrics with different global and local holomorphic pinching constants on Hirzebruch manifolds ([24]).

In a previous work of Zheng and the second named author ([24]), we observe the following result, which follows from some pinching equality on $H > 0$ due to Berger [3] and recent works on nonnegative orthogonal bisectional curvature ([7], [10], and [23]).

Proposition 1.2 ([24]). *Let (M^n, g) be a compact Kähler manifold with $0 < \lambda \leq H \leq 1$ in the local sense for some constant λ, then the following holds:*

(1) If $\lambda > \frac{1}{2}$, then M^n is bibolomorphic to $\mathbb{CP}^n$.

(2) If $\lambda = \frac{1}{2}$, then M^n satisfies one of the following

(i) *M^n is biholomorphic to $\mathbb{CP}^n$.*

(ii) *M^n is holomorphically isometric to $\mathbb{CP}^k \times \mathbb{CP}^{n-k}$ with a product metric of Fubini-Study metrics. Moreover, each factor must have the same constant H.*

(iii) *M^n is holomorphically isometric to an irreducible compact Hermitian symmeric space of rank 2 with its canonical Kähler-Einstein metric.*

Let us remark that in the case that Kähler manifold in Proposition 1.2 is projective and endowed with the induced metric from the Fubini-Study metric of the ambient projective space, a complete characterization of such a projective manifold and the corresponding embedding has been proved by Ros [20].

Comparing with Proposition 1.2, we may view Theorem 1.1 as a rigidity result on compact Kähler manifolds with almost one half-pinched $H > 0$. For example, Hirzebruch manifolds can not admit Kähler metrics whose global pinching constants are arbitrarily close to $\frac{1}{2}$.

It is very interesting to find the next threshold for holomorphic pinching constants and prove some characterization of Kähler manifolds with such a threshold pinching constant. Before making any reasonable speculation, it is helpful to understand examples on such holomorphic constants of some canonical Kähler metrics. In this regard, the Kähler-Einstein metric on a irreducible compact Hermitian symmetric space has its holomorphic pinching constant exactly the reciprocal of its rank ([6]). The Kähler-Einstein metrics on many simply-connected compact homogeneous Kähler manifolds (Kähler C-spaces) also have $H > 0$, and it seems very tedious to work with corresponding Lie algebras carefully to determine these holomorphic pinching constants except in lower dimensions. It was observed in [24] that the flag 3-manifold, the only Kähler C-space in dimesion 3 which is not Hermitian symmetric, has $\frac{1}{4}$-holomorphic pinching for its canonical Kähler-Einstein metric. Note that Alvarez-Chaturvedi-Heier [1] studied pinching constants of Hitchin's examples of Kähler metrics with $H > 0$ on a Hirzebruch surface. However, it remains unknown what is the best pinching constant among all Kähler metrics with $H > 0$ on such a surface. We refer the interested reader to [1] and [24] for more discussions.

2. The proof

The proof is motivated by the work of Petersen-Tao [18] on Riemannian manifolds with almost quarter-pinched sectional curvature.

Assume for some complex dimension $n \geq 2$, there exists a sequence of compact Kähler manifolds (M_k, J_k, g_k) $(k \geq 1)$ whose holomorphic sectional curvature satisfies $\frac{1}{2} - \frac{1}{4k} \leq H(M_k, g_k) \leq 1$, and none of (M_k, J_k) is biholomorphic to any of the three listed in the conclusion of Theorem 1.1. In the following steps, $c(n)$, maybe different from line to line, are all constants which only depend on n.

Step 1: (A uniform lower bound for the maximal existence time of the Kähler-Ricci flow)

It is well-known ([16] for example) that bounds on holomorphic sectional curvature lead to bounds on Riemannian sectional curvature and the full curvature tensor. In particular, for any unit orthogonal vectors X and Y we have

$$\begin{aligned} Sec(X,Y) = R(X,Y,Y,X) = &\frac{1}{8}\Big[3H(\frac{X+JY}{\sqrt{2}}) + 3H(\frac{X-JY}{\sqrt{2}}) \\ &- H(\frac{X+Y}{\sqrt{2}}) - H(\frac{X-Y}{\sqrt{2}}) - H(X) - H(Y)\Big] \end{aligned}$$

From the works of Hamilton and Shi ([11], [12], and [21], and Cor 7.7 in [9] for an exposition of these results on compact manifolds), we conclude that for any $k \geq 1$, there exists a constant $T(n) > 0$ such that the Kähler-Ricci flow $(M_k, J_k, g_k(t))$ with the initial metric g_k is well-defined on the time interval $[0, T(n)]$ for any $k \geq 1$. Moreover, we have $|Rm(M_k, g_k(t))|_{g(t)} \leq c(n)$ for some constant $c(n)$ and all $t \in [0, T(n)]$ and $k \geq 1$.

Step 2: (An improved curvature bound on a smaller time interval)

This step is due to Ilmanen, Shi, and Rong (Proposition 2.5 in [19]). Namely, there exists constants $\delta(n) < T(n)$ and $c(n)$ such that for any $t \in [0, \delta(n)]$

$$\begin{aligned} \min_{p, V\subset T_p(M_k)} Sec(M_k, g_k(t), p, V) - c(n)t \leq\ & Sec(M_k, g_k(t), p, P) \\ &\leq \max_{p, V\subset T_p(M_k)} Sec(M_k, g_k(t), p, V) + c(n)t. \end{aligned}$$

It is direct to see that a similar estimate holds for holomorphic sectional curvature. Indeed, there exists $\delta(n)$ and $c(n)$ such that for any $t \in [0, \delta(n)]$

$$\frac{1}{2} - c(n)\, t \leq H(M_k, g_k(t)) \leq 1 + c(n)\, t.$$

Step 3: (An injective radius bound on $g_k(t_0)$ for some fixed $t_0 \in [0, \delta]$)

We observe that Klingenberg's injectivity radius estimates on even-dimensional Riemannian manifolds with positive sectional curvature (Theorem 5.9 in [5] or p178 of [17] for example) can be adapted to show that for any $t \in [0, \delta_1(n)]$ such that $inj(M_k, g_k(t)) \geq c(n)$ for some constant $c(n) > 0$. Indeed it will follow from the claim below:

Claim 2.1. *Let (M^n, g) be a compact Kähler manifold with positive holomorphic sectional curvature $H \geq \delta > 0$ and $Sec \leq K$ where the constant $K > 0$, then the injectivity radius $inj(M^n, g) \geq c(K)$ for some constant $c(K)$.*

The proof of the above claim goes along as the proof of Theorem 5.9 in [5] except that we need to use the variational vector field as $J\gamma'(t)$ where $\gamma(t)$ is a closed geodesic. This is where we use $H > 0$. In fact such a kind of estimate has been proved in [8] assuming positive bisectional curvature.

Step 4: (An lower bound on orthogonal bisectional curvatures of $(M_k, g_k(t))$

This step is motivated by Peterson-Tao [18], where they derived a similar lower bound estimate for isotropic curvatures of almost quarter-pinched Riemannian manifolds along Ricci flow.

Claim 2.2. *There exists some constant $\delta_2(n)$ such that any $t \in [0, \delta_2(n)]$, the orthogonal bisectional curvature of $(M_k, g_k(t))$ has a lower bound $-\frac{1}{k}e^{c(n)t}$ for some constant $c(n) > 0$.*

Proof of Claim 2.2. Note that Berger's inequality [3] (see also Lemma 2.5 in [24] for an exposition) implies the orthogonal bisectional curvature of $(M_k, g_k(0))$ is bounded from below by $-\frac{1}{4k}$. Now the proof is based on a maximum principle developed in [12]. In the setup of orthogonal bisectional curvature, it is proved (in [7], [10], and [23]) that nonnegative orthogonal bisectional curvature is preserved under Kähler-Ricci flow. For the sake of convenience, we simply write $(M, J, g(t))$ where $t \in [0, T(n)]$ instead of the sequence $(M_k, J_k, g_k(t))$.

Following [12], one may use Uhlenbeck's trick. Consider a fixed complex vector bundle $E \to M$ isomorphic to $TM \to M$, with a suitable choice of bundle isomorphims $\iota_t : E \to TM$, one obtain a fixed metric $\iota_t^* g(t)$ on E. Now choose an unitary frame $\{e_\alpha\}$ on $T^{1,0}(E)$ which corresponds to an evolving unitary frame on TM via ι_t, let $R_{\alpha\overline{\alpha}\beta\overline{\beta}}$ denote $R(\iota_t^* g(t), e_\alpha, \overline{e_\alpha}, e_\beta, \overline{e_\beta})$, the evolution equation of bisectional curvature reads ([10] for example)

$$\frac{\partial}{\partial t} R_{\alpha\overline{\alpha}\beta\overline{\beta}} = \Delta_{g(t)} R_{\alpha\overline{\alpha}\beta\overline{\beta}} + \sum_{\mu,\nu} (R_{\alpha\overline{\alpha}\mu\overline{\nu}} R_{\beta\overline{\beta}\mu\overline{\nu}} - |R_{\alpha\overline{\mu}\beta\overline{\nu}}|^2 + |R_{\alpha\overline{\beta}\mu\overline{\nu}}|^2) \tag{1}$$

Now assume $m(t) = \min_{U \perp V \in T^{1,0}(E)} R(U, \overline{U}, V, \overline{V})$, and assume $m(t_0) = R_{\alpha\overline{\alpha}\beta\overline{\beta}}$ for some t_0 and some point $p \in M$. Consider the first and the second variation of $R_{\alpha\overline{\alpha}\beta\overline{\beta}}$, follow the proof of Proposition 1.1 in [10] and the curvature bounds in Step 1 we conclude from (1) that $\frac{d^- m(t)}{dt}|_{t=t_0} \geq c(n)m(t_0)$ whenever $m(t_0) < 0$. Therefore there exists some time interval $[0, \delta_2(n)]$ either $m(t) \geq 0$ or $\frac{d^+(-m(t))}{dt} \leq c(n)(-m(t))$ if $m(t) < 0$. Recall $m(0) \geq -\frac{1}{4k}$, in any case we end up with $m(t) \geq -\frac{1}{k}e^{c(n)t}$ for all $t \in [0, \delta_2(n)]$. □

Step 5: (A contradiction after taking the limit of $(M_k, , J_k, g_k(t))$)

Let us consider $(M_k, J_k, g_k(t))$ where $t \in (0, \delta_2(n)]$ be a sequence of Kähler-Ricci flow, from previous steps we conclude that there exist $\delta_3(n)$ and $c(n)$ such that

(i) $|Rm|_{g_k(t)}(M_k, g_k(t)) \leq c(n)$ for and $k \geq 1$ and $t \in [0, \delta_3(n)]$.
(ii) $inj(M_k, g_k(t_0)) \geq \frac{1}{c(n)}$ for some $t_0 \in [0, \delta_3(n)]$
(iii) $Ric(M_k, g_k(t_0)) \geq \frac{1}{c(n)}$ for any k, this follows from Step 2 and 4.

It follows from Hamilton's compactness theorem of Ricci flow [13] that $(M_k, J_k, g_k(t))$ converges to a compact limiting Kähler-Ricci flow $(M_\infty, J_\infty, g_\infty(t))$ where $t \in (0, \delta_2(n)]$. It follows from Step 4 that $g_\infty(t)$ has nonnegative orthogonal bisectional curvature and $H(g_\infty(t)) > 0$ for any $0 < t \leq t_0$. Note that all M_k and M_∞ are simply-connected ([22]), it follows from [7], [10], and [23] that $(M_\infty, g_\infty(t_0))$ must be of the following form.

$$(\mathbb{CP}^{k_1}, g_{k_1}) \times \cdots \times (\mathbb{CP}^{k_r}, g_{k_r}) \times (N^{l_1}, h_{l_1}) \times \cdots (N^{k_r}, h_{l_s}). \tag{2}$$

Where each of $(\mathbb{CP}^{k_i}, g_{k_i})$ has nonnegative bisectional curvature and each of (N^{l_i}, h_{l_i}) is a compact irreducible Hermitian symmetric spaces of rank ≥ 2 with its canonical Kähler-Einstein metric. Now consider a time $t_1 < t_0$ close to $t = 0$, it follows from Step 2 that $g_\infty(t_1)$ is close to $\frac{1}{2}$-holmorphic pinching and also have the same decomposition as (2). Indeed the decomposition (2) is reduced to exactly the list in the conclusion of Proposition 1.2. To see it one may also apply the formula of piching constants of a product of metrics with $H > 0$ ([1]). However (M_k, J_k) is not biholomorphic to any of the three three listed in the conclusion of Theorem 1.1. Now we have a sequence of Kähler manifolds $(M_\infty, \phi_k^* J_k, \phi_k^* g_k(t_1))$ converging to $(M_\infty, J_\infty, g_\infty(t_1))$ where $\phi_k : M_\infty \to M_k$ are the diffeomorphisms from Hamilton's compactness theorem. This is a contradiction since any compact Hermitian symmetric space is infinitesimally rigid, i.e. $H^1(M_\infty, \Theta_{M_\infty}) = 0$ where Θ is the sheaf of holomorphic vector fields on M_∞ (see Bott [4]). This finishes the proof of Theorem 1.1.

3. A remark

Note that Proposition 1.2 works in the case of the local one half pinching, therefore it seems natural to ask:

Question 3.1. *Does Theorem 1.1 hold if we replace the global almost one half pinching to the local one?*

Another optimistic hope is that $H > 0$ is preserved along Kähler-Ricci flow as long as the initial metric has a suitable large holomorphic pinching constant. We refer to [24] for more discussions.

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