Minimal Graded Betti Numbers and Stable Ideals

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Abstract

Let k be a field, and let $R = k[x_1, x_2, x_3]$. Given a Hilbert function H for a cyclic module over R, we give an algorithm to produce a stable ideal I such that R/I has Hilbert function H and uniquely minimal graded Betti numbers among all R/J with the same Hilbert function, where J is another stable ideal in R. We also show that such an algorithm is impossible in more variables and disprove a related conjecture Deery makes in [4].

1 Background

Research on the interplay between the graded free resolution of a polynomial ring modulo a homogeneous ideal and its Hilbert function dates back to Hilbert's 1890 paper [8]. In this amazing work, he uses the free resolution to compute the Hilbert function for graded modules over $S = k[x_1, \ldots, x_n]$, where k is a field. If I is a homogeneous ideal, given a graded free resolution

$$0 \to \bigoplus_j S^{\beta_{nj}}(-j) \to \cdots \to \bigoplus_j S^{\beta_{1j}}(-j) \to S \to S/I \to 0,$$

one can easily read off the Hilbert function $H_{S/I}$ of S/I from the graded Betti numbers β_{ij}. Work in the last decade has focused on what information we can harvest knowing only the Hilbert function. That is, we seek to know the possible graded Betti numbers that can appear in the resolution of a module with a given Hilbert function.

It is convenient to put a partial order on the graded Betti numbers of resolutions of modules with the same Hilbert function: If M and N are modules over a polynomial ring S with the same Hilbert function and with graded Betti numbers β_{ij}^M and β_{ij}^N respectively, we say that $\beta^M \geq \beta^N$ if and only if $\beta_{ij}^M \geq \beta_{ij}^N$ for each i and j.

Using this partial order, one of the first major breakthroughs came in the papers of Bigatti [1] and Hulett [9]. They prove independently that in characteristic zero there is a resolution that has the unique maximal graded Betti numbers for a given Hilbert function, and Pardue [12] generalizes the result to positive characteristic. The lexicographic ideal yields that largest resolution. We recall that an ideal $I \subset S$ is called *lexicographic* if for each d, I is generated in degree d by the first $\dim_k I_d$ monomials of degree d in descending lexicographic order; Macaulay shows in [10] that for each valid Hilbert function for a cyclic module, there exists a quotient corresponding to a lexicographic ideal attaining it.

Alas, there is no result for minimal graded Betti numbers corresponding to that of Bigatti, Hulett, and Pardue. Charalambous and Evans [3] provide a Hilbert function for which there are incomparable minimal graded Betti numbers in $k[x_1, \ldots, x_n]$ for all $n \geq 3$. In the same paper, they are able to characterize all the possible graded resolutions in the dimension two finite length cyclic module case, and one would like to be able to do the same in higher dimensions.

Unfortunately, determining the sets of graded Betti numbers that can occur is much harder beginning in $k[x_1, x_2, x_3]$. Deery proposes in [4] restricting to the class of stable ideals to overcome the incomparability problem that exists when considering all ideals in $k[x_1, x_2, x_3]$. Stable ideals are of particular interest because they have proven tremendously useful in reduction arguments in the areas of Hilbert functions and graded Betti numbers recently. We show that for a given Hilbert function, a unique minimal set of graded Betti numbers does exist among all stable ideals in three variables with that Hilbert function. Deery conjectured that a slightly weaker statement would be true in more variables, but in the final section, we give a family of counterexamples.

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2 Definitions and some previous work

For the rest of the paper, let $R = k[x_1, x_2, x_3]$, where k is a field. We wish to explore some earlier work on computing minimal graded Betti numbers, and this requires some definitions and notation.

Definition 2.1 *For a monomial m, define $\max(m)$ to be the largest index i such that x_i divides m.*

For example, $\max(x_1^2) = 1$, and $\max(x_1 x_2^3 x_4) = 4$. We use this notion to define stable ideals.

Definition 2.2 *An ideal I is said to be **stable** if I is a monomial ideal, and for each monomial m in I and all $i < \max(m)$, $x_i m / x_{\max(m)}$ is also in I. (We define a stable set of monomials analogously.)*

To check whether a monomial ideal is stable, it is enough to check the condition on minimal generators, and an ideal is stable if and only if the set of monomials in each degree is stable. Clearly every lexicographic ideal is stable, but there are many others as well. The combinatorial structure of stable ideals makes them easier to study than typical monomial ideals, and their graded free resolution is known; we even have a convenient formula for their graded Betti numbers. Eliahou and Kervaire describe the minimal free resolution in [5], and the formula

$$\beta_{qj}^{R/I} = \sum_{\substack{m \in G(I) \\ deg(m)=j-q+1}} \binom{\max(m)-1}{q-1}$$

from [9] gives the graded Betti numbers, where $G(I)$ is the minimal monomial generating set of a stable ideal I. We seek to minimize this quantity.

To do this, we shall consider some special stable ideals. In [4], Deery defines the notions of reverse-lex ideals and weakly reverse-lex ideals (Deery calls them "almost reverse-lex" ideals) to prove some minimality properties. Recall that if $x_1^{a_1} \cdots x_n^{a_n}$ and $x_1^{b_1} \cdots x_n^{b_n}$ are monomials of the same degree, then $x_1^{a_1} \cdots x_n^{a_n} > x_1^{b_1} \cdots x_n^{b_n}$ in the reverse-lex order if and only if the last nonzero entry of $(a_1 - b_1, \ldots, a_n - b_n)$ is negative.

Definition 2.3 *Call a monomial ideal I a **reverse-lex** ideal if in each degree, I is generated by an initial reverse-lex segment.*

For example, the ideal $(x_1^3, x_1^2x_2, x_1x_2^2, x_2^3) \subset k[x_1, x_2, x_3]$ is a reverse-lex ideal. Of course, if one replaces "reverse-lex" with "lexicographic," one obtains the definition of a lexicographic ideal. In [4], Deery shows that if I is a reverse-lex ideal in $S = k[x_1, \ldots, x_n]$, and J is a stable ideal with the same Hilbert function, then the graded Betti numbers in the resolution of S/I are at most those of S/J. Marinari and Ramella have a similar result in [11]. Unfortunately, Hilbert functions admitting a reverse-lex ideal are rather uncommon, and it is beneficial to consider a generalization introduced in [4].

Definition 2.4 *Let I be a monomial ideal, and for each degree d, let m_d be the minimal monomial generator of I in degree d that is smallest with respect to reverse-lex order. We say that I is a **weakly reverse-lex ideal** if all monomials of degree d greater than m_d in the reverse-lex order are in I.*

Obviously any reverse-lex ideal is a weakly reverse-lex ideal, and it is easy to see that weakly reverse-lex ideals are stable. The ideal $(x_1) + (x_1, x_2, x_3)^3$ is not a reverse-lex ideal since in degree 2, x_2^2 is not in the ideal, while x_1x_3 is. However, it is a weakly reverse-lex ideal.

In [4], Deery modifies combinatorial methods that Bigatti uses in [1] to show that in three variables, the weakly reverse-lex ideal has uniquely minimal graded Betti numbers among all stable ideals with the same Hilbert function. This is a nice generalization since weakly reverse-lex ideals can attain many more Hilbert functions than reverse-lex ideals. Nevertheless, there are still many Hilbert functions for which there is no quotient corresponding to a weakly reverse-lex ideal in R. Consider, for example, the Hilbert function {1,3,3,4}. We need to put three monomials into the ideal in degree 2, and they must be x_1^2, x_1x_2, and x_2^2. But this puts seven monomials into the ideal in degree three, and there can be at most six. Thus there is no weakly reverse-lex ideal such that the quotient can have Hilbert function {1,3,3,4}.

Our goal is to find a class of stable ideals in three variables with their quotients attaining all possible Hilbert functions for cyclic modules; this class should include the weakly reverse-lex ideals, and its members should have uniquely minimal graded Betti numbers among stable ideals with a given Hilbert function. The algorithm in Section 5 produces the desired ideals.

3 Some lemmas and formulas

In this section, we gather some results and formulas about stable ideals that we shall need in the proof that our algorithm works. Many of the techniques in the next two sections come from [1] and [4]. Following their notation, if T is a finite set of monomials in $k[x_1, \ldots, x_n]$, let $m_i(T)$ be the number of monomials u in T such that $\max(u) = i$. Similarly, let $m_{\leq i}(T)$ be the number of monomials u in T such that $\max(u) \leq i$. Finally, let $\mathbf{X}_nT$ denote the set of monomials $\{x_iu | 1 \leq i \leq n \text{ and } u \in T\}$. We have the following proposition from [1] and [4]:

Proposition 3.1 *Let T be a stable set of monomials of degree d. Then: (i) $\boldsymbol{X}_nT$ is stable. (ii) $m_i(\boldsymbol{X}_nT) = m_{\leq i}(T)$. (iii) $|\boldsymbol{X}_nT| = \sum_{i=1}^{n} m_{\leq i}(T)$.*

Given a stable ideal, this result is extremely useful in helping determine how many monomials are in the ideal in a high degree when one knows what the ideal looks like in low degrees. We are particularly interested in the case $n = 3$. Suppose a stable ideal $I \subset R$ is generated in degree d and lower, and suppose we form a new stable ideal $J = (I, m)$ by adding to I a minimal monomial generator m of degree d. If $\max(m) = 2$, then Proposition 3.1 tells us that, comparing J to I, J has one more monomial in degree d, two more in degree $d + 1$, and, in general, $k + 1$ more monomials in degree $d + k$. If instead $\max(m) = 3$, then J has exactly one more monomial than I in each degree d and higher.

In the proof that our algorithm is correct, we shall have to use some information about the growth of a Hilbert function for a quotient of R. Macaulay characterizes the valid Hilbert functions for cyclic modules in [10], and one can express this characterization using sums of binomial coefficients. It is not hard to show that, if a and d are positive integers, a has a unique decomposition as

$$a = \binom{b_d}{d} + \binom{b_{d-1}}{d-1} + \cdots + \binom{b_2}{2} + \binom{b_1}{1},$$

where $b_d > b_{d-1} > \cdots > b_1 \geq 0$. Green [7] calls this the *d-th Macaulay representation of a*. We can then define a new object $a^{<d>}$ to be

$$a^{<d>} = \binom{b_d + 1}{d + 1} + \binom{b_{d-1} + 1}{d} + \cdots + \binom{b_2 + 1}{3} + \binom{b_1 + 1}{2}.$$

Macaulay's characterization says that $H = \{h_0, h_1, \ldots\}$ is a valid Hilbert function if and only if $h_0 = 1$ and for all $d \geq 1$, $h_{d+1} \leq h_d^{<d>}$. (See, e.g., Chapter 4 of [2].) We are interested in Hilbert functions for quotients in three variables, so $h_d \leq \binom{d+2}{d} = \binom{d+2}{2}$ for each d. In order to determine whether we have a valid Hilbert function, the next lemma is helpful.

Lemma 3.2 *Let $a \geq 1$ be an integer, and let l be an integer such that $1 \leq l \leq \binom{a+2}{a}$. If m_l is the unique integer such that $1+\cdots+m_l < l \leq 1+\cdots+(m_l+1)$ (equivalently, $\binom{m_l+1}{2} < l \leq \binom{m_l+2}{2}$), then $[\binom{a+2}{a}-l]^{<a>} = [\binom{a+2}{a}-l]+(a-m_l)$.*

Proof: First suppose that $\binom{a+2}{a} - l \geq a+1$. Since $l \geq 1$, the a-th Macaulay expansion of $\binom{a+2}{a} - l$ begins with $\binom{a+1}{a}$, and each additional term is of the form $\binom{y+1}{y}$, $\binom{y}{y}$, or zero. When passing to $[\binom{a+2}{a} - l]^{<a>}$, it is clear that the values of the latter two types of terms are unchanged, and sending $\binom{y+1}{y}$ to $\binom{y+2}{y+1}$ adds one to the previous sum. To compute $[\binom{a+2}{a} - l]^{<a>}$, we work from the Macaulay expansion

$$\binom{a+2}{a} - l = \binom{a+1}{a} + \binom{a}{a-1} + \cdots + \binom{z+1}{z} + \binom{z-1}{z-1} + \cdots + \binom{b}{b}.$$

Clearly $[\binom{a+2}{a} - l]^{<a>} = [\binom{a+2}{a} - l] + (a - z + 1)$. Therefore it suffices to show that $z = m_l + 1$. Note that

$$(a+1) + a + \cdots + (z+1) + z > \binom{a+2}{2} - l \geq (a+1) + a + \cdots + (z+1).$$

Multiplying by -1 and adding $\binom{a+2}{2}$, we obtain

$$1 + \cdots + (z-1) < l \leq 1 + \cdots + z.$$

Consequently, $m_l = z - 1$ as desired.

If instead $\binom{a+2}{a} - l \leq a$, all the terms in the Macaulay expansion are 1, and so the "upper a" operation should not add anything. Note that $1+\cdots+a < l \leq 1 + \cdots + (a+1)$. Hence $m_l = a$, and this agrees with our formula. □

4 Combinatorial formulas for the Betti numbers

We wish to find a condition under which the graded Betti numbers of a given stable ideal will be minimal among the Betti numbers of all stable ideals with a fixed Hilbert function. It proves helpful to relate the formula from Section 2 for the graded Betti numbers to the $m_{\leq i}$ defined in the previous section. To do this, we again borrow some definitions and results from Bigatti and Deery. First, we compare $m_{\leq i}$ for two stable sets, taking a result from [4].

Lemma 4.1 *Let T and U be stable sets of the same degree such that $m_{\leq i}(T) \leq m_{\leq i}(U)$ for all i. Then $m_{\leq i}(\boldsymbol{X}_n T) \leq m_{\leq i}(\boldsymbol{X}_n U)$.*

Next, we make two definitions from [4] that will allow us to draw conclusions about graded Betti numbers and total Betti numbers (the sum of the graded Betti numbers for a fixed syzygy). For a finite set of monomials T, we define

$$b_{qj}(T) = \sum_{\substack{m \in T \\ deg(m)=j-q+1}} \binom{\max(m)-1}{q-1} \quad \text{and} \quad b_q(T) = \sum_{j=0}^{\infty} b_{qj}(T).$$

The first step in relating the $m_{\leq i}$ to the Betti numbers is the following proposition of Bigatti from [1] and [4]:

Proposition 4.2 *If T is a stable set of monomials of all the same degree in $R = k[x_1, x_2, x_3]$, then*

$$b_q(T) = \binom{2}{q-1}|T| - \sum_{i=1}^{2}\left[m_{\leq i}(T)\binom{i-1}{q-2}\right].$$

We need two more results before we can prove our main lemma. The first is Deery's adaptation in [4] of a formula from [1], and its proof follows from the previous results.

Lemma 4.3 *Let T and U be stable sets of monomials of the same degree such that $|T| = |U|$ and $m_{\leq i}(T) \geq m_{\leq i}(U)$ for each i. Then (i) $b_q(T) \leq b_q(U)$ and (ii) $b_q(\boldsymbol{X}_n T) \geq b_q(\boldsymbol{X}_n U)$.*

Now let $M(I_d)$ be the set of monomials in I of degree d. Our final tool is the following from [1] and [4] (note that the subscripts on I are incorrect in [4]):

Lemma 4.4 *If I is a stable ideal, then for $j \geq q$, $\beta_{qj}(R/I) = b_q(M(I_{j-q+1})) - b_q(\boldsymbol{X}_3 M(I_{j-q}))$.*

Finally, we may state the lemma that we need to prove the minimality of the graded Betti numbers in the following section. This is based almost completely on Deery's proof of Theorem 3.10 in [4].

Lemma 4.5 *Let $I \subset R$ be a stable ideal such that if $J \subset R$ is any stable ideal with the same Hilbert function, then $m_{\leq i}(I_d) \geq m_{\leq i}(J_d)$ for all i and d. Then for all q and j, $\beta_{qj}(R/I) \leq \beta_{qj}(R/J)$.*

Proof: By Lemma 4.3, $b_q(M(I_d)) \leq b_q(M(J_d))$ and $b_q(\mathbf{X}_3 M(I_d)) \geq b_q(\mathbf{X}_3 M(J_d))$ for all d. Now if $j < q$, clearly $\beta_{qj}(R/I) = \beta_{qj}(R/J) = 0$. Suppose $j \geq q$. By the previous lemma,

$$\begin{aligned}
\beta_{qj}(R/I) &= b_q(M(I_{j-q+1})) - b_q(\mathbf{X}_3 M(I_{j-q})) \\
&\leq b_q(M(J_{j-q+1})) - b_q(\mathbf{X}_3 M(J_{j-q})) \\
&= \beta_{qj}(R/J). \ \square
\end{aligned}$$

Since we are working in three variables, if two stable ideals I and J have the same Hilbert function, $m_{\leq 1}(I_d) = m_{\leq 1}(J_d)$ and $m_{\leq 3}(I_d) = m_{\leq 3}(J_d)$ in each degree d. Hence in the previous lemma, it suffices to note that I and J are stable ideals with the same Hilbert function such that $m_{\leq 2}(I_d) \geq m_{\leq 2}(J_d)$. That is, we seek a stable ideal I that has as many monomials in each degree that do not involve x_3 as possible.

5 The algorithm

In this section we present our algorithm to obtain an ideal in R with uniquely minimal graded Betti numbers among all stable ideals with a given Hilbert function. We begin by assuming that our Hilbert function has finite length (that is, that the Hilbert function is eventually zero).

Input: A valid finite length Hilbert function H for a cyclic graded module over $R = k[x_1, x_2, x_3]$.
Output: A stable ideal I in R such that R/I has Hilbert function H and uniquely minimal graded Betti numbers among all quotients of R with Hilbert function H that correspond to stable ideals.

Initialization:
p:=first positive degree in which H is zero;
I:=$(x_1, x_2, x_3)^p$;
$\bar{H}$:=Hilbert function of R/I;
Δ:=$\bar{H} - H$;

```
begin
while Δ ≠ {0,...,0} do
    t:=first degree in which Δ is not zero;
    M:=monomials of degree t in descending reverse-lex order;
    while entry t of Δ is not zero do
        m:=first monomial in M not in I;
        H_0:=Hilbert function of R/(I,m);
        if (max(m)=2 and H_0 − H is not weakly increasing)
            then M := only the monomials of degree t involving x_3;
        else if (max(m)=1 or 2) or (max(m)=3 and (I,m) is stable)
            then
                I := (I,m);
                H̄:=Hilbert function of R/I;
        M := M \ {m};
        Δ := H̄ − H;
return I;
end
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Our algorithm works by adding monomials into the ideal being constructed in reverse-lex order as much as possible. The list Δ indicates how many monomials we still need to add into the ideal in each degree to attain the Hilbert function H. As long as the Hilbert function allows it, we put in generators only involving x_1 and x_2. At some point, it is possible that adding in such a generator will make the Hilbert function too small in a higher degree; by Proposition 3.1, this occurs when Δ is no longer weakly increasing. Then we must resort to adding generators with nonzero powers of x_3 while being careful to maintain the stability of the ideal. The next theorem asserts that the algorithm gives the ideal that we want.

Theorem 5.1 *Given a finite length Hilbert function H for a cyclic graded module over $R = k[x_1, x_2, x_3]$, the algorithm above yields a stable ideal I such that $H_{R/I} = H$, and given any stable ideal $J \subset R$ with $H_{R/J} = H$, $\beta^{R/I} \leq \beta^{R/J}$.*

Proof: The main work in the proof is showing that if we cannot add a generator involving only x_1 and x_2, then there exists a monomial involving x_3 that we can add that keeps the ideal stable. We prove this by supposing that no such monomial exists and enumerating the monomials that must already be in the ideal. This allows us to compute a formula for the Hilbert function in each degree. Consequently, we can use the bound on the growth of the Hilbert function from Lemma 3.2 to show that the Hilbert function grows faster than is allowed, so our input was not a valid Hilbert function.

For each Hilbert function H, there is a quotient corresponding to a lexicographic ideal attaining that Hilbert function. Thus Proposition 3.1 shows that Δ, the difference vector of the Hilbert function of R/I at a point in the algorithm and H, must initially be weakly increasing. The first generator is forced, and it must be a power of x_1. If one needs to add a second generator, it must be x_1 to one fewer power times a power of x_2 to ensure stability. Clearly Δ stays weakly increasing since the ideal is currently a lexicographic ideal.

Suppose now that we have added in $w > 1$ generators, maintaining a stable ideal with weakly increasing Δ and putting in the monomials in the order dictated by the algorithm. If we need not add in more monomials (that is, if Δ is the zero vector), we are completely finished, for the ideal has the desired Hilbert function. Otherwise, we need to put in a new monomial in, say, degree t. If not all $x_1^{t-d}x_2^d$ are in I, we attempt to add the first one (in reverse-lex order) that is not already in. If Δ remains weakly increasing, then we are done since the ideal clearly remains stable. If not, then we must add in a monomial with nonzero power of x_3. Proposition 3.1 shows that in this case, there is some $s \geq 0$ such that $\Delta_{t+s} = \Delta_{t+s+1} = y$ for some $y > 0$. We wish to show that we can find a monomial m of degree t involving x_3 to add to the ideal that keeps the ideal stable; if so, Δ will remain weakly increasing by the formulas in Proposition 3.1.

We claim that we can always do this. Suppose not, and suppose that x_1^t, $x_1^{t-1}x_2$, ..., and $x_1^{t-d}x_2^d$ are in I. Under our assumption, there does not exist a monomial m of degree t such that (I, m) is stable and the Hilbert function is not too small. Hence we may conclude that $x_1^{t-1}x_3$, $x_1^{t-2}x_2x_3$, ..., $x_1^{t-d}x_2^{d-1}x_3$, $x_1^{t-2}x_3^2$, ..., $x_1^{t-d}x_2^{d-2}x_3^2$, ..., $x_1^{t-d}x_3^d$ are all in I. Therefore in degree t, there is one monomial involving only x_1; there are d monomials u with $\max(u) = 2$, and there are $\binom{d+1}{2} + r$ monomials u with $\max(u) = 3$, where $r \geq 0$. By (ii) of Proposition 3.1, it is easy to compute that there are $d+s$ monomials involving x_2 but not x_3 in degree $t+s$, and there are $d+s+1$ in degree $t+s+1$. Similarly, we can count the monomials in each degree in I that have nonzero powers of x_3. There are $\binom{s+1}{2}$ $+sd+$ $\binom{d+1}{2}$ $+r$ in degree $t+s$ and $\binom{s+2}{2}$ $+(s+1)d+\binom{d+1}{2}$ $+r$ in degree $t+s+1$.

Thus in I,

$$\#\text{ of monomials in degree } t+s = \binom{s+2}{2} + (s+1)d + \binom{d+1}{2} + r,$$

and

$$\#\text{ of monomials in degree } t+s+1 = \binom{s+3}{2} + (s+2)d + \binom{d+1}{2} + r.$$

Consequently, we obtain formulas for the Hilbert function in degrees $t+s$ and $t+s+1$:

$$h_{t+s} = \binom{t+s+2}{2} - \left[\binom{s+2}{2} + (s+1)d + \binom{d+1}{2} + r + y\right],$$

and

$$h_{t+s+1} = \binom{t+s+3}{2} - \left[\binom{s+3}{2} + (s+2)d + \binom{d+1}{2} + r + y\right].$$

As a result, we have $h_{t+s+1} - h_{t+s} = t+s+2-(s+2)-d = t-d$. Thus $h_{t+s} + t - d = h_{t+s+1} \leq h_{t+s}^{<t+s>}$. We wish to find $h_{t+s}^{<t+s>}$. Let l be the number of monomials in the desired final ideal in degree $t+s$. By Lemma 3.2,

$$h_{t+s}^{<t+s>} = h_{t+s} + t + s - m_l, \quad \binom{m_l+1}{2} < l \leq \binom{m_l+2}{2}.$$

Hence $h_{t+s} + t - d \leq h_{t+s}$ $+t+s-m_l$. Therefore $s + d \geq m_l$.

Thus if we cannot find a monomial with nonzero power of x_3 to add that keeps the ideal stable, then $s + d \geq m_l$. We have

$$\begin{aligned} l &= \binom{s+2}{2} + (s+1)d + \binom{d+1}{2} + r + y \\ &= \binom{s+d+2}{2} + r + y \\ &\leq \binom{m_l+2}{2} \\ &\leq \binom{s+d+2}{2}. \end{aligned}$$

This is a contradiction because $r + y > 0$. Hence by induction, we can always find the monomial we need to keep the ideal stable and to ensure that Δ stays weakly increasing.

The algorithm will clearly terminate because the Hilbert function is eventually zero. The minimality of the graded Betti numbers is immediate since the algorithm maximizes the number of monomials in each degree not involving x_3 subject to the constraints of the Hilbert function and keeping the ideal stable. $\square$

We can use the finite length case now to generalize the result to Hilbert functions that are never zero in positive degree.

Corollary 5.2 *Fix a Hilbert function H for a cyclic graded module over $R = k[x_1, x_2, x_3]$. Then there exists a unique minimal set of graded Betti numbers among all quotients of R with Hilbert function H corresponding to stable ideals. That is, there are no incomparable minimals in this class.*

Proof: The finite length case is in the previous theorem, so we now consider a Hilbert function H for a cyclic graded module over R that is never zero in positive degree. The idea is to truncate the Hilbert function at an appropriate place so that we may work with something of finite length. Since R modulo the lexicographic ideal has the most generators in each degree of any cyclic module with a given Hilbert function, we first compute the highest degree d of a minimal generator for the lexicographic ideal L whose quotient has Hilbert function H. Writing $H = \{1, h_1, h_2, \ldots, h_d, h_{d+1}, \ldots\}$, let $H' = \{1, h_1, h_2, \ldots, h_d, h_{d+1}\}$. Then H' has finite length, so we can use H' as the input in the finite length algorithm, obtaining output I'. Now the Hilbert function of R/I' is H', so it agrees with H through one degree beyond the degree of the maximal generator of L. We then remove all the generators of I' of degree higher than d, forming a new ideal I. Both L and I have generators of degree no higher than d, and the Hilbert functions of R/L and R/I agree through degree $d+1$. By the Gotzmann Persistence Theorem, since R/L has Hilbert function H, R/I must also, and its graded Betti numbers obviously have the minimality property we want. $\square$

The ideal we obtain from our algorithm actually satisfies a more restrictive condition than stability. Recall that we call a monomial ideal *strongly stable* if whenever a monomial $m \in I$, then $x_i m / x_j \in I$ for each $i < j$.

Proposition 5.3 *The above algorithm produces a strongly stable ideal.*

Proof: It suffices to check the strongly stable property on minimal generators. Suppose $x_1^a x_2^b x_3^c$ is a minimal generator of I. We need to show that $x_1^{a+1} x_2^{b-1} x_3^c \in I$, for the other monomials to check are automatically in I by stability. We proceed by induction on c. If $c = 0$, $x_1^{a+1} x_2^{b-1} \in I$ because it comes before $x_1^a x_2^b$ in reverse-lex order, so we would have chosen it first. Now suppose that $c \geq 1$ and that the strongly stable property holds for all monomials with power of x_3 at most $c - 1$. Stability implies that both $x_1^{a+1} x_2^b x_3^{c-1}$ and $x_1^a x_2^{b+1} x_3^{c-1}$

are in I. Note that $x_1^{a+1}x_2^{b-1}x_3^c >_{rlex} x_1^a x_2^b x_3^c$, so if $x_1^{a+1}x_2^{b-1}x_3^c$ is not in I, we must not have chosen it in our algorithm because its inclusion in I would have ruined stability. Thus either $x_1^{a+2}x_2^{b-1}x_3^{c-1}$ or $x_1^{a+1}x_2^b x_3^{c-1}$ is not in I. But by the induction hypothesis, $x_1^{a+1}x_2^b x_3^{c-1} \in I$ implies that $x_1^{a+2}x_2^{b-1}x_3^{c-1} \in I$, and similarly, $x_1^a x_2^{b+1}x_3^{c-1} \in I$ implies that $x_1^{a+1}x_2^b x_3^{c-1} \in I$. This is a contradiction, so $x_1^{a+1}x_2^{b-1}x_3^c \in I$ after all. □

These results mean that one must go beyond the class of stable ideals in three variables to find the incomparability behavior that Charalambous and Evans exhibit with monomial ideals in three variables that are not stable. We consider stable ideals in more variables next, where the combinatorics is harder to unravel.

6 Incomparability in more variables

We now turn to rings of higher dimension, motivated by Deery's conjecture in [4] that whenever H is a Hilbert function such that there exists a weakly reverse-lex ideal I with $H_{S/I} = H$, then $\beta^{S/I} \leq \beta^{S/J}$ for all stable J with $H_{S/J} = H$.

Initially, one might hope that we could find a algorithm similar to that of Section 5 for Hilbert functions in $S = k[x_1, \ldots, x_n]$, where $n \geq 4$. Unfortunately, if one tries the analogous procedure, it is quickly apparent that it fails. Consider, for example, the Hilbert function $\{1, 4, 6, 9\}$ in $k[x_1, x_2, x_3, x_4]$. Putting in the first three monomials of degree two in reverse-lex order keeps the difference of the Hilbert functions vector weakly increasing, but then we can add only one more monomial in degrees two and three. No choice of degree two monomial will satisfy this condition. Minor adjustments to account for the change in combinatorics from three to four variables also do not seem to help much since one can no longer simply worry about the number of monomials not involving the last variable.

In lieu of such an algorithm, we would at least like to find some minimality result that applies to Hilbert functions for which there is no quotient corresponding to a reverse-lex ideal. In presenting the above conjecture on weakly reverse-lex ideals, Deery notes that one cannot use the same methods of proof in four or more variables as in three variables by exhibiting an example for which the techniques of Section 4 fail. We show that Deery's example is actually a counterexample to his conjecture (also modifying it to use finite length modules).

Let $I = (x_1^4, x_1^3x_2, x_1^2x_2^2, x_1x_2^3, x_2^6) + (x_1, x_2, x_3, x_4)^8$, and let $J = (x_1^4, x_1^3x_2, x_1^3x_3, x_1^3x_4, x_1^2x_2^3, x_1x_2^4, x_2^5) + (x_1, x_2, x_3, x_4)^8$ be ideals in $S = k[x_1, x_2, x_3, x_4]$. It is easy to check that S/I and S/J both have Hilbert function $\{1, 4, 10, 20, 31, 43, 55, 67\}$. Moreover, I is a weakly reverse-lex ideal, and J is stable. (Both ideals are actually strongly stable.) To display the graded Betti numbers, we use the notation of the computer algebra system Macaulay 2 [6]. The rows and columns are numbered starting with zero, and one can find β_{ij} in column i and row $j - i$. We use a period in place of the number zero. The graded free resolutions of S/I and S/J are as follows:

S/I:	total:	1	84	229	213	67
	0:	1	.	.	.	.
	1:	.	.	.	.	.
	2:	.	.	.	.	.
	3:	.	4	3	.	.
	4:	.	.	.	.	.
	5:	.	1	1	.	.
	6:	.	.	.	.	.
	7:	.	79	225	213	67

S/J:	total:	1	86	234	217	68
	0:	1	.	.	.	.
	1:	.	.	.	.	.
	2:	.	.	.	.	.
	3:	.	4	6	4	1
	4:	.	3	3	.	.
	5:	.	.	.	.	.
	6:	.	.	.	.	.
	7:	.	79	225	213	67

Obviously these sets of graded Betti numbers are incomparable. We argue that there is no set of graded Betti numbers for a quotient with the same Hilbert function corresponding to a stable ideal that lies below both of these sequences. For if there were, then $\beta_{00} = 1$, $\beta_{14} = 4$, $\beta_{25} \leq 3$, and all other graded Betti numbers above row seven are zero. It is clear that in this case, β_{25} must be at least three, so it is exactly three. There are no second syzygies in row three, so by the Eliahou-Kervaire formula, the generators in degree four have to involve only x_1 and x_2. But this gives us the generators of I in degree four. To get the necessary Hilbert function, we would need an additional generator in degree six, a contradiction.

We can construct a similar family of finite length examples in more variables by putting all but the last four variables in each ideal and then forming the ideals corresponding to I and J using the final four variables. Stability is obvious, and it is not hard to see that we obtain incomparable graded Betti diagrams with no set of Betti numbers for that Hilbert function beneath both. Hence if we have at least four variables, the weakly reverse-lex ideal does not necessarily have uniquely minimal graded Betti numbers among stable ideals with a given Hilbert function. The above family of examples shows that we can actually have incomparable minimal elements whenever our ring has more than three variables, and therefore we cannot hope for an algorithm similar to that of Section 5 in higher dimensions.

References

[1] A. M. Bigatti, *Upper Bounds for the Betti Numbers of a Given Hilbert Function*, Comm. Algebra, **21** (1993), no. 7, 2317-2334.

[2] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1993.

[3] H. Charalambous and E. G. Evans, *Resolutions with a Given Hilbert Function*, Contemp. Math., **159** (1994), 19-26.

[4] T. Deery, *Rev-Lex Segment Ideals and Minimal Betti Numbers*, Queen's Papers in Pure and Appl. Math., **102** (1995), 193-219.

[5] S. Eliahou and M. Kervaire, *Minimal Resolutions of Some Monomial Ideals*, J. Algebra, **129** (1990), 1-25.

[6] D. Grayson and M. Stillman, *Macaulay 2, a Software System for Research in Algebraic Geometry.* Available at http://www.math.uiuc.edu/Macaulay2.

[7] M. Green, Restrictions of Linear Series to Hyperplanes, and Some Results of Macaulay and Gotzmann. In E. Ballico and C. Ciliberto (eds.), *Algebraic Curves and Projective Geometry*, LNM **1389**, Springer, 1989, pp. 76-86.

[8] D. Hilbert, *Über die Theorie der Algebraischen Formen*, Math. Annalen, **36** (1890), 473-534.

[9] H. A. Hulett, *Maximum Betti Numbers of Homogeneous Ideals with a Given Hilbert Function*, Comm. Algebra, **21** (1993), no. 7, 2335-2350.

[10] F. S. Macaulay, *Some Properties of Enumeration in the Theory of Modular Systems*, Proc. London Math. Soc., **26** (1927), 531-555.

[11] M. Marinari and L. Ramella, *Some properties of Borel ideals*, J. Pure Appl. Algebra, **139** (1999), no. 1-3, 183-200.

[12] K. Pardue, *Deformation Classes of Graded Modules and Maximal Betti Numbers*, Illinois J. Math., **40** (1996), 564-585.