A Four-Parameter Partition Identity

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Abstract. We present a new partition identity and give a combinatorial proof of our result. This generalizes a result of Andrews in which he considers the generating function for partitions with respect to size, number of odd parts, and number of odd parts of the conjugate.

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1. Introduction

In [1], Andrews considers partitions with respect to size, number of odd parts, and number of odd parts of the conjugate. He derives the following generating function

$$\sum_{\lambda \in \mathrm{Par}} r^{\theta(\lambda)} s^{\theta(\lambda')} q^{|\lambda|} = \prod_{j=1}^{\infty} \frac{(1+rsq^{2j-1})}{(1-q^{4j})(1-r^2q^{4j-2})(1-s^2q^{4j-2})} \tag{1}$$

where Par denotes the set of all partitions, $|\lambda|$ denotes the size (sum of the parts) of λ, $\theta(\lambda)$ denotes the number of odd parts in the partition λ, and $\theta(\lambda')$ denotes the number of odd parts in the conjugate of λ. In this paper, we generalize this result and provide a bijective proof of our generalization. This provides a simple combinatorial proof of Andrews' result. Other combinatorial proofs of (1) have been found by Sills in [2] and Yee in [4].

2. Main Result

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$ and $\lambda_i \in \mathbb{Z}_+$ for $i = 1, 2, 3, \ldots$ be a partition. Consider the following weight functions on the set of all partitions:

$$\begin{aligned} \alpha(\lambda) &= \sum \lceil \lambda_{2i-1}/2 \rceil \\ \beta(\lambda) &= \sum \lfloor \lambda_{2i-1}/2 \rfloor \end{aligned}$$

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$$\begin{aligned}\gamma(\lambda) &= \sum \lceil \lambda_{2i}/2 \rceil \\ \delta(\lambda) &= \sum \lfloor \lambda_{2i}/2 \rfloor.\end{aligned}$$

Also, let a, b, c, d be (commuting) indeterminants, and define

$$w(\lambda) = a^{\alpha(\lambda)} b^{\beta(\lambda)} c^{\gamma(\lambda)} d^{\delta(\lambda)}.$$

For instance, if $\lambda = (5, 4, 4, 3, 2)$ then $\alpha(\lambda)$ is the number of a's in the following diagram for λ, $\beta(\lambda)$ is the number of b's in the diagram, $\gamma(\lambda)$ is the number of c's in the diagram, and $\delta(\lambda)$ is the number of d's in the diagram. Moreover, $w(\lambda)$ is the product of the entries of the diagram.

$$\begin{array}{lllll} a & b & a & b & a \\ c & d & c & d & \\ a & b & a & b & \\ c & d & c & & \\ a & b & & & \end{array}$$

These weights were first suggested by Stanley in [3].

Let $\Phi(a, b, c, d) = \sum w(\lambda)$, where the sum is over all partitions λ, and let $\Psi(a, b, c, d) = \sum w(\lambda)$, where the sum is over all partitions λ with distinct parts. We obtain the following product formulas for $\Phi(a, b, c, d)$ and $\Psi(a, b, c, d)$:

Theorem 1.

$$\Phi(a,b,c,d) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^{j-1} c^{j-1} d^{j-1})(1 + a^j b^j c^j d^{j-1})}{(1 - a^j b^j c^j d^j)(1 - a^j b^j c^{j-1} d^{j-1})(1 - a^j b^{j-1} c^j d^{j-1})}$$

Corollary 2.

$$\Psi(a,b,c,d) = \prod_{j=1}^{\infty} \frac{(1 + a^j b^{j-1} c^{j-1} d^{j-1})(1 + a^j b^j c^j d^{j-1})}{(1 - a^j b^j c^{j-1} d^{j-1})}$$

Andrews' result follows easily from Theorem 1. Note that we can express number of odd parts of λ, number of odd parts of λ' and size of λ in terms of the number of a's, b's, c's, and d's in the diagram for λ as follows:

$$\begin{aligned}\theta(\lambda) &= \alpha(\lambda) - \beta(\lambda) + \gamma(\lambda) - \delta(\lambda) \\ \theta(\lambda') &= \alpha(\lambda) + \beta(\lambda) - \gamma(\lambda) - \delta(\lambda) \\ |\lambda| &= \alpha(\lambda) + \beta(\lambda) + \gamma(\lambda) + \delta(\lambda).\end{aligned}$$

Thus we transform $\Phi(a, b, c, d)$ by sending $a \mapsto rsq$, $b \mapsto r^{-1}sq$, $c \mapsto rs^{-1}q$, and $d \mapsto r^{-1}s^{-1}q$. A straightforward computation gives (1).

Our main result is a generalization of Theorem 1 and Corollary 2. It is the corresponding product formula in the case where we restrict the parts to some congruence class (mod k) and we restrict the number of times those parts can occur. Let R be a subset of positive integers congruent to $i(\text{mod } k)$ and let ρ be a map from R to the even positive integers. Let $\text{Par}(i,k;R,\rho)$ be the set of all partitions with parts congruent to $i(\text{mod } k)$ such that if $r \in R$, then r appears as a part less than $\rho(r)$ times. Let $\Phi_{i,k;R,\rho}(a,b,c,d) = \sum_\lambda w(\lambda)$ where the sum is over all partitions in $\text{Par}(i,k;R,\rho)$.

For example, $\text{Par}(0,1;\emptyset,\rho))$ is Par, the set of all partitions. Also, if we let R be the set of all positive integers and ρ map every positive integer to 2, then $\text{Par}(1,1;\mathbb{Z}_+,\rho)$ is the set of all partitions with distinct parts. These are the two cases found in Theorem 1 and Corollary 2.

Theorem 3.

$$\Phi_{i,k;R,\rho}(a,b,c,d) = ST$$

where

$$S = \prod_{j=1}^{\infty} \frac{(1 + a^{\lceil\frac{(j+1)k+i}{2}\rceil} b^{\lfloor\frac{(j+1)k+i}{2}\rfloor} c^{\lceil\frac{jk+i}{2}\rceil} d^{\lfloor\frac{jk+i}{2}\rfloor})}{(1 - a^{\lceil\frac{jk+i}{2}\rceil} b^{\lfloor\frac{jk+i}{2}\rfloor} c^{\lceil\frac{jk+i}{2}\rceil} d^{\lfloor\frac{jk+i}{2}\rfloor})(1 - a^{jk} b^{(j-1)k} c^{jk} d^{(j-1)k})}$$

and

$$T = \prod_{r\in R} (1 - a^{\lceil\frac{r}{2}\rceil\frac{\rho(r)}{2}} b^{\lfloor\frac{r}{2}\rfloor\frac{\rho(r)}{2}} c^{\lceil\frac{r}{2}\rceil\frac{\rho(r)}{2}} d^{\lfloor\frac{r}{2}\rfloor\frac{\rho(r)}{2}})$$

3. Combinatorial Proof of these Results

The proof of Theorem 3 is a slight modification of the proofs of Theorem 1 and Corollary 2. For clarity, we will first give the argument in the special case where we consider all partitions and partitions with distinct parts and then we will mention how the proof can be modified to work in general.

Proof of Theorem 1.
Consider the following class of partitions:

$$\mathcal{R} = \{\lambda \in \text{Par} : \lambda_{2i-1} - \lambda_{2i} \leq 1\}.$$

We are restricting the difference between a part of λ which is at an odd level and the following part of λ to be at most 1.

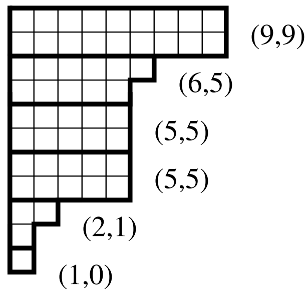

Figure 1. $\lambda = (9,9,6,5,5,5,5,5,2,1,1)$ decomposes into blocks $\{(9,9),(6,5),(5,5),(5,5),(2,1),(1,0)\}$

To find the generating function for partitions in $\mathcal{R}$ under weight $w(\lambda)$, we decompose $\lambda \in \mathcal{R}$ into blocks of height 2, $\{(\lambda_1,\lambda_2),(\lambda_3,\lambda_4),\ldots\}$. (In order to do this if we have an odd number of parts, add one part equal to 0.) Since the difference of parts is restricted to either 0 or 1 at odd levels, we can only get two types of blocks. For any $k \geq 1$, we can have a block with two parts of length k, i.e. (k,k). Call this Type I. In addition, for any $k \geq 1$, we can have a block with one part of length k and then other of length $k-1$, i.e. $(k,k-1)$. Call this Type II.

In fact, partitions in $\mathcal{R}$ correspond uniquely to a multiset of blocks of Type I and II with at most one block of Type II for each length k. Figure 1 shows an example of such a decomposition.

To calculate the generation function for $\mathcal{R}$, it remains to calculate the weights of our blocks. The blocks of Type I get filled as follows:

$$\begin{array}{cccccccc} a & b & a & b & \ldots & a & b \\ c & d & c & d & \ldots & c & d \end{array} \quad \text{or} \quad \begin{array}{ccccccccc} a & b & a & b & \ldots & a & b & a \\ c & d & c & d & \ldots & c & d & c \end{array}$$

depending on the length of the blocks. Therefore they have weights $a^jb^jc^jd^j$ or $a^jb^{j-1}c^jd^{j-1}$.

The blocks of Type II get filled as follows:

$$\begin{array}{cccccccc} a & b & a & b & \ldots & a & b & a \\ c & d & c & d & \ldots & c & d & \end{array} \quad \text{or} \quad \begin{array}{ccccccc} a & b & a & b & \ldots & a & b \\ c & d & c & d & \ldots & c & \end{array}$$

depending on the length of the blocks. Therefore they have weights $a^jb^{j-1}c^{j-1}d^{j-1}$ or $a^jb^jc^jd^{j-1}$.

So we have the following generating function:

$$\sum_{\lambda\in\mathcal{R}} w(\lambda) = \prod_{j=1}^{\infty} \frac{(1+a^jb^{j-1}c^{j-1}d^{j-1})(1+a^jb^jc^jd^{j-1})}{(1-a^jb^jc^jd^j)(1-a^jb^{j-1}c^jd^{j-1})}.$$

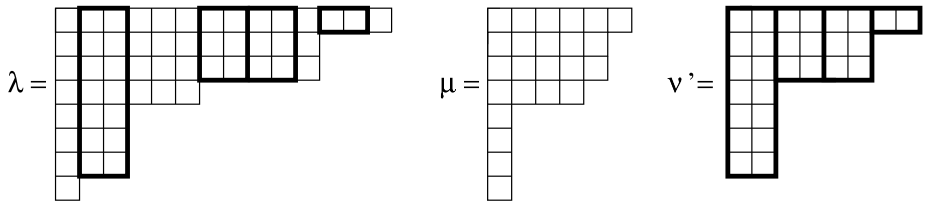

Figure 2. $\lambda = (14, 11, 11, 6, 3, 3, 3, 1)$ and $f(\lambda) = (\mu, \nu')$ where $\nu = (7, 7, 3, 3, 3, 3, 1, 1)$ and $\mu = (6, 5, 5, 4, 1, 1, 1, 1)$

Notice that $\sum_{\lambda \in \mathcal{R}} w(\lambda)$ contains all the factors in $\Phi(a, b, c, d)$ except for

$$\prod_{j=1}^{\infty} \frac{1}{1 - a^j b^j c^{j-1} d^{j-1}}.$$

Let $\mathcal{S}$ be the set of partitions whose conjugates have only odd parts each of which is repeated an even number of times. We give a bijection $f : \mathrm{Par} \to \mathcal{R} \times \mathcal{S}$, such that $\mathcal{S}$ contributes exactly the missing factors.

Given a partition λ, let ν be the partition with $\lambda_{2i-1} - \lambda_{2i}$ parts equal to $2i - 1$ if $\lambda_{2i-1} - \lambda_{2i}$ is even and $\lambda_{2i-1} - \lambda_{2i} - 1$ parts equal to $2i - 1$ if $\lambda_{2i-1} - \lambda_{2i}$ is odd. Also, let μ be the partition defined by $\mu_i = \lambda_i - \nu'_i$. Then we let $f(\lambda) = (\mu, \nu')$. In other words, the map f removes as many blocks of width 2 and odd height as possible from λ. (Call these blocks of Type III.) These blocks are joined together to give ν'. The boxes which are left behind form μ. By its definition, ν has only odd parts repeated an even number of times, which implies that $\nu' \in \mathcal{S}$. Moreover, since we are removing as many pairs of columns of odd height as possible from λ, μ must have $\lambda_{2i-1} - \lambda_{2i} \leq 1$. To see that f is a bijection, we note that its inverse is simply taking the sum of μ and ν' since $\lambda_i = \mu_i + \nu'_i$. An example is shown in Figure 2.

Now we examine the relationship between $w(\lambda)$, $w(\mu)$, and $w(\nu')$. Consider the blocks of Type III in λ. They always have weight $a^j b^{j-1} c^j d^{j-1}$ regardless of whether their first column contains a's and c's or b's and d's. This is also the weight of the blocks when they are placed in ν'. Hence $w(\nu')$ is the product of the entries in the diagram of λ which are removed to get μ.

Moreover, since we are removing columns of width 2, the entries in the squares of the diagram of λ that correspond to squares in the diagram on μ do not change when ν' is removed. This implies that $w(\lambda) = w(\mu)w(\nu')$ and the result follows.

Proof of Corollary 2.
Let $\mathcal{D}$ denote the set of partitions with distinct parts and let $\mathcal{E}$ denote the set of partitions whose parts appear an even number of times. Then

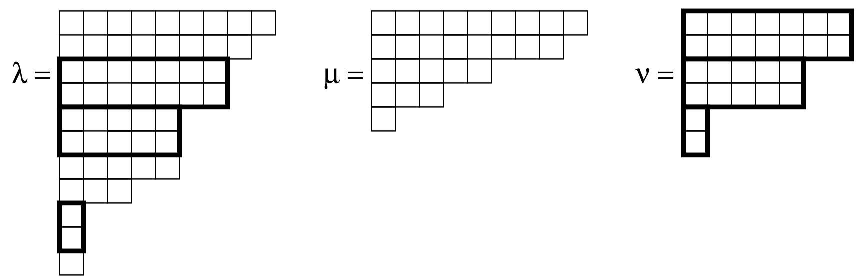

Figure 3. $\lambda = (9,8,7,7,5,5,5,3,1,1,1)$ and $g(\lambda) = (\mu,\nu)$ where $\mu = (9,8,5,3,1)$ and $\nu = (7,7,5,5,1,1)$

we define the following map $g : \mathrm{Par} \to \mathcal{D} \times \mathcal{E}$. Suppose λ has k parts equal to i. If k is even then ν has k parts equal to i, and if k is odd then ν has $k-1$ parts equal to i. The parts of λ which are not removed to form ν, at most one of each length, give μ. We let $g(\lambda) = (\mu,\nu)$. An example is shown in Figure 3. The map g is a bijection since its inverse is taking the union of the parts of μ and ν. Similarly to the situation in the proof of Theorem 1, since we are removing an even number of rows of each length, we see that $w(\lambda) = w(\mu)w(\nu)$.

Now using the decompostion from the proof of Theorem 1, partitions in $\mathcal{E}$ have a decomposition which only uses blocks of Type I. Hence we get that

$$\Phi(a,b,c,d) = \Psi(a,b,c,d) \prod_{j=1}^{\infty} \frac{1}{(1-a^jb^jc^jd^j)(1-a^jb^{j-1}c^jd^{j-1})}$$

and the result follows.

The proof of our main result follows by the same argument with a modification to the sizes of the blocks.

Proof of Theorem 3.
First we find the generation function $S = \Phi_{i,k;\emptyset,\rho}(a,b,c,d)$ without any restriction on the number of times each part may occur. This is done by using Type I blocks with two parts each of length $jk+i$, for $j \geq 1$, Type II blocks with two parts, one of length $jk+i$ and one of length $(j-1)k+i$, for $j \geq 1$, and Type III blocks which are rectangular with width $2k$ and odd height.

Next, we notice a bijection, analogous to the one in the proof of Corollary 2, between $\mathrm{Par}(i,k;\emptyset,\rho)$ and $\mathrm{Par}(i,k;R,\rho) \times \mathcal{T}$ where $\mathcal{T}$ is the set of all partitions with parts, $r \in R$ and occuring a multiple of

$\rho(r)$ times. Since the generating function for $\mathcal{T}$ is

$$T^{-1} = \prod_{r \in R} \frac{1}{1 - a^{\lceil \frac{r}{2} \rceil \frac{\rho(r)}{2}} b^{\lfloor \frac{r}{2} \rfloor \frac{\rho(r)}{2}} c^{\lceil \frac{r}{2} \rceil \frac{\rho(r)}{2}} d^{\lfloor \frac{r}{2} \rfloor \frac{\rho(r)}{2}}}$$

we get that $S = \Phi_{i,k;R,\rho}(a,b,c,d)T^{-1}$ and the result follows.

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