A COMBINATORIAL PROOF OF THE ROGERS-RAMANUJAN AND SCHUR IDENTITIES

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Abstract. We give a combinatorial proof of the first Rogers-Ramanujan identity by using two symmetries of a new generalization of Dyson's rank. These symmetries are established by direct bijections.

Introduction

The Roger-Ramanujan identities are perhaps the most mysterious and celebrated results in partition theory. They have a remarkable tenacity to appear in areas as distinct as enumerative combinatorics, number theory, representation theory, group theory, statistical physics, probability and complex analysis [4, 6]. The identities were discovered independently by Rogers, Schur, and Ramanujan (in this order), but were named and publicized by Hardy [20]. Since then, the identities have been greatly romanticized and have achieved nearly royal status in the field. By now there are dozens of proofs known, of various degree of difficulty and depth. Still, it seems that Hardy's famous comment remains valid: *"None of the proofs of* [the Rogers-Ramanujan identities] *can be called "simple" and "straightforward"* [...]; *and no doubt it would be unreasonable to expect a really easy proof"* [20].

In this paper we propose a new combinatorial proof of the first Rogers-Ramanujan identity with a minimum amount of algebraic manipulation. Almost completely bijective, our proof would not satisfy Hardy as it is neither "simple" nor "straightforward". On the other hand, the heart of the proof is the analysis of two bijections and their properties, each of them elementary and approachable. In fact, our proof gives new generating function formulas (see (✠) in Section 1) and is amenable to advanced generalizations which will appear elsewhere (see [8]).

We should mention that on the one hand, our proof is heavily influenced by the works of Bressoud and Zeilberger [10, 11, 12, 13], and on the other hand by Dyson's papers [14, 15], which were further extended by Berkovich and Garvan [7] (see also [19, 21]). In fact, the basic idea to use a generalization of Dyson's rank was explicit in [7, 19]. We postpone historical and other comments until Section 3.

Let us say a few words about the structure of the paper. We split the proof of the first Rogers-Ramanujan identity into two virtually independent parts. In the first, *the algebraic part*, we use the Jacobi triple product identity to derive the identity from two *symmetry equations*. The latter are proved in *the combinatorial part* by direct bijections. Our presentation is elementary and completely self-contained, except for the use of the classical Jacobi triple product identity. We conclude with the final remarks section.

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1. The algebraic part

We consider the first *Rogers-Ramanujan identity*:

$$(\blacklozenge) \qquad 1+\sum_{k=1}^{\infty}\frac{t^{k^2}}{(1-t)(1-t^2)\cdots(1-t^k)} \;=\; \prod_{i=0}^{\infty}\frac{1}{(1-t^{5i+1})(1-t^{5i+4})}\,.$$

Our first step is standard. Recall the *Jacobi triple product identity* (see e.g. [4]):

$$\sum_{k=-\infty}^{\infty} z^k q^{\frac{k(k+1)}{2}} \;=\; \prod_{i=1}^{\infty}(1+zq^i)\,\prod_{j=0}^{\infty}(1+z^{-1}q^j)\,\prod_{i=1}^{\infty}(1-q^i).$$

Set $q \leftarrow t^5$, $z \leftarrow (-t^{-2})$ and rewrite the right hand side of $(\blacklozenge)$ as follows:

$$\prod_{r=0}^{\infty}\frac{1}{(1-t^{5r+1})(1-t^{5r+4})} \;=\; \sum_{m=-\infty}^{\infty}(-1)^m\, t^{\frac{m(5m-1)}{2}}\;\prod_{i=1}^{\infty}\frac{1}{(1-t^i)}\,.$$

This gives us *Schur's identity*, which is equivalent to $(\blacklozenge)$:

$$(\lozenge) \qquad \left(1+\sum_{k=1}^{\infty}\frac{t^{k^2}}{(1-t)(1-t^2)\cdots(1-t^k)}\right) \;=\; \prod_{i=1}^{\infty}\frac{1}{(1-t^i)}\;\sum_{m=-\infty}^{\infty}(-1)^m\, t^{\frac{m(5m-1)}{2}}\,.$$

To prove Schur's identity we need several combinatorial definitions. Denote by $\mathcal{P}_n$ the set of all partitions λ of n, and let $\mathcal{P} = \cup_n \mathcal{P}_n$, $p(n) = |\mathcal{P}_n|$. Denote by $\ell(\lambda)$ and $e(\lambda)$ the number of parts and the smallest part of the partition, respectively. By definition, $e(\lambda) = \lambda_{\ell(\lambda)}$. We say that λ is a *Rogers-Ramanujan partition* if $e(\lambda) \geq \ell(\lambda)$. Denote by $\mathcal{Q}_n$ the set of Rogers-Ramanujan partitions, and let $\mathcal{Q} = \cup_n \mathcal{Q}_n$, $q(n) = |\mathcal{Q}_n|$. Recall that

$$P(t) \;:=\; 1 \,+\, \sum_{n=1}^{\infty} p(n)\, t^n \;=\; \prod_{i=1}^{n}\frac{1}{1-t^i}\,,$$

and

$$Q(t) \;:=\; 1 \,+\, \sum_{n=1}^{\infty} q(n)\, t^n \;=\; 1 \,+\, \sum_{k=1}^{\infty}\frac{t^{k^2}}{(1-t)(1-t^2)\cdots(1-t^k)}\,.$$

We consider a statistic on $\mathcal{P} \smallsetminus \mathcal{Q}$, the set of non-Rogers-Ramanujan partitions, which we call the $(2,0)$-*rank* of a partition, and denote by $r_{2,0}(\lambda)$, for $\lambda \in \mathcal{P} \smallsetminus \mathcal{Q}$. Similarly, for $m \geq 1$ we consider a statistic on $\mathcal{P}$ which we call the $(2,m)$-*rank* of a partition, and denote by $r_{2,m}(\lambda)$, for $\lambda \in \mathcal{P}$. We formally define and study these statistics in the next section. Denote by $h(n,m,r)$ the number of partitions λ of n with $r_{2,m}(\lambda) = r$. Similarly, let $h(n,m,\leq r)$ and $h(n,m,\geq r)$ be the number of partitions with the $(2,m)$-rank $\leq r$ and $\geq r$, respectively. The following is apparent from the definitions:

$$(\circledast) \qquad \begin{aligned} h(n,m,\leq r) \,+\, h(n,m,\geq r+1) \;&=\; p(n), \ \text{ for } \ m>0, \ \text{ and} \\ h(n,0,\leq r) \,+\, h(n,0,\geq r+1) \;&=\; p(n)-q(n), \end{aligned}$$

for all $r \in \mathbb{Z}$ and $n \geq 1$. The following two equations are the main ingredients of the proof. We have:

(first symmetry) $\qquad h(n,0,r) \;=\; h(n,0,-r)$, and

(second symmetry) $\qquad h(n,m,\leq -r) \;=\; h(n-r-2m-2,\,m+2,\,\geq -r)$.

The first symmetry holds for any r and the second symmetry holds for $m, r > 0$ and for $m = 0$ and $r \geq 0$.

Both symmetry equations will be proved in the next section. For now, let us continue to prove Schur's identity. For every $j \geq 0$ let

$$\begin{aligned} a_j &= h\left(n - jr - 2jm - j(5j-1)/2, m+2j, \leq -r-j\right), \quad \text{and} \\ b_j &= h\left(n - jr - 2jm - j(5j-1)/2, m+2j, \geq -r-j+1\right). \end{aligned}$$

The equation ($\circledast$) gives us $a_j + b_j = p(n - jr - 2jm - j(5j-1)/2)$, for all $r, j > 0$. The second symmetry equation gives us $a_j = b_{j+1}$. Applying these multiple times we get:

$$\begin{aligned} h(n, m, \leq -r) &= a_0 = b_1 \\ &= b_1 + (a_1 - b_2) - (a_2 - b_3) + (a_3 - b_4) - \ldots \\ &= (b_1 + a_1) - (b_2 + a_2) + (b_3 + a_3) - (b_4 + a_4) + \ldots \\ &= p(n - r - 2m - 2) - p(n - 2r - 4m - 9) + p(n - 3r - 6m - 21) - \ldots \\ &= \sum_{j=1}^{\infty} (-1)^{j-1} p(n - jr - 2jm - j(5j-1)/2). \end{aligned}$$

In terms of the generating functions

$$H_{m,\leq -r}(t) := \sum_{n=1}^{\infty} h(n, m, \leq -r)\, t^n,$$

this gives (for $m, r > 0$ and for $m = 0$ and $r \geq 0$)

$$(\maltese) \qquad H_{m,\leq -r}(t) = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)} \sum_{j=1}^{\infty} (-1)^{j-1}\, t^{jr+2jm+j(5j-1)/2}.$$

In particular, we have:

$$\begin{aligned} H_{0,\leq 0}(t) &= \prod_{n=1}^{\infty} \frac{1}{(1-t^n)} \sum_{j=1}^{\infty} (-1)^{j-1}\, t^{\frac{j(5j-1)}{2}}, \\ H_{0,\leq -1}(t) &= \prod_{n=1}^{\infty} \frac{1}{(1-t^n)} \sum_{j=1}^{\infty} (-1)^{j-1}\, t^{\frac{j(5j+1)}{2}}. \end{aligned}$$

From the first symmetry equation and ($\circledast$) we have:

$$H_{0,\leq 0}(t) + H_{0,\leq -1}(t) = H_{0,\leq 0}(t) + H_{0,\geq 1}(t) = P(t) - Q(t).$$

We conclude:

$$\begin{aligned} &\prod_{n=1}^{\infty} \frac{1}{(1-t^n)} \left(\sum_{j=1}^{\infty} (-1)^{j-1}\, t^{\frac{j(5j-1)}{2}} + \sum_{j=1}^{\infty} (-1)^{j-1}\, t^{\frac{j(5j+1)}{2}} \right) \\ &\quad = \prod_{n=1}^{\infty} \frac{1}{(1-t^n)} - \left(1 + \sum_{k=1}^{\infty} \frac{t^{k^2}}{(1-t)(1-t^2)\ldots(1-t^k)} \right), \end{aligned}$$

which implies ($\diamondsuit$) and completes the proof of ($\blacklozenge$).

2. The combinatorial part

2.1. **Definitions.** Let $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$, $\lambda_1 \geq \ldots \geq \lambda_{\ell(\lambda)} > 0$, be an integer partition of $n = \lambda_1 + \ldots + \lambda_{\ell(\lambda)}$. We will say that $\lambda_j = 0$ for $j > \ell(\lambda)$. We graphically represent the partition λ by a Young diagram $[\lambda]$ as in Figure 1. Denote by λ' the *conjugate partition* of λ obtained by reflection upon main diagonal (see Figure 1).

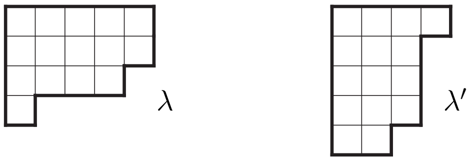

Figure 1. Partition $\lambda = (5, 5, 4, 1)$ and conjugate partition $\lambda' = (4, 3, 3, 3, 2)$.

For $m \geq 0$, define an *m-rectangle* to be a rectangle whose height minus its width is m. Define the *first m-Durfee rectangle* to be the largest m-rectangle which fits in diagram $[\lambda]$. Denote by $s_m(\lambda)$ the height of the first m-Durfee rectangle. Define the *second m-Durfee rectangle* to be the largest m-rectangle which fits in diagram $[\lambda]$ below the first m-Durfee rectangle, and let $t_m(\lambda)$ be its height. We will allow an m-Durfee rectangle to have width 0 but never height 0. Finally, denote by α, β, and γ the three partitions to the right of, in the middle of and below the m-Durfee rectangles (see Figures 2 and 3). Notice that if $m > 0$ and we have an m-Durfee rectangle of width 0, as in Figure 3, then γ must be the empty partition.

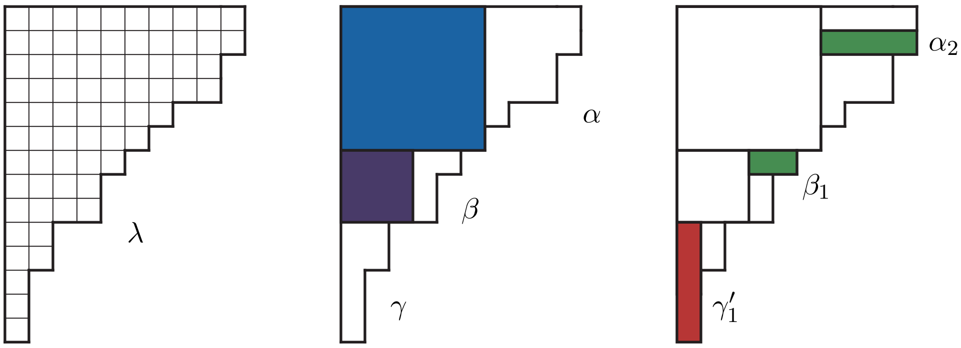

Figure 2. Partition $\lambda = (10, 10, 9, 9, 7, 6, 5, 4, 4, 2, 2, 1, 1, 1)$, the first Durfee square of height $s_0(\lambda) = 6$, and the second Durfee square of height $t_0(\lambda) = 3$. Here the remaining partitions are $\alpha = (4, 4, 3, 3, 1)$, $\beta = (2, 1, 1)$, and $\gamma = (2, 2, 1, 1, 1)$. In this case, the $(2, 0)$-rank is $r_{2,0}(\lambda) = \beta_1 + \alpha_2 - \gamma'_1 = 2 + 4 - 5 = 1$.

We define *$(2, m)$-rank*, $r_{2,m}(\lambda)$, of a partition λ by the formula:

$$r_{2,m}(\lambda) := \beta_1 + \alpha_{s_m(\lambda) - t_m(\lambda) - \beta_1 + 1} - \gamma'_1 .$$

Note that $(2, 0)$-rank is only defined for non-Rogers-Ramanujan partitions because otherwise β_1 does not exist, while $(2, m)$-rank is defined for all partitions for all $m > 0$. Again, see Figures 2 and 3 for examples.

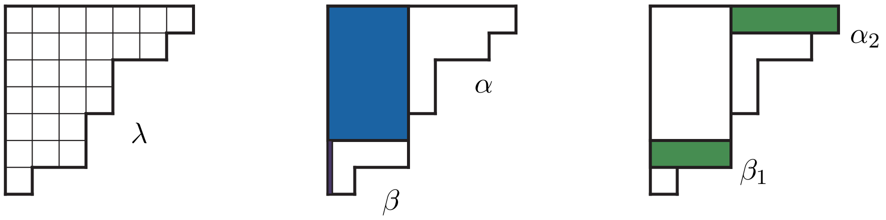

FIGURE 3. Partition $\lambda = (7, 6, 4, 4, 3, 3, 1)$, the first 2-Durfee rectangle of height $s_2(\lambda) = 5$ and width 3, and the second 2-Durfee square of height $t_2(\lambda) = 2$ and width 0. Here the remaining partitions are $\alpha = (4, 3, 1, 1)$, $\beta = (3, 1)$, and γ which is empty. In this case, we have $(2, 2)$-rank $r_{2,2}(\lambda) = \beta_1 + \alpha_1 - \gamma'_1 = 3 + 4 - 0 = 7$.

Let $\mathcal{H}_{n,m,r}$ be the set of partitions of n with $(2, m)$-rank r. In the notation above, $h(n, m, r) = \left|\mathcal{H}_{n,m,r}\right|$. Define $\mathcal{H}_{n,m,\leq r}$ and $\mathcal{H}_{n,m,\geq r}$ similarly.

2.2. **Proof of the first symmetry.** In order to prove the first symmetry we present an involution φ on $\mathcal{P} \smallsetminus \mathcal{Q}$ which preserves the size of partitions as well as their Durfee squares, but changes the sign of the rank:

$$\varphi : \mathcal{H}_{n,0,r} \to \mathcal{H}_{n,0,-r} \, .$$

Let λ be a partition with two Durfee square and partitions α, β, and γ to the right of, in the middle of, and below the Durfee squares. This map φ will preserve the Durfee squares of λ whose sizes we denote by

$$s = s_0(\lambda) \qquad \text{and} \qquad t = t_0(\lambda) \, .$$

We will describe the action of $\varphi : \lambda \mapsto \widehat{\lambda}$ by first mapping (α, β, γ) to a 5-tuple of partitions $(\mu, \nu, \pi, \rho, \sigma)$, and subsequently mapping that 5-tuple to different triple $(\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma})$ which goes to the right of, in the middle of, and below the Durfee squares in $\widehat{\lambda}$.

(1) First, let $\mu = \beta$.
Second, remove the following parts from α: $\alpha_{s-t-\beta_j+j}$ for $1 \leq j \leq t$. Let ν be the partition comprising of parts removed from α and π be the partitions comprising of the parts which were not removed.
Third, for $1 \leq j \leq t$, let

$$k_j = \max\{k \leq s - t \mid \gamma'_j - k \geq \pi_{s-t-k+1}\} \, .$$

Let ρ be the partition with parts $\rho_j = k_j$ and σ be the partition with parts $\sigma_j = \gamma'_j - k_j$.

(2) First, let $\widehat{\gamma}' = \nu + \mu$ be the *sum* of partitions, defined to have parts $\widehat{\gamma}'_j = \nu_j + \mu_j$.
Second, let $\widehat{\alpha} = \sigma \cup \pi$ be the *union* of partitions, defined as a union of parts in σ and π.[1]
Third, let $\widehat{\beta} = \rho$.

Figure 4 shows an example of φ and the relation between these two steps.

[1]Alternatively, the union can be defined via the sum: $\sigma \cup \pi = (\sigma' + \pi')'$.

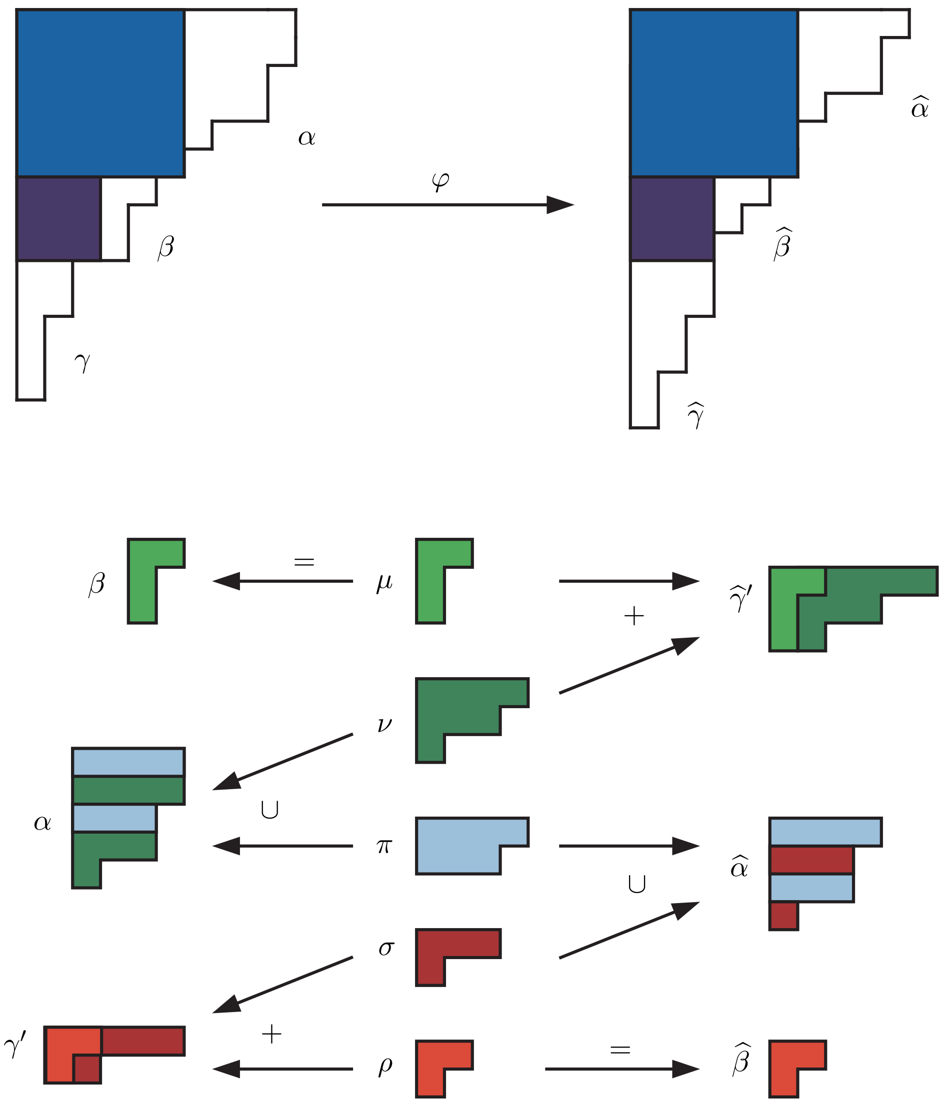

FIGURE 4. An example of the first symmetry involution $\varphi : \lambda \mapsto \widehat{\lambda}$, where $\lambda \in \mathcal{H}_{n,0,r}$ and $\widehat{\lambda} \in \mathcal{H}_{n,0,-r}$ for $n = 71$, and $r = 1$. The maps are defined by the following rules: $\beta = \mu$, $\alpha = \nu \cup \pi$, $\gamma' = \sigma + \rho$, while $\widehat{\beta} = \rho$, $\widehat{\alpha} = \pi \cup \sigma$, $\widehat{\gamma}' = \mu + \nu$. Also, $\lambda = (10, 10, 9, 9, 7, 6, 5, 4, 4, 2, 2, 1, 1, 1)$ and $\widehat{\lambda} = (10, 9, 9, 7, 6, 6, 5, 4, 3, 3, 3, 2, 2, 2, 1, 1)$.

Remark 2.1. The key to understanding the map φ is the definition of k_j. By considering $k = 0$, we see that k_j is defined for all $1 \le j \le t$. Moreover, one can check that k_j is the *unique* integer k which satisfies

$$(\dagger) \qquad \pi_{s-t-k+1} \le \gamma'_j - k \le \pi_{s-t-k}\,.$$

(We do not consider the upper bound for $k = s - t$.) This characterization of k_j can also be taken as its definition. Equation ($\dagger$) is used repeatedly in our proof of the next lemma.

Lemma 2.2. *The map φ defined above is an involution.*

Proof. Our proof is divided into five parts; we prove that

(1) ρ is a partition, **(2)** σ is a partition, **(3)** $\widehat{\lambda} = \varphi(\lambda)$ is a partition,

(4) φ^2 is the identity map, and **(5)** $r_{2,0}(\widehat{\lambda}) = -r_{2,0}(\lambda)$.

(1) Considering the bounds (†) for j and $j+1$, we note that, if $k_j \le k_{j+1}$, then

$$\pi_{s-t-k_j+1} + k_j \le \pi_{s-t-k_{j+1}+1} + k_{j+1} \le \gamma'_{j+1} \le \gamma'_j \le \pi_{s-t-k_j} + k_j .$$

This gives us

$$\pi_{s-t-k_j+1} \le \gamma'_{j+1} - k_j \le \pi_{s-t-k_j}$$

and uniqueness therefore implies that $k_j = k_{j+1}$. We conclude that $k_j \ge k_{j+1}$ and that ρ is a partition.

(2) If $k_j > k_{j+1}$, then we have $s-t-k_j+1 \le s-t-k_{j+1}$ and therefore

$$\pi_{s-t-k_{j+1}} \le \pi_{s-t-k_j+1} .$$

Again, by considering (†) for j and $j+1$, we conclude that

$$\gamma'_j - k_j \ge \gamma'_{j+1} - k_{j+1} .$$

If $k_j = k_{j+1}$, then we simply need to recall that γ' is a partition to see that

$$\gamma'_j - k_j \ge \gamma'_{j+1} - k_{j+1} .$$

This implies that σ is a partition.

(3) By their definitions, it is clear that μ, ν, and π are partitions. Since we just showed that ρ and σ are all partition, it follows that $\widehat{\alpha}$, $\widehat{\beta}$, and $\widehat{\gamma}$ are also partitions. Moreover, by their definitions, we see that μ, ν, and σ have at most t parts, π has at most $s-t$, and ρ has at most t parts each of which is less than or equal to $s-t$. This implies that $\widehat{\alpha}$ has at most s parts, $\widehat{\beta}$ has at most t parts each of which is less than or equal to $s-t$, and $\widehat{\gamma}'$ has parts at most t. Therefore, $\widehat{\alpha}$, $\widehat{\beta}$, and $\widehat{\gamma}$ fit to the right of, in the middle of, and below Durfee squares of sizes s and t and so $\varphi(\lambda)$ is a partition.

(4) We will apply φ twice to a non-Rogers-Ramanujan partition λ with α, β, and γ to the right of, in the middle of, and below its two Durfee squares. As usual, let μ, ν, π, ρ, σ be the partitions occurring in the intermediate stage of the first application of φ to λ and let $\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\gamma}$ be the partitions to the right of, in the middle of, and below the Durfee squares of $\widehat{\lambda} = \varphi(\lambda)$. Similarly, let $\widehat{\mu}$, $\widehat{\nu}$, $\widehat{\pi}$, $\widehat{\rho}$, $\widehat{\sigma}$ be the partitions occurring in the intermediate stage of the second application of φ and let α^*, β^*, and γ^* be the partitions to the right of, in the middle of and below the Durfee squares of $\varphi^2(\lambda) = \varphi(\widehat{\lambda})$.

We need several observations. First, note that $\widehat{\mu} = \widehat{\beta} = \rho$. Second, by (†) we have:

$$\pi_{s-t-k_j+1} \le \gamma'_j - k_j = \sigma_j \le \pi_{s-t-k_j} .$$

Since σ is a partition, this implies that $\widehat{\alpha}_{s-t-k_j+j} = \sigma_j$. On the other hand, since $\widehat{\beta}_j = \rho_j = k_j$, the map φ removes the rows $\widehat{\alpha}_{s-t-k_j+j} = \sigma_j$ from $\widehat{\alpha}$. From here we conclude that $\widehat{\nu} = \sigma$ and $\widehat{\pi} = \pi$. Third, define

$$\widehat{k}_j = \max\{\widehat{k} \le s-t \mid \gamma'_j - \widehat{k} \ge \pi_{s-t-\widehat{k}+1}\} .$$

By Remark 2.1, we know that $\widehat{k}_j$ as above is the unique integer $\widehat{k}$ which satisfies:

$$\widehat{\pi}_{s-t-\widehat{k}+1} \le \widehat{\gamma}'_j - \widehat{k} \le \widehat{\pi}_{s-t-\widehat{k}} .$$

On the other hand, recall that $\widehat{\gamma}'_j = \mu_j + \nu_j$ and $\beta_j = \mu_j$. This implies $\widehat{\gamma}'_j - \beta_j = \nu_j$. Also, by the definition of ν, we have $\nu_j = \alpha_{s-t-\beta_j+j}$. Therefore, by the definition of π, we have:

$$\pi_{s-t-\beta_j+1} \leq \alpha_{s-t-\beta_j+j} = \nu_j = \widehat{\gamma}'_j - \beta_j \leq \pi_{s-t-\beta_j}\,.$$

Since, $\widehat{\pi} = \pi$, by the uniqueness in Remark 2.1 we have $\widehat{k}_j = \beta_j = \mu_j$. This implies that $\widehat{\rho} = \mu$ and $\widehat{\sigma} = \nu$.

Finally, the second step of our bijection gives $\alpha^* = \nu \cup \pi = \alpha$, $\beta^* = \mu = \beta$, and $(\gamma^*)' = \rho + \sigma = \gamma'$. This implies that φ^2 is the identity map.

(5) Using the results from **(4)**, we have:

$$r_{2,0}(\lambda) = \beta_1 + \alpha_{s-t-\beta_1+1} - \gamma'_1 = \mu_1 + \nu_1 - \rho_1 - \sigma_1\,.$$

On the other hand,

$$r_{2,0}(\widehat{\lambda}) = \widehat{\beta}_1 + \widehat{\alpha}_{s-t-\widehat{\beta}_1+1} - \widehat{\gamma}'_1 = \rho_1 + \sigma_1 - \mu_1 - \nu_1\,.$$

We conclude that $r_{2,0}(\widehat{\lambda}) = -r_{2,0}(\lambda)$. □

2.3. **Proof of the second symmetry.** In order to prove the second symmetry we present a bijection

$$\psi_{m,r} : \mathcal{H}_{n,m,\leq -r} \to \mathcal{H}_{n-r-2m-2,m+2,\geq -r}\,.$$

This map will only be defined for $m, r > 0$ and for $m = 0$ and $r \geq 0$ and in both of these cases the first and second m-Durfee rectangles of a partition $\lambda \in \mathcal{H}_{n,m,\leq -r}$ have non-zero width. For $m = 0$, $(2, 0)$-rank is only defined for partitions in $\mathcal{P} \smallsetminus \mathcal{Q}$ which by definition have two Durfee squares of non-zero width. For $m > 0$, since we also have $r > 0$, a partition $\lambda \in \mathcal{H}_{n,m,\leq -r}$ must have

$$r_{2,m}(\lambda) = \beta_1 + \alpha_{s_m(\lambda)-t_m(\lambda)-\beta_1+1} - \gamma'_1 \leq -r < 0\,.$$

This forces $\gamma'_1 > 0$ and so both m-Durfee rectangles must have non-zero width.

We describe the action of $\psi := \psi_{m,r}$ by giving the sizes of the Durfee rectangles of $\widehat{\lambda} := \psi_{m,r}(\lambda) = \psi(\lambda)$ and the partitions $\widehat{\alpha}$, $\widehat{\beta}$, and $\widehat{\gamma}$ which go to the right of, in the middle of, and below those Durfee rectangles in $\widehat{\lambda}$.

(1) If λ has two m-Durfee rectangles of height

$$s := s_m(\lambda) \qquad \text{and} \qquad t := t_m(\lambda)$$

then $\widehat{\lambda}$ has two $(m+2)$-Durfee rectangles of height

$$s' := s_{m+2}(\widehat{\lambda}) = s + 1 \qquad \text{and} \qquad t' := t_{m+2}(\widehat{\lambda}) = t + 1\,.$$

(2) Let

$$k_1 = \max\{k \leq s - t \mid \gamma'_1 - r - k \geq \alpha_{s-t-k+1}\}\,.$$

Obtain $\widehat{\alpha}$ from α by adding a new part of size $\gamma'_1 - r - k_1$, $\widehat{\beta}$ from β by adding a new part of size k_1, and $\widehat{\gamma}$ from γ by removing its first column.

Figure 5 shows an example of the bijection $\psi = \psi_{m,r}$.

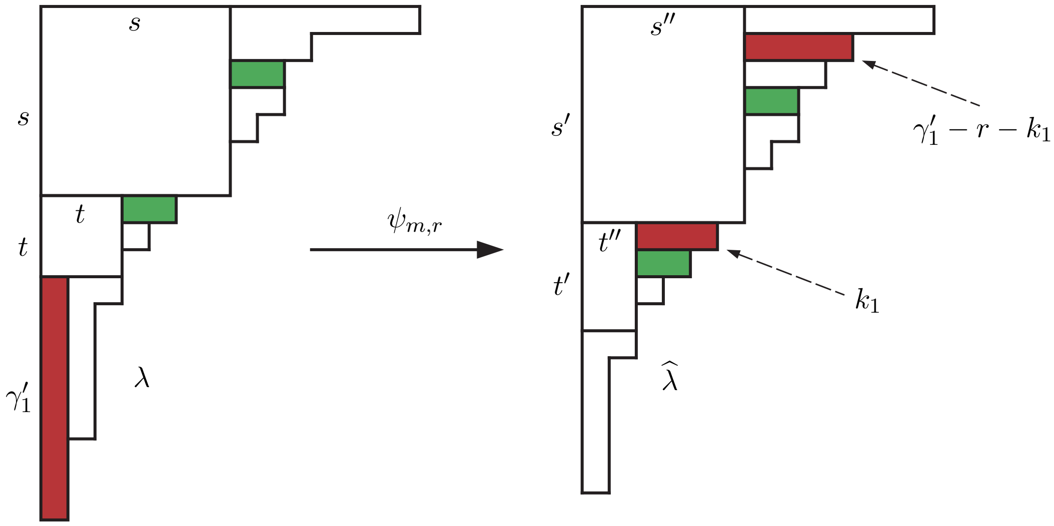

FIGURE 5. An example of the second symmetry bijection $\psi_{m,r} : \lambda \mapsto \widehat{\lambda}$, where $\lambda \in \mathcal{H}_{n,m,\leq -r}$, $\widehat{\lambda} \in \mathcal{H}_{n',m+2,\geq -r}$, for $m = 0$, $r = 2$, $n = 92$, and $n' = n-r-2m-2 = 88$. Here $r_{2,0}(\lambda) = 2+2-9 = -5 \leq -2$ and $r_{2,2}(\widehat{\lambda}) = 3+4-6 = 1 \geq -2$, where $\lambda = (14, 10, 9, 9, 8, 7, 7, 5, 4, 3, 3, 2, 2, 2, 2, 2, 2, 1, 1, 1)$ and $\widehat{\lambda} = (13, 10, 9, 8, 8, 7, 6, 6, 5, 4, 3, 2, 2, 1, 1, 1, 1, 1)$. Also, $s = 7$, $s' = s+1 = 8$, $s'' = s' - m - 2 = 6$, $t = 3$, $t' = 4$, $t'' = 2$, $\gamma_1' = 9$, $k_1 = 3$, and $\gamma_1' - r - k_1 = 4$.

Remark 2.3. As in Remark 2.1, by considering $k = \beta_1$ we see that k_1 is defined and indeed we have $k_1 \geq \beta_1$. Moreover, it follows from its definition that k_1 is the *unique* k such that

$$(\ddagger) \qquad \alpha_{s-t-k+1} \leq \gamma_1' - r - k \leq \alpha_{s-t-k}\,.$$

(If $k = s - t$ we do not consider the upper bound.)

Lemma 2.4. *The map $\psi = \psi_{m,r}$ defined above is a bijection.*

Proof. Our proof has four parts:

(1) we prove that $\widehat{\lambda} = \psi(\lambda)$ is a partition,
(2) we prove that the size of $\widehat{\lambda}$ is $n - r - 2m - 2$,
(3) we prove that $r_{2,m+2}(\widehat{\lambda}) \geq -r$, and
(4) we present the inverse map ψ^{-1}.

(1) To see that $\widehat{\lambda}$ is a partition we simply have to note that since λ has m-Durfee rectangles of non-zero width, $\widehat{\lambda}$ may have $(m+2)$-Durfee rectangles of width $s-1$ and $t-1$. Also, the partitions $\widehat{\alpha}$ and $\widehat{\beta}$ have at most $s+1$ and $t+1$ parts, respectively, while the partitions $\widehat{\beta}$ and $\widehat{\gamma}$ have parts of size at most $s-t$ and $t-1$, respectively. This means that they can sit to the right of, in the middle of, and below the two $(m+2)$-Durfee rectangles of $\widehat{\lambda}$.

(2) To prove that the above construction gives a partition $\widehat{\lambda}$ of $n-r-2m-2$, note that the sum of the sizes of the rows added to α and β is r less than the size of the column removed from γ, and that both the first and second $(m+2)$-Durfee rectangles of $\widehat{\lambda}$ have size $m+1$ less than the size of the corresponding m-Durfee rectangle of λ.

(3) By Remark 2.3, the part we inserted into β will be the largest part of the resulting partition, i.e. $\widehat{\beta}_1 = k_1$. By equation (‡) we have:

$$\alpha_{s-t-k_1+1} \leq \gamma'_1 - r - k_1 \leq \alpha_{s-t-k_1} \,.$$

Therefore, we must have:

$$\widehat{\alpha}_{s'-t'-\widehat{\beta}_1+1} = \widehat{\alpha}_{s-t-k_1+1} = \gamma'_1 - r - k_1 \,.$$

Indeed, we have chosen k_1 in the *unique* way so that the rows we insert into α and β are $\widehat{\alpha}_{s'-t'-\widehat{\beta}_1+1}$ and $\widehat{\beta}_1$ respectively.

Having determined $\widehat{\alpha}_{s'-t'-\widehat{\beta}_1+1}$ and $\widehat{\beta}_1$ allows us to bound the $(2, m+2)$-rank of $\widehat{\lambda}$:

$$r_{2,m+2}(\widehat{\lambda}) = \widehat{\alpha}_{s'-t'-\widehat{\beta}_1+1} + \widehat{\beta}_1 - \widehat{\gamma}'_1 \;=\; \gamma'_1 - r - k_1 + k_1 - \widehat{\gamma}'_1 \;\geq\; -r \,,$$

where the last inequality follows since $\widehat{\gamma}'_1$ is the size of the second column of γ whereas γ'_1 is the size of the first column of γ.

(4) The above characterization of k_1 also shows us that to recover α, β, and γ from $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\gamma}$, we remove part $\widehat{\alpha}_{s'-t'-\widehat{\beta}_1+1}$ from $\widehat{\alpha}$, remove part $\widehat{\beta}_1$ from $\widehat{\beta}$, and add a column of height $\widehat{\alpha}_{s'-t'-\widehat{\beta}_1+1} + \widehat{\beta}_1 + r$ to $\widehat{\gamma}$. Since we can also easily recover the sizes of the previous m-Durfee rectangles, we conclude that ψ is a bijection between the desired sets. □

3. Final remarks

3.1. Of the many proofs of Rogers-Ramanujan identities only a few can be honestly called "combinatorial". We would like to single out [3] as an interesting example. Perhaps, the most important combinatorial proof was given by Schur in [24] where he deduced his identity by a direct involutive argument. The celebrated bijection of Garsia and Milne [18] is based on this proof and the involution principle. In [11], a different involution principle proof was obtained (see also [13]) based on a short proof of Bressoud [10]. We refer to [22] for further references, historical information, and combinatorial proofs of other partition identities.

3.2. Dyson's rank $r_1(\lambda) = \lambda_1 - \lambda'_1$ was defined in [14] for the purposes of finding a combinatorial interpretation of Ramanujan's congruences. Dyson used the rank to obtain a simple combinatorial proof of Euler's pentagonal theorem in [15] (see also [16, 21]). It was shown in [21] that this proof can be converted into a direct involutive proof, and such a proof in fact coincides with the involution obtained by Bressoud and Zeilberger [12].

Roughly speaking, our proof of Schur's identity is a Dyson-style proof with a modified Dyson's rank, where the definition of the latter was inspired by [11, 12, 13]. Unfortunately, reverse engineering the proofs in [13] is not straightforward due to the complexity of that paper. Therefore, rather than giving a formal connection, we will only say that, for some m and r, our map $\psi_{m,r}$ is similar to the maps φ in [11] and Φ in [13].

It would be interesting to extend our Dyson-style proof to the generalization of Schur's identity found in [17]. This would give a new combinatorial proof of the generalizations of the Rogers-Ramanujan identities found in that paper and, in a special case, provide a new combinatorial proof of the second Rogers-Ramanujan identity (see e.g. [4, 6, 20, 22]).

3.3. The idea of using iterated Durfee squares to study the Rogers-Ramanujan identities and their generalizations is due to Andrews [5]. The $(2, m)$-rank of a partition is a special case of a general (but more involved) notion of (k, m)-rank which is presented in [8]. It leads to combinatorial proofs of some of Andrews' generalizations of Rogers-Ramanujan identities mentioned above.

Garvan [19] defined a generalized notion of a rank to partitions with iterated Durfee squares, that is different from ours, but still satisfies equation (✠) (for $m = 0$). In [7], Berkovich and Garvan asked for a Dyson-style proof of (✠) but unfortunately, they were unable to carry out their program in full as the combinatorial symmetry they obtain seem to be hard to establish bijectively. (This symmetry is somewhat different from our second symmetry.) The first author was able to relate the two generalizations of rank by a bijective argument. This also appears in [8].

3.4. Yet another generalization of Dyson's rank was kindly brought to our attention by George Andrews. The notion of successive rank can also be used to give a combinatorial proof of the Rogers-Ramanujan identities and their generalizations by a sieve argument (see [2, 9]). However, this proof involves a different combinatorial description of the partitions on the left hand side of the Rogers-Ramanujan identities than the proof presented here.

3.5. Finally, let us note that the Jacobi triple product identity has a combinatorial proof due to Sylvester (see [22, 25]). We refer to [1] for an elementary algebraic proof.

Also, while our proof is mostly combinatorial it is by no means a direct bijection. The quest for a direct bijective proof is still under way, and as recently as this year Zeilberger lamented on the lack of such proof [26]. The results in [23] seem to discourage any future work in this direction.

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