Twisted Poisson manifolds and their almost symplectically complete isotropic realizations

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Integrable systems

Definition

A completely integrable Hamiltonian system is a symplectic manifold (M^{2n}, ω) with n independent smooth functions in involution, i.e. $f_1, \cdots, f_n$ such that $\{f_i, f_j\} := \omega(X_{f_i}, X_{f_j}) = 0$ for $1 \leq i, j \leq n$.

We assume that $f_1^{-1}(c_1) \cap \cdots \cap f_n^{-1}(c_n)$ are compact and connected for any $c_1, \cdots, c_n \in \mathbb{R}$. The fibers are then tori, called the *invariant tori*.

Example (Harmonic oscillator)

$M = T^*\mathbb{R}$, $\omega = dp \wedge dq$, $f =$ energy $= \frac{1}{2}(p^2 + q^2)$.

Example (Pendulum moving under gravity)

$M = T^*S^2 \subset T^*\mathbb{R}^3$, $\omega = \iota_{S^2}^* \sum_{i=1}^3 dp_i \wedge dq_i$.
$f_1 =$ vertical angular momentum $= q_1 p_2 - q_2 p_1$.
$f_2 =$ energy $= \frac{1}{2}\|p\|^2 + q_3$.

Integrable systems

Theorem (Liouville-Arnold Theorem)

A completely integrable Hamiltonian system admits local action-angle coordinates $(f_1, \cdots, f_n, \theta_1, \cdots, \theta_n)$ *such that, locally, the symplectic form can be expressed as*

$$\omega = \sum_{i=1}^{n} df_i \wedge d\theta_i$$

Example (Harmonic oscillator)

Let $\theta = \tan^{-1} \frac{q}{p}$. Then $\omega = dp \wedge dq = df \wedge d\theta$.

Example (Pendulum)

The angle coordinates are the spherical coordinates (φ, θ), corresponding to the total energy and vertical angular momentum respectively.

More general contexts, and some problems

One can generalize the notion of completely integrable Hamiltonian systems in two contexts.

1) (Super)integrable systems, i.e. the number of independent smooth functions in pairwise involution is less than half of the dimension of the system.
2) The system is almost symplectic, i.e. is equipped with a non-degenerate 2-form which may not be closed.

Problems of interest:

A. Find obstructions of existence of *global* action and angle coordinates, and
B. Classify those systems.

More general contexts, and some problems

Definition

We call $\pi : (M^{2d}, \omega) \to B^k$ ($k \geq d$) a regular integrable symplectic Hamiltonian system if ω is symplectic, and

1. π is a surjective submersion with compact and connected fibers,
2. fibers of π are isotropic, i.e. ω vanishes on restriction to fibers, and
3. for any $b \in B$, there exists a π-saturated neighborhood U such that there exist linearly independent Hamiltonian vector fields $Y_1, \cdots, Y_n$ in U tangent to fibers (here $n = 2d - k$).

M is then a torus bundle over B.

Remark

In the example of a harmonic oscillator, if we delete the point $(q, p) = (0, 0)$ from $T^*\mathbb{R}$, then what remains (call it M') is a regular integrable symplectic Hamiltonian system, which is a (trivial) circle bundle over $B = \mathbb{R}^+$. M' admits global action and angle coordinates.

Some history

Duistermaat studied the completely integrable case, i.e. M is a Lagrangian torus bundle over B. He introduced two topological invariants, namely *monodromy* and *Chern class*, the former of which measures the obstruction of the existence of global action coordinate, while the vanishing of the latter with a certain condition on ω amounts to the existence of global angle coordinates.

Later, Dazord-Delzant, based on Duistermaat's work, considered the (super)integrable case with a slightly different approach.

Some history

Definition

B is a *Poisson manifold* if there is a bivector field $\Pi \in \Gamma(\bigwedge^2 TM)$ such that the Poisson bracket $\{f, g\} := \Pi(df, dg)$ satisfies the Jacobi identity. It is regular of rank r if $\Pi^\sharp : T^*M \to TM$ is of constant rank r.

Proposition

If B is a regular Poisson manifold of rank r, then the distribution $\Pi^\sharp(T^*M)$ induces a foliation where each leaf is an r-dimensional symplectic manifold.

Some history

Proposition

$\pi : (M^{2d}, \omega) \to B^k$ is a regular integrable symplectic Hamiltonian system iff

1. *π is a surjective submersion with compact and connected fibers, and*
2. *B^k is a regular Poisson manifold of rank $2(k-d)$ and π is a Poisson map, i.e. $\pi_* \Pi_M = \Pi_B$.*

This prompts the

Definition

Let (B, Π_B) be a regular Poisson manifold. A *symplectically complete isotropic realization* (SCIR) (M, ω) over B is a surjective submersion $\pi : M \to B$ whose fibers are compact and connected and $\pi_* \Pi_M = \Pi_B$.

Some history

Dazord-Delzant classified SCIRs over (B, Π) using *Lagrangian class*, an invariant finer than the Chern class and the vanishing of which is equivalent to the existence of global angle coordinates. They also give a sufficient and necessary topological condition for a torus bundle over (B, Π) to have a compatible symplectic form so as to be an SCIR.

Goal: To generalize Duistermaat/Dazord-Delzant's work to the almost symplectic case, which is motivated by the study of nonholonomic systems.

The completely integrable almost symplectic case was first addressed by Sjamaar.

Integrable almost symplectic Hamiltonian systems

Definition

A regular *integrable almost symplectic Hamiltonian system* $\pi : (M^{2d}, \omega) \to B^k$ satisfies all the conditions of an integrable symplectic Hamiltonian system except that

1. ω is only assumed to be almost symplectic, and
2. the vector fields $Y_1, \cdots, Y_n$ are *strongly Hamiltonian*, i.e. for $1 \leq i \leq n$, there exist locally defined functions f_i on B such that $Y_i = X_{f_i}$ and $\mathcal{L}_{Y_i}\omega = 0$.

Proposition (Fassò-Sansonetto)

1. *M is a torus bundle over B.*
2. *There exist local action-angle coordinates $(f_1, \cdots, f_n, \theta_1, \cdots, \theta_n)$. Together with other local coordinates $b_1, \cdots, b_{2(k-d)}$, ω can be written as $\sum_{i=1}^n df_i \wedge d\theta_i + \sum_{i,j} A_{ij} df_i \wedge df_j + \sum_{i,j} B_{ij} df_i \wedge db_j + \sum_{i,j} C_{ij} db_i \wedge db_j$. In particular, $d\omega = \pi^*\eta$ for some closed 3-form η.*
3. *B is an η-twisted regular Poisson manifold of rank $2(k-d)$, and $\pi_*\Pi_M = \Pi_B$.*

Integrable almost symplectic Hamiltonian systems

Definition

B is a *twisted Poisson manifold* if there is a bivector field $\Pi \in \Gamma(\bigwedge^2 TM)$ such that the Poisson bracket $\{f, g\} := \Pi(df, dg)$ satisfies the η-twisted Jacobi identity for some closed 3-form η.

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} + \eta(X_f, X_g, X_h) = 0,$$

where $X_f := -\Pi^\sharp(df)$. It is regular of rank r if $\Pi^\sharp : T^*M \to TM$ is of constant rank r.

Remark

1. *Similarly, a twisted Poisson manifold admits an almost symplectic foliation.*
2. *A twisted Poisson bivector field Π may correspond to two different twisting forms η_1 and η_2. $\eta_1 - \eta_2$ then vanishes on restriction to each leaf.*

Definition

Θ is a *characteristic form* of (B, Π) if its restriction to any almost symplectic leaf is the almost symplectic form induced by Π.

Almost symplectically complete isotropic realizations

Proposition

$\pi : (M^{2d}, \omega) \to B^k$ is a regular integrable almost symplectic Hamiltonian system iff

1. *π is a surjective submersion with compact and connected fibers, and*
2. *$d\omega = \pi^*\eta$, B is an η-twisted Poisson manifold, and $\pi_*\Pi_M = \Pi_B$.*

Definition

Let (B, Π) be a regular twisted Poisson manifold. An *almost symplectically complete isotropic realization* (ASCIR) (M, ω) over B is an almost symplectic manifold (M, ω) with surjective submersion $\pi : M \to B$ whose fibers are compact and connected, $\pi_*\Pi_M = \Pi_B$, and $d\omega = \pi^*\eta$ for some closed 3-form η, which is called the *twisting form* of the ASCIR.

Goal: Classify ASCIRs over (B, Π).

Period bundles and action coordinates

Definition

Let $\mathcal{F}$ be the almost symplectic foliation of (B, Π). The conormal bundle with respect to $\mathcal{F}$ is defined to be

$$\nu^*\mathcal{F} := \{(b, \alpha) \in T_b^*B | \ \alpha|_{T_b\mathcal{F}_b} = 0, b \in B\}$$

Let (M, ω) be an ASCIR over (B, Π). For any $b \in B$, a form $\alpha \in (\nu^*\mathcal{F})_b$ gives rise to a vector field $\omega^\sharp(\pi^*\alpha) \in \Gamma(TM|_{\pi^{-1}(b)})$ of the torus fiber over b. This vector field integrates to a time 1 flow (denoted by $\varphi_b(\alpha) : \pi^{-1}(b) \to \pi^{-1}(b)$). Putting these fiberwise action leads to an action of $\nu^*\mathcal{F}$ on M.

Definition

Let P be the stabilizer bundle of this action. P is called the *period bundle* of (M, ω).

Period bundles and action coordinates

P is a $\mathbb{Z}^n$-bundle. $\nu^*\mathcal{F}/P$ acts on M freely and transitively, making it a $\nu^*\mathcal{F}/P$-*torsor*.

Proposition

1. $\varphi(\alpha)^*\omega = \omega + \pi^* d\alpha$.
2. *Sections of P are closed 1-forms which vanish on restriction to the almost symplectic leaves.*

Definition

P is also called a *period bundle* of (B, Π) if it is a $\mathbb{Z}^n$-subbundle of $\nu^*\mathcal{F}$ whose sections are closed 1-forms.

So for (B, Π) to have an ASCIR, it necessarily has a period bundle. Conversely, if (B, Π) has a period bundle P, then $(\nu^*\mathcal{F}/P, \omega_{\text{can}} + \pi^*\Theta)$ is an ASCIR.

For $b \in B$, let U be its trivializing neighborhood for P. Let $\alpha_1, \cdots, \alpha_n$ be a local frame of P on U. Restricting U further if necessary, we can write $\alpha_i = df_i$ by Poincaré lemma. f_i are local action coordinates for any ASCIR over B with period bundle P.

Proposition

Global action coordinates exist iff P is trivial.

Chern classes of torsors

Refined goal: Classify ASCIRs over (B, Π) with a given period bundle P. Any such ASCIR is a $\nu^*\mathcal{F}/P$-torsor.

Definition

1. $(T, \omega_T) := (\nu^*\mathcal{F}/P, \omega_{\text{can}} + \pi^*\Theta)$.
2. $\mathcal{T}$:=sheaf of smooth sections of T.
3. For $k > 1$, $\Omega^k_{\mathcal{F}}$:=sheaf of differential k-forms on B which vanish on each almost symplectic leaf.
4. $\mathcal{P}$:=sheaf of locally constant sections of P.

Note that $H^1(B, \mathcal{T})$ classifies the isomorphism classes of T-torsors over B. The long exact sequence of cohomology associated with the short exact sequence

$$0 \longrightarrow \mathcal{P} \longrightarrow \Omega^1_{\mathcal{F}} \longrightarrow \mathcal{T} \longrightarrow 0$$

and the fact that $\Omega^1_{\mathcal{F}}$ is a fine sheaf give an isomorphism

$$\delta : H^1(B, \mathcal{T}) \to H^2(B, \mathcal{P})$$

Chern classes of torsors

Definition

Let $[M] \in H^1(B, \mathcal{T})$ be the isomorphism class of M as a T-torsor over B. The *Chern class* of M is defined to be $c(M) := \delta([M])$.

Chern class provides the complete invariant of the classification of T-torsors over B.

A preliminary classification

We shall first classify those ASCIRs over (B, Π) whose twisting form is $d\Theta$ and period bundle P.

Definition

Let (M, ω) be an ASCIR over (B, Π) with twisting form $d\Theta$.

1. $\mathcal{O}_P$:=sheaf of 'local action functions', defined by

$$\mathcal{O}_P(U) := \{f \in C^\infty(U) | df \in \mathcal{P}(U)\}$$

2. $\mathcal{A}\mathrm{ut}^M_{(B,\Pi,P,\Theta)}$:=sheaf of possible transition functions of M defined by

$$\mathcal{A}\mathrm{ut}^M_{(B,\Pi,P,\Theta)}(U) = \{f \in \mathrm{Diff}(\pi^{-1}(U)) | f \circ \pi = f, f^*\omega = \omega\}$$

$\mathcal{A}\mathrm{ut}^M_{(B,\Pi,P,\Theta)}$ is independent of M because locally, (M, ω) is symplectomorphic to $(T, \omega_{\mathrm{can}} + \pi^*\Theta)$. The ASCIRs over (B, Π) whose twisting form is $d\Theta$ and period bundle P are classified by $H^1(B, \mathcal{A}ut_{(B,\Pi,P,\Theta)})$. The short exact sequence

$$0 \longrightarrow \mathcal{O}_P \longrightarrow \Omega^0_{\mathcal{F}} \xrightarrow{\varphi \circ d} \mathcal{A}ut_{(B,\Pi,P,\Theta)} \longrightarrow 0$$

gives $H^1(B, \mathcal{A}\mathrm{ut}_{(B,\Pi,P,\Theta)}) \cong H^2(B, \mathcal{O}_P)$.

Picard group over twisted Poisson manifolds

For $i = 1, 2$, let (M_i, ω_i) be ASCIRs over (B, Π, P, Θ) with $d\omega_i = \pi^*\eta_i$. The fiber product $M_1 \times_B M_2$ is a $T \times_B T$-torsor. Let $\pi_i : M_1 \times_B M_2 \to M_i$ and $\pi' : M_1 \times_B M_2 \to B$ be natural projections. Let T^- be the Lie group bundle over B whose fibers are anti-diagonal of the fibers of $T \times_B T$. Then $M_1 \otimes M_2 := M_1 \times_B M_2 / T^-$ is a T-torsor. Since $\pi_1^*\omega_1 + \pi_2^*\omega_2$ vanishes on restriction to the T^--orbits it pushes down to the form $\omega_1 \oplus \omega_2$ on $M_1 \otimes M_2$.

Definition

Let $\mathrm{Pic}(B, \Pi, P, \Theta)$ be the group of isomorphism classes of ASCIRs over (B, Π) with period bundles P, where $(M_1, \omega_1) \otimes (M_2, \omega_2) = (M_1 \times_B M_2 / T^-, \omega_1 \oplus \omega_2 - \pi^*\Theta)$ with twisting form $\eta_1 + \eta_2 - d\Theta$. The identity element is $[(T, \omega_{\mathrm{can}} + \pi^*\Theta)]$ and the inverse of $[(M, \omega)]$ is $[(M, -\omega + 2\pi^*\Theta)]$.

Picard group over twisted Poisson manifolds

Theorem (F-Sjamaar)

1. *We have the exact sequence*

$$0 \longrightarrow H^2(B, \mathcal{O}_P) \longrightarrow Pic(B, \Pi, P, \Theta) \longrightarrow Z^3_{\mathcal{F}}(B)$$

where the last map sends (M, ω) *to* $d\Theta - \eta$.

2. *We have the short exact sequence*

$$0 \longrightarrow \Omega^2_{\mathcal{F}}/d\Omega^1_{\mathcal{F}} \longrightarrow Pic(B, \Pi, P, \Theta) \longrightarrow H^2(B, \mathcal{P}) \longrightarrow 0$$

where the first map sends $\gamma \in \Omega^2_{\mathcal{F}}/d\Omega^1_{\mathcal{F}}$ *to* $[(T, \omega_{can} + \pi^*(\Theta + \gamma))]$.

Picard group over twisted Poisson manifolds

Remark

1. *$\Omega^2_{\mathcal{F}}/d\Omega^1_{\mathcal{F}}$ is to the subgroup of Picard group consisting of isomorphism classes of degree 0 invertible sheaves in algebraic geometry what $H^2(B, \mathcal{P})$ is to the Néron-Severi group.*
2. *Any T-torsor can be made an ASCIR over (B, Π, P).*
3. *There is an action of $\Omega^2_{\mathcal{F}}$ on Pic(B, Π, P, Θ) by $\gamma \cdot [(M, \omega)] = [(M, \omega + \pi^*\gamma)]$, called the coarse gauge transformation. We have that there is a bijection between the coarse gauge equivalence classes and the isomorphism classes of T-torsors, and any coarse gauge equivalence class is a principal homogeneous space of $\Omega^2_{\mathcal{F}}/d\Omega^1_{\mathcal{F}}$.*

Arigato gozaimasu!