THE REAL K-THEORY OF COMPACT LIE GROUPS

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Let G be a compact, connected, and simply-connected Lie group, equipped with a Lie group involution σ_G and viewed as a G-space via the conjugation action. In this thesis we compute Atiyah's Real K-theory of G in several contexts.

We first obtain a complete description of the algebra structure of the equivariant KR-theory of both (G, σ_G) and $(G, \sigma_G \circ \mathrm{inv})$, where inv means group inversion, by utilizing the notion of Real equivariant formality and drawing on previous results on the module structure of the KR-theory and the ring structure of the equivariant K-theory.

The Freed-Hopkins-Teleman Theorem (FHT) asserts a canonical link between the equivariant twisted K-homology of G and its Verlinde algebra. In the latter part of the thesis we give a partial generalization of FHT in the presence of a Real structure of G. Along the way we develop preliminary materials necessary for this generalization, which are of independent interest in their own right. These include the definitions of Real Dixmier-Douady bundles, the Real third cohomology group which is shown to classify the former, and Real Spin^c structures.

BIOGRAPHICAL SKETCH

Hailing from Hong Kong, Chi-Kwong Fok has developed a penchant for working on mathematical problems since a young age. He was offered a glimpse of modern geometry and analysis and mesmerized by them when he took part, as a high school student, in the Enrichment Program for Young Mathematical Talents held by the Chinese University of Hong Kong (CUHK) from 2002 to 2003. As a result he matriculated at CUHK to study mathematics in 2003. In the summers of 2005 and 2006 he had the valuable opportunities to do research at Caltech and Cornell University on group theory and analysis on fractals, respectively. These wonderful research experiences bolstered his determination to pursue further studies and led to his enrollment in the mathematics PhD program at Cornell University in 2007, after he received a Bachelor of Science degree in mathematics from CUHK in the same year. At Cornell he did his doctoral work on symplectic geometry and K-theory under the supervision of his thesis advisor, Reyer Sjamaar. He is expected to assume a postdoctoral position in the National Center for Theoretical Sciences in Hsinchu, Taiwan, after completing his PhD studies in August, 2014.

To my parents, Kam Fok and Siu Chun Lau, and my brother, Chi Sum Fok, for their unconditional love and unswerving support

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CHAPTER 1

BACKGROUND

1.1 Overview

This thesis addresses generalizations to Atiyah's Real K-theory (or KR-theory) of complex K-theory of compact Lie groups in several contexts.

It has been well-known, since 40s, that any generalized cohomology theory of a Lie group is a Hopf algebra, with comultiplication induced by group multiplication, while the complex K-theory of compact connected Lie groups with torsion-free fundamental groups was explicitly worked out by Hodgkin in the 60s (cf. [Ho] and Theorem 2.1.4). Hodgkin's result asserts that the K-theory ring is the $\mathbb{Z}_2$-graded exterior algebra over $\mathbb{Z}$ on the module of primitive elements of the Hopf algebra, which are of degree -1 and associated with the representations of the Lie group. For the elegant proof of Hodgkin's result in the special case where G is simply-connected, see [At2].

In [At3], Atiyah introduced KR-theory, which is basically a version of topological K-theory for the category of Real spaces, i.e. topological spaces equipped with an involution. KR-theory can be regarded as a hybrid of KO-theory, complex K-theory and KSC-theory (cf. [At3]). One can also consider equivariant KR-theory, where a certain compatibility condition between the group action and the involution is assumed. For definitions and some basic properties, see Definitions 1.3.1

and 1.3.6, [At3] and [AS].

Since Atiyah and Hodgkin's work, there have appeared three kinds of generalizations of K-theory of compact Lie groups. The first such is KR-theory of compact Lie groups, which was first studied by Seymour (cf. [Se]). He obtained the $KR^*(\mathrm{pt})$-module structure of $KR^*(G)$, where G is a compact, connected and simply-connected Lie group equipped with a Lie group involution, using his structure theorem of KR-theory of a certain type of spaces (cf. Theorems 2.3.1, 2.3.5 and 2.3.8). He was unable to obtain a complete description of the ring structure, however, and could only make some conjectures about it.

The second one is the equivariant K-theory of compact Lie groups. In [BZ], Brylinski and Zhang showed that, for a compact connected Lie group G with torsion free fundamental group, the equivariant K-theory, $K^*_G(G)$, where G acts on itself by conjugation, is isomorphic to the ring of Grothendieck differentials of the complex representation ring $R(G)$ (for definition, see [BZ]). It is noteworthy that G satisfies the property of being weakly equivariantly formal *à la* Harada and Landweber (cf. Definition 4.1 of [HL] and Remark 2.1.8).

The third one is a deep result by Freed-Hopkins-Teleman, which relates the twisted equivariant K-homology of a compact Lie group with Verlinde algebra, the representation group of positive energy representations of its loop group equipped with an intricately defined ring structure called fusion product (cf. [Fr], [FHT1], [FHT2], [FHT3]). Verlinde algebra is an object of great interest in mathematical physics and algebraic geometry. One of the remarkable aspects of Freed-

Hopkins-Teleman Theorem is that it provides an algebro-topological approach to interpreting the fusion product, which is usually defined using conformal blocks or moduli spaces of G-bundles on Riemann surfaces (cf. [Be], [BL]). Moreover, Freed-Hopkins-Teleman also provides the framework for a formulation of geometric quantization of q-Hamiltonian spaces (cf. [M3] and [M4]). We will discuss Freed-Hopkins-Teleman Theorem in greater details in Chapter 4.

In this thesis, we give generalizations of Brylinski-Zhang's and Freed-Hopkins-Teleman's results in the context of KR-theory. The organization is as follows.

The rest of Chapter 1 covers the background materials of this thesis, some of which cannot be found by us in the literature and thus are recorded here, for they are of independent interest in their own right. We review KR-theory, K-homology and its twisted version. We find the representing spaces for KR-theory, define Real Dixmier-Douady bundles which are used to realize the twists of KR-theory (homology) and the Real third cohomology groups which is shown to classify the twists. We also carefully develop the notion of Real Spin^c structures and Real orientation twists. We show that KR-theory twisted by the Real orientation twist of $\mathbb{R}^4$ is Quaternionic K-theory (cf. Proposition 1.6.3).

In Chapter 2, based on the previous works of Seymour's and Brylinski–Zhang's and putting both the Real and equivariant structures together, we obtain a description of the ring structure of the equivariant KR-theory of any compact, connected and simply-connected Lie group, which is recorded in Theorem 2.5.33.

We express the ring structure using relations of generators associated to Real representations of G of real, complex and quaternionic type (with respect to the Lie group involution). Our main contribution is twofold. First, we observe that the conditions of Seymour's structure theorem are an appropriate candidate for defining the notion of 'Real formality' in analogy to weakly equivariant formality. These notions together prompt us to introduce the definition of 'Real equivariant formality', which leads to a structure theorem for equivariant KR-theory (Theorem 2.5.5). Any compact, connected and simply-connected Real Lie group falls under the category of Real equivariantly formal spaces and Theorem 2.5.5 enables us to obtain a preliminary description of the equivariant KR-theory as an algebra over the coefficient ring. In fact the notion of Real equivariant formality will be heavily used throughout this thesis. Second, inspired by Seymour's conjecture, we obtain the squares of the real and quaternionic type generators, which in addition to other known relations complete the description of the ring structure. These squares are non-zero 2-torsions in general. Hence the equivariant KR-theory in general is not a ring of Grothendieck differentials, as in the case for equivariant K-theory. Despite this, we remark that, in certain cases, if we invert 2 in the equivariant KR-theory ring, then the result is an exterior algebra over the localized coefficient ring of equivariant KR-theory.

Chapter 3 is basically an outgrowth of Chapter 2. We equip G instead with an anti-involution, and observe that it is also a Real equivariantly formal space. We obtain the algebra structure of the corresponding equivariant KR-theory using the strategy in Chapter 2. We find that the squares of the algebra generators are

0 (cf. Theorem 3.2.8), in stark contrast to the case considered in Chapter 2. In the special case where G does not have any Real representations of complex type, the equivariant KR-theory is simply the ring of Grothendieck differentials of the coefficient ring of KR-theory. This suggests that anti-involutions are the right type of involutions needed to generalize Brylinski-Zhang's result.

The benefit of considering the anti-involution does not end there. We devote Chapter 4 to a partial generalization of Freed-Hopkins-Teleman Theorem. As a matter of fact all the work in this thesis is motivated by our desire to obtain such a generalization. Using the preliminary materials we develop in the latter sections in Chapter 1, we find that G equipped with an anti-involution, instead of an involutive automorphism, is the right candidate for generalizing Freed-Hopkins-Teleman in KR-homology. Following the idea in [M2] of applying Segal's spectral sequence to a simplicial description of G, we find that the twisted equivariant KR-homology of G endowed with anti-involution in the special case that G does not have compelex type representations is isomorphic to the tensor product of Verlinde algebra and the KR-homology coefficient ring (cf. Theorem 4.6.4). We also make a conjecture about the general case (Conjecture 4.6.6) and draft a future research direction along the line of geometric quantization of q-Hamiltonian spaces expounded in [M2], [M3] and [M4].

1.2 Notations and conventions

Throughout this thesis, we use $\Gamma = \{1, \gamma\}$ to denote the group of order 2, and σ_X to denote the involutive homeomorphism induced by γ on the Real space X. If X is a group G, σ_G always means an involutive automorphism, while a_G means the corresponding anti-involution $\sigma_G \circ \mathrm{inv}$, where inv means group inversion. We sometimes suppress the notations for involution if there is no danger of confusion about the involutive homeomorphism. For example, we simply use G with the understanding that it is equipped with σ_G, while we use G^- to mean the group G equipped with a_G. Following the convention in [At3], we use the word 'Real' (with a capital R) in all contexts involving involutions, so as to avoid confusion with the word 'real' with the usual meaning. For example, 'Real K-theory' is used interchangeably with KR-theory, whereas 'real K-theory' means KO-theory. Moreover, we do not discriminate between the meanings of the terms 'Real structure' and 'involution'.

1.3 KR-theory

Motivated by the index theory of real elliptic operators, Atiyah introduced KR-theory in [At3]. It is a version of topological K-theory for the category of Real spaces (see below for definition), and used by Atiyah to derive the 8-periodicity of KO-theory from the 2-periodicity of complex K-theory. In recent years KR-

theory has found applications in string theory, as it classifies the D-brane charges in orientifold string theory (cf. [DMR]).

Definition 1.3.1. 1. A *Real space* is a pair (X, σ_X) where X is a topological space equipped with an involutive homeomorphism σ_X, i.e. $\sigma_X^2 = \mathrm{Id}_X$. A *Real pair* is a pair (X, Y) where Y is a closed subspace of X invariant under σ_X.

2. Let $\mathbb{R}^{p,q}$ be the Euclidean space $\mathbb{R}^{p+q}$ equipped with the involution which is identity on the first q coordinates and negation on the last p-coordinates. Let $B^{p,q}$ and $S^{p,q}$ be the unit ball and sphere in $\mathbb{R}^{p,q}$ with the inherited involution.

3. A *Real vector bundle* (to be distinguished from the usual real vector bundle) over X is a complex vector bundle E over X which itself is also a Real space with involutive homeomorphism σ_E satisfying

   (a) $\sigma_X \circ p = p \circ \sigma_E$, where $p : E \to X$ is the projection map,

   (b) σ_E maps E_x to $E_{\sigma_X(x)}$ anti-linearly.

   A *Quaternionic vector bundle* (to be distinguished from the usual quaternionic vector bundle) over X is a complex vector bundle E over X equipped with an anti-linear lift σ_E of σ_X such that $\sigma_E^2 = -\mathrm{Id}_E$.

4. A complex of Real vector bundles over the Real pair (X, Y) is a complex of Real vector bundles over X

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots \longrightarrow E_n \longrightarrow 0$$

   which is exact on Y.

5. Let X be a Real space. The ring $KR(X)$ is the Grothendieck group of the isomorphism classes of Real vector bundles over X, equipped with the usual product structure induced by tensor product of vector bundles over $\mathbb{C}$. The relative KR-theory for a Real pair $KR(X, Y)$ can be similarly defined using complexes of Real vector bundles over (X, Y), modulo homotopy equivalence and addition of acyclic complexes (cf. [S2]). In general, the graded KR-theory ring of the Real pair (X, Y) is given by

$$KR^*(X, Y) := \bigoplus_{q=0}^{7} KR^{-q}(X, Y),$$

where

$$KR^{-q}(X, Y) := KR(X \times B^{0,q}, X \times S^{0,q} \cup Y \times B^{0,q}). \tag{1.1}$$

The ring structure of KR^* is extended from that of KR, in a way analogous to the case of complex K-theory. The number of graded pieces, which is 8, is a result of Bott periodicity for KR-theory (cf. [At3]).

Note that when $\sigma_X = \mathrm{Id}_X$, then $KR(X) \cong KO(X)$. On the other hand, if $X \times \mathbb{Z}_2$ is given the involution which swaps the two copies of X, then $KR(X \times \mathbb{Z}_2) \cong K(X)$. Also, if X is equipped with the trivial involution, then $KR(X \times S^{2,0}) \cong KSC(X)$, the Grothendieck group of homotopy classes of self-conjugate bundles over X (cf. [At3]). In this way, it is natural to view KR-theory as a unifying thread of KO-theory, K-theory and KSC-theory. Keeping this in mind, we find below the representing spaces of KR^{-q} based on those of the K-theory and KO-theory. Recall that

q	Representing space for KO^{-q}
0	$\mathbb{Z} \times O(2\infty)/(O(\infty) \times O(\infty))$
1	$O(\infty)$
2	$O(2\infty)/U(\infty)$
3	$U(2\infty)/Sp(\infty)$
4	$\mathbb{Z} \times Sp(2\infty)/(Sp(\infty) \times Sp(\infty))$
5	$Sp(\infty)$
6	$Sp(\infty)/U(\infty)$
7	$U(\infty)/O(\infty)$

Table 1.1: Representing spaces for KO-theory

Definition 1.3.2. A *representing space* Ω_q for the graded generalized cohomology theory E^q is a space such that for any X, there is a natural bijective correspondence between $E^q(X)$ and $[X, \Omega_q]$, the set of homotopy equivalence classes of maps from X to Ω_q.

The representing spaces of K^0 and K^{-1} are $\mathbb{Z}\times U(2\infty)/U(\infty)\times U(\infty))$ (i.e. product of $\mathbb{Z}$ and the infinite complex Grassmannian) and $U(\infty)$ respectively, while those of the various degree pieces of KO-theory are listed in Table 1.1 (cf. [Bot]). Here $O(\infty) := \lim_{n\to\infty} O(n)$ and $O(2\infty) := \lim_{n\to\infty} O(2n)$, etc. $O(\infty) \times O(\infty)$ is the direct limit of $O(n) \times O(n)$, which is embedded into $O(2n)$ as block diagonal matrices. $U(\infty) \times U(\infty)$ and $Sp(\infty) \times Sp(\infty)$ are similarly defined.

Definition 1.3.3. Let K be a maximal rank subgroup of $U(n)$ or a direct limit of such subgroups. Let $\sigma_{\mathbb{R}}$ be the complex conjugation on $U(n)$, $U(n)/K$ or $U(\infty)$. Let $\sigma_{\mathbb{H}}$ be the symplectic type involution on $U(2m)$ (given by $g \mapsto J_m\overline{g}J_m^{-1}$), $U(2\infty)$ or the involution $gK \mapsto J_m\overline{g}K$ on $U(2m)/K$. Let $a_{\mathbb{R}}$ and $a_{\mathbb{H}}$ be the cor-

q	Representing space for KR^{-q}
0	$(\mathbb{Z} \times U(2\infty)/(U(\infty) \times U(\infty)), \mathrm{Id} \times \sigma_{\mathbb{R}})$
1	$(U(\infty), \sigma_{\mathbb{R}})$
2	$(U(2\infty)/H, \sigma_{\mathbb{R}})$
3	$(U(2\infty), a_{\mathbb{H}})$
4	$(\mathbb{Z} \times U(2\infty)/(U(\infty) \times U(\infty)), \mathrm{Id} \times \sigma_{\mathbb{H}})$
5	$(U(2\infty), \sigma_{\mathbb{H}})$
6	$(U(2\infty)/H, \sigma_{\mathbb{H}})$
7	$(U(\infty), a_{\mathbb{R}})$

Table 1.2: Representing spaces for KR-theory

responding anti-involutions on the unitary groups. Let $H = \lim_{n\to\infty} H_n$, where $H_n = \left\{ \left. \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \right| A, B \in M_n(\mathbb{C}) \right\} \cap U(2n)$.

Remark 1.3.4. Note that H and H_n are isomorphic to $U(\infty) \times U(\infty)$ and $U(n) \times U(n)$ respectively.

Proposition 1.3.5. *The representing spaces of various degree pieces of KR-theory are listed in Table 1.2. Here Ω_q is a representing space for KR^q if it is a Real space such that for any Real locally compact Hausdorff space X, there is a bijective correspondence between $KR^q(X)$ and $[X, \Omega_q]_{\mathbb{R}}$, where $[X, Y]_{\mathbb{R}}$ means the set of Real homotopy equivalence classes of Real maps from X to Y. Here Real homotopy equivalence is the one witnessed by a family of Real maps.*

Proof. The representing space for KR^{-q} is a Real space which is homeomorphic to the representing space for K^{-q} and whose subspace fixed under the Real structure is homeomorphic to that for KO^{-q}. By the definition of Real vector bundles and KR-theory, and noting that the representing spaces for K-theory and KO-theory are homogeneous spaces constructed out of various infinite dimensional matrix groups (orthogonal, unitary, and symplectic), the representing spaces for KR-theory are obtained from those for KO-theory by 'complexifying' the relevant matrix groups (to be distinguished from the usual complexification of Lie groups). For example, $O(\infty)$ is transformed to $(U(\infty), \sigma_{\mathbb{R}})$ (analogous to the fact that $\mathbb{R}$ is complexified to $\mathbb{C}$), $Sp(\infty)$ to $(U(2\infty), \sigma_{\mathbb{H}})$ (analogous to the fact that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{M}_2(\mathbb{C})$), and $U(\infty)$ to $(U(\infty) \times U(\infty), (g_1, g_2) \mapsto (\overline{g}_2, \overline{g}_1))$ (analogous to the fact that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$). We shall show the case for KR^{-7}. The other cases are similar. The representing space for KR^{-7} is $(U(\infty) \times U(\infty)/U(\infty)_\Delta, (g_1, g_2)U(\infty)_\Delta \mapsto (\overline{g}_2, \overline{g}_1)U(\infty)_\Delta)$, obtained from that for KO^{-7}, which is $U(\infty)/O(\infty)$. We have that $(U(\infty), a_{\mathbb{R}})$ is Real diffeomorphic to $(U(\infty) \times U(\infty)/U(\infty)_\Delta, (g_1, g_2)U(\infty)_\Delta \mapsto (\overline{g_2}, \overline{g_1})U(\infty)_\Delta)$ by the map $g \mapsto (g, e)U(\infty)$. $\square$

On top of the Real structure, we may further add compatible group actions and define equivariant KR-theory.

Definition 1.3.6.

1. A *Real G-space X* is a quadruple $(X, G, \sigma_X, \sigma_G)$ where a group G acts on X and σ_G is an involutive automorphism of G such that

$$\sigma_X(g \cdot x) = \sigma_G(g) \cdot \sigma_X(x).$$

2. A *Real G vector bundle E* over a Real G-space X is a Real vector bundle and a G-bundle over X, and it is also a Real G-space.

3. In a similar spirit, one can define equivariant KR-theory $KR^*_G(X, Y)$. Notice

that the G-actions on $B^{0,q}$ and $S^{0,q}$ in the definition of $KR_G^{-q}(X,Y)$ (cf. Equation (1.1)) are trivial.

Definition 1.3.7. 1. Let $K^*(+)$ be the complex K-theory of a point extended to a $\mathbb{Z}_8$-graded algebra over $K^0(\mathrm{pt}) \cong \mathbb{Z}$, i.e. $K^*(+) \cong \mathbb{Z}[\beta]\big/\beta^4 - 1$. Here $\beta \in K^{-2}(+)$, called the *Bott class*, is the class of the reduced canonical line bundle on $\mathbb{CP}^1 \cong S^2$.

2. Let $\overline{\sigma_X^*}$ be the map defined on (equivariant) vector bundles on X by $\overline{\sigma_X^*}E := \sigma_X^*\overline{E}$. The involution induced by $\overline{\sigma_X^*}$ on $K_G^*(X)$ is also denoted by $\overline{\sigma_X^*}$ for simplicity.

In the following proposition, we collect, for reader's convenience, some basic results of KR-theory (cf. [Se, Section 2]), some of which are stated in the more general context of equivariant KR-theory.

Proposition 1.3.8. *1. We have*

$$KR^*(pt) \cong \mathbb{Z}[\eta,\mu]/(2\eta,\eta^3,\mu\eta,\mu^2-4),$$

where $\eta \in KR^{-1}(pt)$, $\mu \in KR^{-4}(pt)$ represents the reduced Hopf bundles of $\mathbb{RP}^1$ and $\mathbb{HP}^1$ respectively.

2. Let $c : KR_G^*(X) \to K_G^*(X)$ be the homomorphism which forgets the Real structure of Real vector bundles, and $r : K_G^*(X) \to KR_G^*(X)$ be the realification map defined by $[E] \mapsto [E \oplus \sigma_G^\circ\sigma_X^*\overline{E}]$, where σ_G° means twisting the original G-action on E by σ_G. Then we have the following relations

(a) $c(1) = 1$, $c(\eta) = 0$, $c(\mu) = 2\beta^2$, *where* $\beta \in K^{-2}(pt)$ *is the Bott class,*

(b) $r(1) = 2$, $r(\beta) = \eta^2$, $r(\beta^2) = \mu$, $r(\beta^3) = 0$,

(c) $r(xc(y)) = r(x)y$, $cr(x) = x + \sigma_G^\circ \overline{\sigma_X^*} x$ *and* $rc(y) = 2y$ *for* $x \in K_G^*(X)$ *and* $y \in KR_G^*(X)$, *where* $K_G^*(X)$ *is extended to a* $\mathbb{Z}_8$*-graded algebra by Bott periodicity.*

Proof. (1) is given in [Se, Section 2]. The proof of (2) is the same as in the nonequivariant case, which is given in [At3]. □

Definition 1.3.9. A Quaternionic G-vector bundle over a Real space X is a complex vector bundle E equipped with an anti-linear vector bundle endomorphism J on E such that $J^2 = -\mathrm{Id}_E$ and $J(g \cdot v) = \sigma_G(g) \cdot J(v)$. Let $KH_G^*(X)$ be the corresponding K-theory constructed using Quaternionic G-bundles over X.

By generalizing the discussion preceding in [Se, Lemma 5.2] to the equivariant and graded setting, we define a natural transformation

$$t: \; KH_G^{-q}(X) \to KR_G^{-q-4}(X)$$

which sends

$$0 \longrightarrow E_1 \xrightarrow{f} E_2 \longrightarrow 0$$

to

$$0 \longrightarrow \pi^*(\mathbb{H} \otimes_{\mathbb{C}} E_1) \xrightarrow{g} \pi^*(\mathbb{H} \otimes_{\mathbb{C}} E_2) \longrightarrow 0.$$

Here

1. E_i, $i = 1, 2$ are equivariant Quaternionic vector bundles on $X \times \mathbb{R}^{0,q}$ equipped with the Quaternionic structures J_{E_i},

2. f is an equivariant Quaternionic vector bundle homomorphism which is an isomorphism outside $X \times \{0\}$,

3. $\pi : X \times \mathbb{R}^{0,q+4} \to X \times \mathbb{R}^{0,q}$ is the projection map,

4. $\mathbb{H} \otimes_{\mathbb{C}} E_i$ is the equivariant Real vector bundles equipped with the Real structure $J \otimes J_{E_i}$,

5. g is an equivariant Real vector bundle homomorphism defined by $g(v, w \otimes e) = (v, vw \otimes f(e))$.

The discussion in the last section of [AS] can be extended to the equivariant setting and yields

Proposition 1.3.10. *t is an isomorphism.*

1.4 (Real) K-homology

K-homology is a homology theory dual to K-theory through the K-theory version of Poincaré duality, where a manifold is oriented in K-theory if it has a Spin^c structure. Inspired by the Atiyah-Singer index theorem, which he used to realize the K-theoretic Poincaré duality, Kasparov gave the first definition of K-homology (cf. [Kas]). In this Section and the next, we follow Kasparov's definition, give a quick review of K-homology and introduce its Real and twisted version. Most of the materials in this Section are directly culled from [BHS], [M2] and [HR], to the latter of which we refer the reader for more details on the subject.

Definition 1.4.1. Let A be a separable $\mathbb{Z}_2$-graded $G-C^*$-algebra. A *Fredholm module* over A is a triple $(\rho, \mathcal{H}, F)$ where

1. $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ is a separable $\mathbb{Z}_2$-graded G-Hilbert space,
2. $\rho : \mathsf{A} \to B(\mathcal{H})$ is a representation of A by bounded linear operators on $\mathcal{H}$ which preserves the grading, and
3. F is an odd graded operator on $\mathcal{H}$ such that

$$(F^2 - 1)\rho(a) \sim 0, (F - F^*)\rho(a) \sim 0, [F, \rho(a)] \sim 0$$

for all $a \in \mathsf{A}$, where $\sim$ means equality modulo compact operators.

Definition 1.4.2. 1. We say the two Fredholm modules $(\rho, \mathcal{H}, F)$ and $(\rho', \mathcal{H}', F')$ are *unitarily equivalent* if there is a unitary isomorphism $U : \mathcal{H}' \to \mathcal{H}$ such that

$$(\rho', \mathcal{H}', F') = (U^* \rho U, \mathcal{H}', U^* F U)$$

2. Two Fredholm modules (ρ, H, F_0) and (ρ, H, F_1) are *operator homotopic* if there exists a norm continuous function $t \mapsto F_t$ for $t \in [0, 1]$.

Definition 1.4.3 (Kasparov's K-homology)**.** The equivariant K-homology group of a $\mathbb{Z}_2$-graded $G-C^*$-algebra A, $K^0_G(\mathsf{A})$, is the abelian group with one generator $[x]$ for each unitary equivalent class of Fredholm modules over A with the following relations

1. if x_0 and x_1 are operator homotopic then $[x_0] = [x_1]$ in $K^0_G(\mathsf{A})$, and

2. $[x_0 \oplus x_1] = [x_0] + [x_1]$

We define $K_G^{-q}(\mathsf{A}) = K_G^0(\mathsf{A}\widehat{\otimes}\mathrm{Cl}(\mathbb{R}^q))$. If A is an ungraded $G - C^*$-algebra, its K-homology is defined to be the one for $\mathsf{A} \oplus \mathsf{A}$ with the obvious $\mathbb{Z}_2$-grading.

Like K-theory, K-homology is also 2-periodic (cf. [HR, Proposition 8.2.13]). The use of superscript to denote the grading of K-homology of $G - C^*$-algebra is due to the fact that the K-homology functor on the category of $G - C^*$-algebras is contravariant. For if $(\rho, \mathcal{H}, F)$ is a Fredholm module over A, and $\alpha : \mathsf{A}' \to \mathsf{A}$ is a $G - C^*$-algebra homomorphism, then $(\rho \circ \alpha, \mathcal{H}, F)$ is a Fredholm module over A'.

Definition 1.4.4. The equivariant K-homology of a locally compact G-space X is defined to be the equivariant K-homology of the $G - C^*$-algebra of space of continuous functions on X vanishing at infinity.

$$K_q^G(X) := K_G^{-q}(C_0(X))$$

The K-homology functor on the category of topological spaces is covariant, as opposed to the contravariance of its counterpart on the category of $G - C^*$-algebras, hence the use of subscript to denote the grading.

The Poincaré duality pairing between K-homology and K-theory is realized by the index pairing, which produces the Fredholm index of a certain operator on a Hilbert space constructed out of the Fredholm module and the vector bundle representing the relevant K-homology and K-theory classes respectively. For instance, for a compact Spin^c manifold M, if a K-homology class is represented by the Fredholm module where

1. $\mathcal{H} = L^2(M, S)$, S is the complex spinor bundle,
2. ρ is the representation of $C(M)$ on $\mathcal{H}$ by left multiplication, and
3. F is the Dirac operator D suitably normalized so that it becomes a bounded operator,

then the index pairing of $(\rho, \mathcal{H}, F)$ and a vector bundle E gives the (analytic) index of the twisted Dirac operator D_E on the twisted spinor bundle $S \otimes E$ (cf. [HR, Lemma 11.4.1]). The K-homology class defined by the above Fredholm module is in fact the fundamental class of M, pairing with which is tantamount to the wrong way map in K-theory induced by the map collapsing M to a point.

As one would expect, the equivariant KR-homology can be defined by adding natural Real structures throughout the definition of equivariant K-homology. We refer the reader to [HR, Appendix B] for a brief discussion on the set-up of KR-homology. Nevertheless, we shall point out notable changes brought by the addition of Real structures. First, the KR-homology is 8-periodic. Second, in defining the Poincaré duality pairing between KR-homology and KR-theory, parallel to that between K-homology and K-theory, we shall need the notion of Real Spin^c structure as the KR-theoretic orientation. As we are unable to find a precise definition of Real Spin^c structure in the literature, we shall hereby give one.

Let $V \to X$ be a Γ-equivariant, orientable real vector bundle of rank $2n$ (caveat: this should be distinguished from Real vector bundles). Equip each fiber with an inner product such that σ_V is orthogonal. Let P be the oriented orthonormal frame bundle of V. P has an $SO(2n)$-action defined by

$$g \cdot (v_1, \cdots, v_{2n}) = \left(\sum_{j=1}^{2n} g_{1,j} v_j, \cdots, \sum_{j=1}^{2n} g_{2n,j} v_j \right)$$

making it a principal $SO(2n)$-bundle. We make the structure group $SO(2n)$ a Real Lie group by assigning the involutive automorphism

$$g \mapsto \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix} g \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}$$

We choose this involution because it restricts to complex conjugation on the special unitary group $SU(n)$ embedded in $SO(2n)$ as a subgroup. We also assign $\mathrm{Spin}^c(2n) := \mathrm{Spin}(2n) \times_{\mathbb{Z}_2} U(1)$, a $U(1)$-central extension of $SO(2n)$, with the involutive automorphism which descends to the one on $SO(2n)$ defined above and restricts to complex conjugation on the central circle. P is then a Real principal $SO(2n)$-bundle if

1. the Real structure on it is defined by

$$\sigma_P(v_1, v_2, \cdots, v_{2n}) = (\sigma_V(v_1), -\sigma_V(v_2), \cdots, (-1)^{i-1}\sigma_V(v_i), \cdots, -\sigma_V(v_{2n}))$$

2. P is preserved by σ_P.

Note that another way of saying condition (2) is that the (unoriented) frame bundle of V with Real structure given by condition (1) is the trivial double cover of P in the Real sense[1](a trivial double cover is trivial in the Real sense if the involution preserves the two connected components). Recall that an ordinary real vector bundle is orientable if its (unoriented) frame bundle is the trivial double cover of the oriented frame bundle. This prompts the following

Definition 1.4.5. Suppose V is orientable. V is *Real orientable* if the oriented frame bundle P is preserved by σ_P defined above.

V is Real orientable if and only if either σ_V is orientation-preserving and n is even, or σ_V is orientation-reversing and n is odd.

Example 1.4.6. Consider $\mathbb{R}^{p,q} \to \mathrm{pt}$, where $p+q$ is even. It is Real orientable if and only if $p-q$ is divisible by 4.

Definition 1.4.7. Let V be a Γ-equivariant, orientable real vector bundle over a Real space X. Suppose further that V is Real orientable. We say V is *Real* Spin^c if there exists, over X, a Real principal $\mathrm{Spin}^c(2n)$-bundle $\widetilde{P}$ whose structure group lifts that of P as the Real $U(1)$-central extension as defined above.

[1]We cannot resist to point out that we thought of using the phrase 'Really trivial double cover' but quickly realized that it was really a bad pun.

1.5 Equivariant Real Dixmier-Douady bundles and their classification

The study of local coefficient systems for K-theory was pioneered in [DK]. Since then various models for the local coefficient systems of K-theory and K-homology have been proposed, namely, bundle gerbes (cf. [BCMMS], [CCM], [CW], [Mu], [MS]), projective Hilbert space bundles ([AS2]) and Dixmier-Douady bundles, fiber bundles with fibers being the C^*-algebra $\mathcal{K}(\mathcal{H})$, the space of compact operators on a separable complex Hilbert space, and the structure group being $\mathrm{Aut}(\mathcal{K}(\mathcal{H})) = PU(\mathcal{H})$ (cf. [M2], [M3], [M4], [R]). In this paper we shall follow the convention in [M3] to twist K-homology by Dixmier-Douady bundles (DD bundles for short). To adapt to our context of equivariant KR-homology, we shall define equivariant Real DD bundles and a Real version of third equivariant cohomology group which classifies equivariant Real DD bundles (up to Morita isomorphism, to be explained below).

Definition 1.5.1. Let X be a locally compact Real G-space. A *G-equivariant Real DD bundle* $\mathcal{A}$ is a G-equivariant, locally trivial $\mathcal{K}(\mathcal{H})$-bundle with structure group $PU(\mathcal{H})$, equipped with an involution $\sigma_{\mathcal{A}}$ such that

1. $(\mathcal{A}, \sigma_{\mathcal{A}})$ is a Real G-space,
2. $\sigma_{\mathcal{A}}$ descends to σ_X on X, and
3. $\sigma_{\mathcal{A}}$ maps fiber to fiber anti-linearly.

Definition 1.5.2. 1. An equivariant Real DD bundle $\mathcal{A}$ is *Morita trivial* if there exists an equivariant Real Hilbert space bundle $\mathcal{E}$ such that $\mathcal{A}$ is isomorphic to $\mathcal{K}(\mathcal{E})$. We say $\mathcal{E}$ Morita trivializes $\mathcal{A}$ if $\mathcal{A} \cong \mathcal{K}(\mathcal{E})$.

2. The *opposite DD bundle* of $\mathcal{A}$, denoted by $\mathcal{A}^{\mathrm{opp}}$, is the $\mathcal{K}(\mathcal{H})$-bundle with the same underlying space as that of $\mathcal{A}$ except that it is modeled on the opposite Hilbert space $\mathcal{H}^{\mathrm{opp}}$ with the conjugate complex structure.

3. The (completed) tensor product of two equivariant Real DD bundles $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the equivariant Real DD bundle modeled on $\mathcal{K}(\mathcal{H}_1) \otimes \mathcal{K}(\mathcal{H}_2) \cong \mathcal{K}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

4. Two equivariant Real DD bundles $\mathcal{A}_1$ and $\mathcal{A}_2$ are *Morita isomorphic* if $\mathcal{A}_1 \otimes \mathcal{A}_2^{\mathrm{opp}}$ is isomorphic to a Morita trivial equivariant Real DD bundle. We say $\mathcal{E}$ *witnesses* the Morita isomorphism between $\mathcal{A}_1$ and $\mathcal{A}_2$ if $\mathcal{E}$ Morita trivializes $\mathcal{A}_1 \otimes \mathcal{A}_2^{\mathrm{opp}}$.

5. Suppose $(X_i, \mathcal{A}_i)$, $i = 1, 2$ are two equivariant Real DD bundles modeled on $\mathcal{K}(\mathcal{H}_i)$. A *Morita morphism*

$$(f, \mathcal{E}) : (X_1, \mathcal{A}_1) \to (X_2, \mathcal{A}_2)$$

consists of an equivariant Real proper map $f : X_1 \to X_2$ and an equivariant Real Hilbert space bundle $\mathcal{E}$ on X_1 which is a $(f^*\mathcal{A}_2, \mathcal{A}_1)$-bimodule, locally modeled on the $(\mathcal{K}(\mathcal{H}_2), \mathcal{K}(\mathcal{H}_1))$-bimodule $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$, the space of compact operators mapping from $\mathcal{H}_1$ to $\mathcal{H}_2$.

If $(f, \mathcal{E}_i)$, $i = 1, 2$, are two Morita morphisms from $(X_1, \mathcal{A}_1)$ to $(X_2, \mathcal{A}_2)$, then the $(f^*\mathcal{A}_2, \mathcal{A}_1)$-bimodules $\mathcal{E}_1$ and $\mathcal{E}_2$ differ by an equivariant Real line bundle L. More precisely,

$$\mathcal{E}_2 = \mathcal{E}_1 \otimes L, \text{ with } L = \mathrm{Hom}_{f^*\mathcal{A}_2 - \mathcal{A}_1}(\mathcal{E}_1, \mathcal{E}_2)$$

Definition 1.5.3. We say the two Morita morphisms $(f, \mathcal{E}_i)$, $i = 1, 2$, are *2-isomorphic* if the equivariant Real line bundle L is trivial.

It follows that, if there is a Morita morphism between $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ covering the equivariant Real map $f : X_1 \to X_2$, the set of 2-isomorphism classes of Morita morphisms covering f is a principal homogeneous space acted upon by the equivariant Real Picard group of X_1.

The category of Real DD bundles over a Real space is therefore endowed with a group structure where multiplication is given by the tensor product, and group inversion the opposite bundle construction. It is well-known that, analogous to complex line bundles being classified by the second integral cohomology group up to isomorphism, DD bundles are classified, up to Morita isomorphism, by the third integral cohomology group (cf. [DiDo]). In what follows we will prove an analogous result for equivariant Real DD bundle by adding one additional structure (Real and equivariant structures) at a time to ordinary DD bundles . We first consider the classification of Real DD bundles.

For a Real space X, let $\mathcal{U} = \{U_\alpha\}_{\alpha \in J}$ be a Γ-cover of X where the Γ-action on the index set J is free. For a Real DD bundle $\mathcal{A}$, there are transition functions $r_{\alpha\beta} : U_{\alpha\beta} \to PU(\mathcal{H})$ satisfying $r_{\alpha\beta}(x) = \overline{r_{\Gamma(\alpha)\Gamma(\beta)}(\sigma_X(x))}$. If $\mathcal{U}$ is fine enough, $r_{\alpha\beta}$ can be lifted to a $U(\mathcal{H})$-valued function $\widehat{r}_{\alpha\beta}$. Let $s_{\alpha\beta\gamma} = \widehat{r}_{\alpha\beta}\widehat{r}_{\beta\gamma}\widehat{r}_{\gamma\alpha}$. $\{s_{\alpha\beta\gamma}\}$ defines a Γ-equivariant

$U(1)$-valued 2-cocycle in the Cech cohomology group $H^2(\check{C}(\mathcal{U}, \underline{U(1)}_\Gamma)^\Gamma)$, where $\underline{U(1)}_\Gamma$ is the Γ-sheaf of continuous $U(1)$-valued functions with Γ acting on $U(1)$ by complex conjugation. The short exact sequence

$$1 \longrightarrow \mathbb{Z}_\Gamma \longrightarrow \underline{\mathbb{R}}_\Gamma \longrightarrow \underline{U(1)}_\Gamma \longrightarrow 1$$

where Γ acts on $\mathbb{Z}$ and $\mathbb{R}$ by negation, and the fact that $\underline{\mathbb{R}}_\Gamma$ is a fine Γ-sheaf (i.e. it admits a Γ-equivariant partition of unity), lead to the isomorphism $H^2(\check{C}(\mathcal{U}, \underline{U(1)}_\Gamma)^\Gamma) \cong H^3(\check{C}(\mathcal{U}, \mathbb{Z}_\Gamma)^\Gamma)$ induced by the coboundary map in the long exact sequence of Cech cohomology groups. By the Corollary in Section 5.5 of [Gr], $\varinjlim_{\mathcal{U}} H^3(\check{C}(\mathcal{U}, \mathbb{Z}_\Gamma)^\Gamma)$ is isomorphic to the sheaf cohomology $H^3(X; \Gamma, \mathbb{Z}_\Gamma)$ defined as the third right derived functor of the Γ-invariant global section functor. According to [St], because Γ is a discrete group, $H^3(X; \Gamma, \mathbb{Z}_\Gamma)$ is isomorphic to $H^3_\Gamma(X, \mathbb{Z}_\Gamma)$, which is Borel's equivariant cohomology defined more generally as follows.

Definition 1.5.4. Let G be a topological group with a Γ-action. Define the Real cohomology

$$H^n_\Gamma(X, \underline{G}_\Gamma) := H^n(X \times_\Gamma E\Gamma, \underline{X \times E\Gamma \times G/\Gamma})$$

where $\underline{X \times E\Gamma \times G/\Gamma}$ is the local coefficient system with fiber G over $X \times_\Gamma E\Gamma$.

Definition 1.5.5. The Real DD-class of $\mathcal{A}$, denoted by $DD_{\mathbb{R}}(\mathcal{A})$, is defined to be the image of the 2-cocycle $\{s_{\alpha\beta\gamma}\}$ in $H^3_\Gamma(X, \mathbb{Z}_\Gamma)$ under the various isomorphisms of cohomology groups discussed above, namely $H^3(\check{C}(\mathcal{U}, \mathbb{Z}_\Gamma)^\Gamma) \cong H^3(X; \Gamma, \mathbb{Z}_\Gamma) \cong H^3_\Gamma(X, \mathbb{Z}_\Gamma)$.

Remark 1.5.6. 1. The introduction of the Real cohomology $H^3_\Gamma(X, \mathbb{Z}_\Gamma)$ as the home where the Real DD-classes live is inspired by [Kah], where Kahn introduced Real Chern classes and use $H^2_\Gamma(X,\mathbb{Z}_\Gamma)$ to classify Real line bundles over X.

2. The map $\mathrm{DD}_\mathbb{R}$: category of Real DD bundles over $X \to H^3_\Gamma(X,\mathbb{Z}_\Gamma)$ is a group homomorphism. If $(f,\mathcal{E}) : (X_1,\mathcal{A}_1) \to (X_2,\mathcal{A}_2)$ is a Morita morphism, then $f^*\mathrm{DD}_\mathbb{R}(\mathcal{A}_2) = \mathrm{DD}_\mathbb{R}(\mathcal{A}_1)$.

By definition, one can see that $\mathcal{A}$ is Morita trivial if and only if $\mathrm{DD}_\mathbb{R}(\mathcal{A}) = 0 \in H^3_\Gamma(X,\mathbb{Z}_\Gamma)$. To show that $H^3_\Gamma(X,\mathbb{Z}_\Gamma)$ classifies the Morita isomorphism classes of Real DD bundles over X, we need the following

Proposition 1.5.7. *For any class $\alpha \in H^3_\Gamma(X,\mathbb{Z}_\Gamma)$, there exists a Real DD bundle $\mathcal{A}$ such that $DD_\mathbb{R}(\mathcal{A}) = \alpha$.*

Proof. The coboundary maps of the long exact sequences cohomology groups induced by the short exact sequences give rise to the following string of isomorphisms

$$\begin{aligned} H^3_\Gamma(X,\mathbb{Z}_\Gamma) \cong & H^2_\Gamma(X,\underline{U(1)}_\Gamma) \\ \cong & H^1_\Gamma(X,\underline{PU(\mathcal{H})}_\Gamma) \\ \cong & H^0_\Gamma(X,\underline{BPU(\mathcal{H})}_\Gamma) \\ \cong & [X, BPU(\mathcal{H})]_\mathbb{R} \end{aligned}$$

where $[X,Y]_\mathbb{R}$ is the set of Γ-equivariant homotopy equivalence classes of Real maps from X to Y. Therefore α corresponds to a Real map $f : X \to BPU(\mathcal{H})$. The Real DD bundle $f^*(EPU(\mathcal{H}) \times_{PU(\mathcal{H})} \mathcal{K}(\mathcal{H}))$ can be easily checked to have α as the Real DD-class. ∎

Remark 1.5.8. The proof above also shows that the Real DD bundle $EPU(\mathcal{H})\times_{PU(\mathcal{H})} \mathcal{K}(\mathcal{H})$ is a universal Real DD bundle.

Let $G \rtimes \Gamma$ be the semi-direct product such that conjugation by γ on G amounts to the automorphic involution σ_G. One can define the equivariant version of the Real cohomology similarly by Borel's construction.

Definition 1.5.9. Let X be a Real G-space. Define the equivariant Real cohomology $H^n_{G\rtimes\Gamma}(X, \mathbb{Z}_\Gamma)$ to be $H^n((X \times_G EG) \times_\Gamma E\Gamma, \underline{(X \times_G EG) \times E\Gamma \times \mathbb{Z}/\Gamma})$.

Atiyah and Segal showed in [AS2] that the equivariant cohomology $H^3_G(X, \mathbb{Z})$ classifies the equivariant DD bundles up to Morita isomorphism. Putting together the equivariant and Real structure, one can, with little effort, show that

Proposition 1.5.10. *The Morita isomorphism classes of equivariant Real DD bundles on the Real G-space X is classified by $H^3_{G\rtimes\Gamma}(X, \mathbb{Z}_\Gamma)$.*

The following simple observation is helpful in computing the free part of $H^*_{G\rtimes\Gamma}(X, \mathbb{Z}_\Gamma)$.

Proposition 1.5.11. *$H^n_{G\rtimes\Gamma}(X, \mathbb{R}_\Gamma)$ is isomorphic to $H^n_G(X, \mathbb{R})^\Gamma$, the Γ-invariant subgroup of $H^n_G(X, \mathbb{R})$, where the Γ-action on the cohomology group is defined by $\gamma \cdot \alpha = -\gamma^*\alpha$.*

Proof. $H^n_{G\rtimes\Gamma}(X, \mathbb{R}_\Gamma)$ is isomorphic to the sheaf cohomology $H^n(X \times_G EG; \Gamma, \mathbb{R}_\Gamma)$ (cf. [St]). $\mathbb{R}_\Gamma$ has the following resolution by fine Γ-sheaves

$$\mathbb{R}_\Gamma \xrightarrow{d} \underline{\Omega^0(X \times_G EG)} \xrightarrow{d} \underline{\Omega^1(X \times_G EG)} \xrightarrow{d} \cdots$$

Note that Γ acts on the global sections of $\underline{\Omega^n(X \times_G EG)}$ by $-\gamma^*$. We have that $H^n(X; \Gamma, \mathbb{R}_\Gamma)$ is the n-th cohomology of the cochain

$$\Omega^0(X \times_G EG)^\Gamma \xrightarrow{d} \Omega^1(X \times_G EG)^\Gamma \xrightarrow{d} \cdots$$

An easy averaging argument shows that taking Γ-invariants and taking cohomology commute. Hence $H^n(X \times_G EG; \Gamma, \mathbb{R}_\Gamma) \cong H^n(X \times_G EG, \mathbb{R})^\Gamma \cong H^*_G(X, \mathbb{R})^\Gamma$. $\square$

Example 1.5.12. We shall show that $H^3_\Gamma(\mathrm{pt}, \mathbb{Z}) \cong \mathbb{Z}_2$ and exhibit the DD bundle whose DD class is the nontrivial 2-torsion. Consider the S^0-bundle $E\Gamma \to B\Gamma$. Being non-orientable, this bundle has Euler class e which lives in the first cohomology group $H^1(B\Gamma, \underline{E\Gamma \times_\Gamma \mathbb{Z}})$ with twisted coefficient system. Applying the Gysin sequence

$$\cdots \longrightarrow H^2(E\Gamma, \mathbb{Z}) \xrightarrow{\int} H^2(B\Gamma, \mathbb{Z}) \xrightarrow{\cup e} H^3(B\Gamma, \underline{E\Gamma \times_\Gamma \mathbb{Z}}) \xrightarrow{\pi^*} H^3(E\Gamma, \mathbb{Z}) \longrightarrow \cdots$$

and by its exactness we have that $H^3(B\Gamma, \underline{E\Gamma \times_\Gamma \mathbb{Z}}) \cong H^2(B\Gamma, \mathbb{Z}) \cong \mathbb{Z}_2$. Thus, there is at least a copy of $\mathbb{Z}_2$ as a summand in $H^3_{G \rtimes \Gamma}(X, \mathbb{Z}_\Gamma)$ which is the image of the split-injective pullback map $H^3_\Gamma(\mathrm{pt}, \mathbb{Z}_\Gamma) \to H^3_{G \rtimes \Gamma}(X, \mathbb{Z}_\Gamma)$. The nonzero 2-torsion of $H^3_\Gamma(\mathrm{pt}, \mathbb{Z}_\Gamma)$ is the Real DD class of the $\mathcal{K}(\mathcal{H})$-bundle over a point where $\mathcal{H} = l^2(\mathbb{N})$ and the involution on $\mathcal{K}(\mathcal{H})$ is induced by the 'quaternionic quarter turn' on $\mathcal{H}$, i.e. $(z_1, z_2, z_3, z_4, \cdots) \mapsto (-\overline{z_2}, \overline{z_1}, -\overline{z_4}, \overline{z_3}, \cdots)$. In Section 1.6 we give another interpretation of this non-zero 2-torsion class as the obstruction for a certain Γ-equivariant real vector bundle over a point to possess a Real Spin^c structure or, equivalently, the KR-theory orientation.

1.6 Twisted K-theory (homology)

Twisted K-theory, or K-theory with local coefficient systems, was first studied in [DK], where the case of local coefficient systems with torsion DD-classes was explored. The general case was taken up in [R], where he defined twisted K-theory as homotopy equivalence classes of sections of a twisted bundle of Fredholm operators, with twisting data given by the local coefficient system. In recently years there have been extensive works done on twisted K-theory and its various models (cf. [AS2], [BCMMS], [CW], [Kar] and the references therein) due to its connection with string theory. It has been conjectured that D-branes and Ramond-Ramond field strength are classified by twisted K-theory, where the twist is defined by a B-field. An impetus to the whole enterprise of studying this mathematical physical connection is a deep result by Freed-Hopkins-Teleman, which is explained in more details in Chapter 4. This result is actually the motivation behind all the work done in this thesis.

In what follows, we shall use the following definition of twisted KR-theory (homology) obtained by incorporating the Real structures into the analytic definition of twisted K-theory and homology, as in [M2].

Definition 1.6.1. For an equivariant Real G-space X with an equivariant Real DD bundle $\mathcal{A}$, we define the *twisted KR-homology*

$$KR^G_q(X, \mathcal{A}) := KR^{-q}_G(S_0(\mathcal{A}))$$

where $S_0(\mathcal{A})$ is the Real $G - C^*$-algebra of space of sections of $\mathcal{A}$ vanishing at infinity. Similarly, we define the twisted KR-theory

$$KR^q_G(X,\mathcal{A}) := KR^G_{-q}(S_0(\mathcal{A}))$$

We list some useful features of twisted equivariant (Real) K-homology, adapted from [M2]. Further properties pertaining to representation theory can be found in Section 4.6.2.

1. A Morita morphism $(f,\mathcal{E}) : (X_1,\mathcal{A}_1) \to (X_2,\mathcal{A}_2)$ induces a pushforward map

$$f_* : KR^G_*(X_1,\mathcal{A}_1) \to KR^G_*(X_2,\mathcal{A}_2)$$

   which depends only on the 2-isomorphism class of $(f,\mathcal{E})$. In particular, the push forward map $(\mathrm{Id},(X\times\mathcal{H})\otimes L)_*$ induces an automorphism on $KR^G_*(X,\mathcal{A})$, which depends on the isomorphism class of the equivariant Real line bundle L. In other words, the equivariant Real Picard group of X acts on the automorphism group of $KR^G_*(X,\mathcal{A})$. If the equivariant Real Picard group of X_1, which is $H^2_{G\rtimes\Gamma}(X_1,\mathbb{Z}_\Gamma)$ by straightforwardly adapting the arguments in Section 1.5, is trivial, then there is only one canonical pushforward map independent of $\mathcal{E}$. From now on, we will, for brevity, sometimes use f_* to denote the pushforward map if it is independent of $\mathcal{E}$.

2. One can define the crossed product, which is a special case of Kasparov product (cf. [HR])

$$KR^G_*(X_1,\mathcal{A}_1)\otimes KR^G_*(X_2,\mathcal{A}_2) \to KR^G_*(X_1\times X_2,\mathcal{A}_1\otimes\mathcal{A}_2)$$

   where $\mathcal{A}_1\otimes\mathcal{A}_2$ is the external tensor product.

3. Recall that one of the motivations for introducing local coefficient systems to generalized cohomology theory is to formulate Thom isomorphism and Poincaré duality for spaces which are non-orientable in the sense of the relevant cohomology theory. For an even rank G-equivariant real vector bundle $V \to X$, which is orientable in the usual sense but not necessarily $G - \mathrm{Spin}^c$ (i.e. equivariant K-theoretic orientable), the K-theoretic local coefficient system reflecting the obstruction for V to be $G - \mathrm{Spin}^c$ can be realized by the Clifford bundle $\mathrm{Cl}(V)$, which is a DD bundle with its DD-class being a 2-torsion. We will call $\mathrm{Cl}(V)$ and any other Morita isomorphic DD bundles the orientation twist of V and denote it by o_V. The Thom isomorphism now can be formulated as

$$K_G^*(X, \mathcal{A} \otimes o_V) \cong K_G^*(V, \pi^* \mathcal{A})$$

whereas the Poincaré duality is

$$K_q^G(X, \mathcal{A}) \cong K_G^{-q}(X, \mathcal{A}^{\mathrm{opp}} \otimes o_{TX})$$

The corresponding statement of Thom isomorphism and Poincaré duality in the Real case are completely analogous.

Example 1.6.2. Consider the Γ-equivariant real vector bundle $\mathbb{R}^{0,4} \to \mathrm{pt}$. It is Real orientable according to Example 1.4.6. We shall show that this vector bundle is not Real Spin^c and hence $\mathrm{DD}_{\mathbb{R}}(o_{\mathbb{R}^{0,4}})$ is the nonzero 2-torsion in $H^3_\Gamma(\mathrm{pt}, \mathbb{Z}_\Gamma)$. The Real principal $SO(4)$-bundle P is isomorphic, through the map $(v_1, v_2, v_3, v_4) \mapsto \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$, to

$SO(4) \to \mathrm{pt}$ with involution being

$$g \mapsto \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} g$$

Though obviously $\widetilde{P} = (\mathrm{Spin}^c(4) \to \mathrm{pt})$ is the principal $\mathrm{Spin}^c(4)$-bundle which lifts the structure group of P, making $\mathbb{R}^{0,4}$ a Spin^c vector bundle, there is no way to equip $\widetilde{P}$ with a compatible Real structure so that $\mathbb{R}^{0,4}$ is Real Spin^c. For the best we can do is to give $\widetilde{P}$ with a Quaternionic structure (i.e. the 'quarter turn' bundle automorphism which descends to the involution on P and compatible with the Real $\mathrm{Spin}^c(4)$-action) as follows. If we identify $\mathrm{Spin}^c(4)$ with $(SU(2) \times SU(2)) \times_\Gamma U(1)$, where Γ acts on $U(1)$ by multiplication by -1 and on $SU(2) \times SU(2)$ by multiplication by $(-I_2, -I_2)$, then the Quaternionic structure is given by

$$[(g_1, g_2, z)] \mapsto [(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} g_1, \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} g_2, \overline{z})]$$

Using the fact that $\mathrm{DD}_{\mathbb{R}}(o_{V \oplus W}) = \mathrm{DD}_{\mathbb{R}}(o_V) + \mathrm{DD}_{\mathbb{R}}(o_W)$, we have that $\mathbb{R}^{0,8} = \mathbb{R}^{0,4} \oplus \mathbb{R}^{0,4}$ is Real Spin^c, vindicating the Bott periodicity for KR-theory. Together with the fact that $\mathbb{R}^{n,n} \to \mathrm{pt}$ is Real Spin^c (because it is isomorphic to $\mathbb{C}^n$ with Γ-action being complex conjugation), we get, more generally, the following fact.

Suppose $p + q$ is even. Then

1. *$\mathbb{R}^{p,q} \to pt$ is not Real orientable iff $\frac{p-q}{2}$ is odd.*

2. *$\mathbb{R}^{p,q} \to pt$ is Real orientable but not Real Spin^c iff $\frac{p-q}{4}$ is odd.*

3. *$\mathbb{R}^{p,q} \to pt$ is Real Spin^c iff $p-q$ is divisible by 8.*

The twisted version of Thom isomorphism yields

Proposition 1.6.3. *$KR^*_G(X, \pi^*(o_{\mathbb{R}^{0,4}})) \cong KR^*_G(X \times \mathbb{R}^{0,4})$, which in turn is isomorphic to the Quaternionic K-theory $KH^*_G(X)$ by Proposition 1.3.10.*

So Quaternionic K-theory is Real K-theory twisted by $o_{\mathbb{R}^{0,4}}$.

CHAPTER 2

EQUIVARIANT KR-THEORY OF COMPACT LIE GROUPS

In this Chapter, we compute the algebra structure of the equivariant KR-theory $KR^*_G(G)$ (cf. Theorem 2.5.33). Most of the materials in this Chapter are adapted from the author's published work [F].

2.1 (Equivariant) K-theory of compact connected Lie groups

Recall that the functor K^{-1} is represented by $U(\infty) := \lim_{n\to\infty} U(n)$, i.e. for any X, $K^{-1}(X)$ is the abelian group of homotopy classes of maps $[X, U(\infty)]$ (cf. [Bot]). Such a description of K^{-1} leads to the following

Definition 2.1.1. Let $\delta : R(G) \to K^{-1}(G)$ be the group homomorphism which sends any complex G-representation ρ to the homotopy class of $i \circ \rho$, where ρ is regarded as a continuous map from G to $U(n)$ and $i : U(n) \hookrightarrow U(\infty)$ is the standard inclusion.

In fact, any element in $K^{-q}(X)$ can be represented by a complex of vector bundles on $X \times \mathbb{R}^q$, which is exact outside $X \times \{0\}$ (cf. [At]). This gives another interpretation of $\delta(\rho)$, as shown in the following Proposition, which we find useful for our exposition.

Proposition 2.1.2. *If V is the underlying complex vector space of ρ, then $\delta(\rho)$ is repre-*

sented by

$$0 \to G \times \mathbb{R} \times V \to G \times \mathbb{R} \times V \to 0,$$

$$(g, t, v) \mapsto \ (g, t, -t\rho(g)v) \text{ if } t \geq 0,$$

$$(g, t, v) \mapsto \ (g, t, tv) \text{ if } t \leq 0.$$

Proposition 2.1.3 ([Ho]). *1. δ is a derivation of $R(G)$ taking values in $K^{-1}(G)$ regarded as an $R(G)$-module whose module structure is given by the augmentation map. In other words, δ is a group homomorphism from $R(G)$ to $K^{-1}(G)$ satisfying*

$$\delta(\rho_1 \otimes \rho_2) = \dim(\rho_1)\delta(\rho_2) + \dim(\rho_2)\delta(\rho_1). \tag{2.1}$$

2. If $I(G)$ is the augmentation ideal of $R(G)$, then $\delta(I(G)^2) = 0$.

The main results of [Ho] are stated in the following

Theorem 2.1.4. *Let G be a compact connected Lie group with torsion-free fundamental group. Then*

1. *$K^*(G)$ is torsion-free.*
2. *Let $J(G) := I(G)/I(G)^2$. Then the map $\widetilde{\delta} : J(G) \to K^{-1}(G)$ induced by δ is well-defined, and $K^*(G) = \bigwedge(\mathrm{Im}(\widetilde{\delta}))$.*
3. *In particular, if G is compact, connected and simply-connected of rank l, then $K^*(G) = \bigwedge_{\mathbb{Z}}(\delta(\rho_1), \ldots, \delta(\rho_l))$, where $\rho_1, \ldots, \rho_l$ are the fundamental representations.*

Viewing G as a G-space via the adjoint action, one may consider the equivariant K-theory $K^*_G(G)$. Let $\Omega^*_{R(G)/\mathbb{Z}}$ be the ring of Grothendieck differentials of $R(G)$

over $\mathbb{Z}$, i.e. the exterior algebra over $R(G)$ of the module of Kähler differentials of $R(G)$ over $\mathbb{Z}$ (cf. [BZ]).

Definition 2.1.5. Let $\varphi : \Omega^*_{R(G)/\mathbb{Z}} \to K^*_G(G)$ be the $R(G)$-algebra homomorphism defined by the following

1. $\varphi(\rho_V) := [G \times V] \in K^*_G(G)$, where G acts on $G \times V$ by $g_0 \cdot (g_1, v) = (g_0 g_1 g_0^{-1}, \rho_V(g_0)v)$,
2. $\varphi(d\rho_V) \in K^{-1}_G(G)$ is the complex of vector bundles in Definition 2.1.2 where G acts on $G \times \mathbb{R} \times V$ by $g_0 \cdot (g_1, t, v) = (g_0 g_1 g_0^{-1}, t, \rho_V(g_0)v)$,

We also define $\delta_G : R(G) \to K^{-1}_G(G)$ by $\delta_G(\rho_V) := \varphi(d\rho_V)$.

Remark 2.1.6. The definition of $\delta_G(\rho_V)$ given in [BZ], where the middle map of the complex of vector bundles is defined to be $(g, t, v) \mapsto (g, t, t\rho_V(g)v)$ for all $t \in \mathbb{R}$, is incorrect, as $\delta_G(\rho_V)$ so defined is actually 0 in $K^{-1}_G(G)$. The definition given in Definition 2.1.5 is the correction made by Brylinski and relayed to us by one of the referees for [F].

Theorem 2.1.7 ([BZ])**.** *1. δ_G is a derivation of $R(G)$ taking values in the $R(G)$-module $K^{-1}_G(G)$, i.e. δ_G satisfies*

$$\delta_G(\rho_1 \otimes \rho_2) = \rho_1 \cdot \delta_G(\rho_2) + \rho_2 \cdot \delta_G(\rho_1). \tag{2.2}$$

2. Let G be a compact connected Lie group with torsion-free fundamental group. Then φ is an $R(G)$-algebra isomorphism.

Remark 2.1.8. 1. Although the definition of $\delta_G(\rho_V)$ given by Brylinski–Zhang in [BZ] is incorrect, their proof of Theorem 2.1.7 can be easily corrected by using the correct definition as in Definition 2.1.5, which does not affect the validity of the rest of their arguments, and Theorem 2.1.7 still stands.

2. In [HL], a G-space X is defined to be *weakly equivariantly formal* if the map $K^*_G(X) \otimes_{R(G)} \mathbb{Z} \to K^*(X)$ induced by the forgetful map is a ring isomorphism, where $\mathbb{Z}$ is viewed as an $R(G)$-module through the augmentation homomorphism. Theorem 2.1.4 and 2.1.7 imply that G is weakly equivariantly formal if it is connected with torsion-free fundamental group. We will make use of this property in computing the equivariant KR-theory of G.

3. Let $f : K^*_G(G) \to K^*(G)$ be the forgetful map. Note that $f(\varphi(\rho)) = \dim(\rho)$ and $f(\varphi(d\rho)) = \delta(\rho)$. Applying f to equation (2.2) in Theorem 2.1.7, we get equation (2.1) in Proposition 2.1.3.

4. $I(G)$, being a prime ideal in $R(G)$, can be thought of as an element in $\mathrm{Spec}\, R(G)$, and $K^*(G) \cong \bigwedge_{\mathbb{Z}} T^*_{I(G)}\mathrm{Spec}R(G)$, $K^*_G(G) \cong \bigwedge_{R(G)} T^*_{I(G)}\mathrm{Spec}R(G)$.

2.2 Real representation rings

This section is an elaboration of the part on Real representation rings in [AS] and [Se]. Since the results in this subsection can be readily generalized from those results concerning the special case where σ_G is trivial, we omit most of the proofs and refer the reader to any standard text on representation theory of Lie groups, e.g. [Bour] and [BtD].

Definition 2.2.1. A *Real Lie group* is a pair (G, σ_G) where G is a Lie group and σ_G a Lie group involution on it. A *Real representation V* of a Real Lie group (G, σ_G) is a finite-dimensional complex representation of G equipped with an anti-linear involution σ_V such that $\sigma_V(gv) = \sigma_G(g)\sigma_V(v)$. Let $\mathcal{R}\mathrm{ep}_{\mathbb{R}}(G, \sigma_G)$ be the category of Real representations of (G, σ_G). A morphism between V and $W \in \mathcal{R}ep_{\mathbb{R}}(G, \sigma_G)$ is a linear transformation from V to W which commute with G and respect σ_V and σ_W. We denote $\mathrm{Mor}(V, W)$ by $\mathrm{Hom}_G(V, W)^{(\sigma_V, \sigma_W)}$. An irreducible Real representation is an irreducible object in $\mathcal{R}\mathrm{ep}_{\mathbb{R}}(G, \sigma_G)$. The Real representation ring of (G, σ_G), denoted by $RR(G, \sigma_G)$, is the Grothendieck group of $\mathcal{R}ep_{\mathbb{R}}(G, \sigma_G)$, with multiplication being tensor product over $\mathbb{C}$. Sometimes we will omit the notation σ_G if there is no danger of confusion about the Lie group involution.

Remark 2.2.2. Let V be an irreducible Real representation of G. Then $\mathrm{Hom}_G(V, V)^{\sigma_V}$ must be a real associative division algebra which, according to Frobenius' theorem, is isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Following the convention of [AS], we call $\mathrm{Hom}_G(V, V)^{\sigma_V}$ the *commuting field* of V.

Definition 2.2.3. If V is an irreducible Real representation of G, then we say V is of *real, complex or quaternionic type* according as the commuting field is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Let $RR(G, \mathbb{F})$ be the abelian group generated by the isomorphism classes of irreducible Real representations with $\mathbb{F}$ as the commuting field.

Remark 2.2.4. $RR(G) \cong RR(G, \mathbb{R}) \oplus RR(G, \mathbb{C}) \oplus RR(G, \mathbb{H})$ as abelian groups.

Definition 2.2.5. Let V be a G-representation. We use $\sigma_G^* V$ to denote the G-representation with the same underlying vector space where the G-action is twisted by σ_G, i.e. $\rho_{\sigma_G^* V}(g)v = \rho_V(\sigma_G(g))v$. We will use $\overline{\sigma_G^*}$ to denote the map on $R(G)$ defined by $[V] \mapsto [\sigma_G^* \overline{V}]$.

Proposition 2.2.6. *If V is a complex G-representation, and there exists $f \in \mathrm{Hom}_G(V, \sigma_G^* \overline{V})$ such that $f^2 = \mathrm{Id}_V$, then V is a Real representation of G with f as the anti-linear involution σ_V.*

Proof. If $f \in \mathrm{Hom}_G(V, \sigma_G^* \overline{V})$, then it is anti-linear on V and $f(gv) = \sigma_G(g) f(v)$ for $g \in G$ and $v \in V$. The assumption that $f^2 = \mathrm{Id}_V$ just says that f is an involution. So V together with $\sigma_V = f$ is a Real representation of G. □

Proposition 2.2.7. *Let V be an irreducible complex representation of G and suppose that $V \cong \sigma_G^* \overline{V}$. Let $f \in \mathrm{Hom}_G(V, \sigma_G^* \overline{V})$.Then*

1. *$f^2 = k\mathrm{Id}_V$ for some $k \in \mathbb{R}$.*
2. *There exists $g \in \mathrm{Hom}_G(V, \sigma_G^* \overline{V})$ such that $g^2 = \mathrm{Id}_V$ or $g^2 = -\mathrm{Id}_V$.*

Proof. Note that $f^2 \in \mathrm{Hom}_G(V, V)$. By Schur's lemma, $f^2 = k\mathrm{Id}_V$ for some $k \in \mathbb{C}$. On the other hand,

$$f(kv) = f(f(f(v))) = kf(v).$$

But f is an anti-linear map on V. It follows that $k = \overline{k}$ and hence $k \in \mathbb{R}$. For part (2), we may first simply pick an isomorphism $f \in \mathrm{Hom}_G(V, \sigma_G^* \overline{V})$. Then $f^2 = k\,\mathrm{Id}_V$ for some $k \in \mathbb{R}^\times$. Schur's lemma implies that any $g \in \mathrm{Hom}_G(V, \sigma_G^* \overline{V})$ must be of the form $g = cf$ for some $c \in \mathbb{C}$. Then $g^2 = cf \circ cf = c\bar{c}f^2 = |c|^2 k \,\mathrm{Id}_V$. Consequently, if k is positive, we choose $c = \frac{1}{\sqrt{k}}$ so that $g^2 = \mathrm{Id}_V$; if k is negative, we choose $c = \frac{1}{\sqrt{-k}}$ so that $g^2 = -\mathrm{Id}_V$. $\square$

Proposition 2.2.8. *Let V be an irreducible Real representation of G.*

1. *The commuting field of V is isomorphic to $\mathbb{R}$ iff V is an irreducible complex representation and there exists $f \in \mathrm{Hom}_G(V, \sigma_G^*\overline{V})$ such that $f^2 = \mathrm{Id}_V$.*
2. *The commuting field of V is isomorphic to $\mathbb{C}$ iff $V \cong W \oplus \sigma_G^*\overline{W}$ as complex G-representations, where W is an irreducible complex G-representation and $W \ncong \sigma_G^*\overline{W}$, and $\sigma_V(w_1, w_2) = (w_2, w_1)$.*
3. *The commuting field of V is isomorphic to $\mathbb{H}$ iff $V \cong W \oplus \sigma_G^*\overline{W}$ as complex G-representations, where W is an irreducible complex G-representation and there exists $f \in \mathrm{Hom}_G(V, \sigma_G^*\overline{V})$ such that $f^2 = -\mathrm{Id}_V$, and $\sigma_V(w_1, w_2) = (w_2, w_1)$.*

Proof. One can easily establish the above Proposition by modifying the proof of Proposition 3 in Appendix 2 of [Bour], which is a special case of the above Proposition where σ_G is trivial. $\square$

Proposition 2.2.9.

1. *The map $i : RR(G) \to R(G)$ which forgets the Real structure is injective.*
2. *Any complex G-representation V which is a Real representation can only possess a unique Real structure up to isomorphisms of Real G-representations.*

Proof. Let $\rho : R(G) \to RR(G)$ be the map

$$[V] \mapsto [(V \oplus \iota_G^* \overline{V}, \sigma_{V \oplus \iota_G^* \overline{V}})],$$

where $\sigma_{V \oplus \iota_G^* \overline{V}}(u, w) = (w, u)$. Let $[(V, \sigma_V)] \in RR(G)$. Then

$$\rho \circ i([(V, \sigma_V)]) = [(V \oplus \iota_G^* \overline{V}, \sigma_{V \oplus \iota_G^* \overline{V}})].$$

We claim that $[(V \oplus V, \sigma_V \oplus \sigma_V)] = [(V \oplus \iota_G^* \overline{V}, \sigma_{V \oplus \iota_G^* \overline{V}})]$, which is easily seen to be true because of the Real G-representation isomorphism

$$\begin{aligned} f : V \oplus \iota_G^* \overline{V} &\to V \oplus V, \\ (u, w) &\mapsto (u + \sigma_V(w), i(u - \sigma_V(w))). \end{aligned}$$

It follows that $\rho \circ i$ amounts to multiplication by 2 on $RR(G)$, and is therefore injective because $RR(G)$ is a free abelian group generated by irreducible Real representations. Hence i is injective. (2) is simply a restatement of (1). □

Proposition 2.2.9 makes it legitimate to regard $RR(G)$ as a subring of $R(G)$. From now on we view $[V] \in R(G)$ as an element in $RR(G)$ if V possesses a compatible Real structure.

Proposition 2.2.10. *Let G be a compact Real Lie group. Let V be an irreducible complex representation of G. Then*

1. $[V] \in RR(G, \mathbb{R})$ *iff there exists a G-invariant symmetric nondegenerate bilinear form* $B : V \times \sigma_G^* V \to \mathbb{C}$.

2. $[V \oplus V] \in RR(G, \mathbb{H})$ *iff there exists a* G*-invariant skew-symmetric nondegenerate bilinear form* $B : V \times \sigma_G^* V \to \mathbb{C}$*.*

3. $[V \oplus \sigma_G^* \overline{V}] \in RR(G, \mathbb{C})$ *iff there does not exist any* G*-invariant nondegenerate bilinear form on* $V \times \sigma_G^* V$*.*

Proof. By Proposition 2.2.8, V is a Real representation of real type iff there exists $f \in \mathrm{Hom}(V, \sigma_G^* \overline{V})^G$ such that $f^2 = \mathrm{Id}_V$. One can define a bilinear form $B : V \times \sigma_G^* V \to \mathbb{C}$ by

$$B(v_1, v_2) = \langle v_1, f(v_2) \rangle, \tag{2.3}$$

where $\langle \, , \, \rangle$ is a G-invariant Hermitian inner product on V. It can be easily seen that B is G-invariant, symmetric and non-degenerate. Conversely, given a G-invariant symmetric non-degenerate bilinear form on $V \times \sigma_G^* V$ and using equation (2.3), we can define $f \in \mathrm{Hom}(V, \sigma_G^* \overline{V})^G$, which squares to identity. Part (2) and (3) follow similarly. $\square$

Proposition 2.2.10 leads to the following

Definition 2.2.11. Let G be a compact Real Lie group. Let V be an irreducible complex representation of G. Define, with respect to σ_G,

1. V to be of real type if there exists a G-invariant symmetric nondegenerate bilinear form $B : V \times \sigma_G^* V \to \mathbb{C}$.

2. V to be of quaternionic type if there exists a G-invariant skew-symmetric nondegenerate bilinear form $B : V \times \sigma_G^* V \to \mathbb{C}$.

3. V to be of complex type if $V \not\cong \sigma_G^* \overline{V}$.

The abelian group generated by classes of irreducible complex representation of type $\mathbb{F}$ is denoted by $R(G, \mathbb{F})$.

Definition 2.2.12. If V is a complex G-representation equipped with an anti-linear endomorphism J_V such that $J_V(gv) = \sigma_G(g)J(v)$ and $J^2 = -\mathrm{Id}_V$, then we say V is a *Quaternionic representation* of G. Let $\mathcal{R}\mathrm{ep}_{\mathbb{H}}(G)$ be the category of Quaternionic representations of G. A morphism between V and $W \in \mathcal{R}\mathrm{ep}_{\mathbb{H}}(G)$ is a linear transformation from V to W which commutes with G and respect J_V and J_W. We denote $\mathrm{Mor}(V, W)$ by $\mathrm{Hom}_G(V, W)^{(J_V, J_W)}$. An irreducible Quaternionic representation is an irreducible object in $\mathcal{R}\mathrm{ep}_{\mathbb{H}}(G, \sigma_G)$. The Quaternionic representation group of G, denoted by $RH(G)$, is the Grothendieck group of $\mathcal{R}\mathrm{ep}_{\mathbb{H}}(G)$. Let $RH(G, \mathbb{F})$ be the abelian group generated by the isomorphism classes of irreducible Quaternionic representations with $\mathbb{F}$ as the commuting field.

Remark 2.2.13. The tensor product of two Quaternionic representations V and W is a Real representation as $J_V \otimes J_W$ is an anti-linear involution which is compatible with σ_G. Similarly the tensor product of a Real representation and a Quaternionic representation is a Quaternionic representation. To put it succinctly, $RR(G) \oplus RH(G)$ is a $\mathbb{Z}_2$-graded ring, with $RR(G)$ being the degree 0 piece and $RH(G)$ the degree -1 piece. Later on we will assign $RH(G)$ with a different degree so as to be compatible with the description of the coefficient ring $KR_G^*(\mathrm{pt})$.

Proposition 2.2.14. *$RH(G)$, as an abelian group, is generated by the following*

1. $[V \oplus \sigma_G^* \overline{V}]$ *where* V *is an irreducible complex representation of real type with* $J(u, w) = (-w, u)$. *Its commuting field is* $\mathbb{H}$.

2. $[V \oplus \sigma_G^* \overline{V}]$, *where* V *is an irreducible complex representation of complex type with* $J(u, w) = (-w, u)$. *Its commuting field is* $\mathbb{C}$.

3. $[V]$, *where* V *is an irreducible complex representation of quaternionic type. Its commuting field is* $\mathbb{R}$.

Proof. The proof proceeds in the same fashion as does the proof for Proposition 2.2.8. □

Corollary 2.2.15. 1. *We have that* $RR(G, \mathbb{R}) \cong RH(G, \mathbb{H}) \cong R(G, \mathbb{R})$, $RH(G, \mathbb{R}) \cong RR(G, \mathbb{H}), \cong R(G, \mathbb{H})$ *and* $RR(G, \mathbb{C}) \cong RH(G, \mathbb{C})$ *as abelian groups.*

2. $RR(G)$ *is isomorphic to* $RH(G)$ *as abelian groups.*

Proof. The result follows easily from Proposition 2.2.8, Definition 2.2.11 and Proposition 2.2.14. □

Proposition 2.2.16. 1. *The map* $j : RH(G) \to R(G)$ *which forgets the Quaternionic structure is injective.*

2. *Any complex* G*-representation which is a Quaternionic representation can only possess a unique Quaternionic structure up to isomorphisms of Quaternionic* G*-representations.*

Proof. The proof proceeds in the same fashion as does the proof for Proposition 2.2.9. It suffices to show that, if $\eta : R(G) \to RH(G)$ is the map $[V] \mapsto$

$[(V \oplus \sigma_G^* \overline{V}, \sigma_{V \oplus \iota_G^* \overline{V}})]$, then $\eta \circ j$ amounts to multiplication by 2, i.e. $(V \oplus \sigma_G^* \overline{V}, \sigma_{V \oplus \sigma_G^* \overline{V}}) \cong (V \oplus V, J \oplus J)$, where $\sigma_{V \oplus \sigma_G^* \overline{V}}(u, w) = (-w, u)$. This is true because of the Quaternionic G-representation isomorphism

$$\begin{aligned} f : V \oplus \iota_G^* \overline{V} &\to V \oplus V, \\ (u, w) &\mapsto (u + Jw, i(u - Jw)). \end{aligned}$$

$\square$

Example 2.2.17. We shall illustrate the similarities and differences of the various aforementioned representation groups with an example. Let $G = Q_8 \times C_3$, the direct product of the quaternion group and the cyclic group of order 3, equipped with the trivial involution. There are 5 irreducible complex representations of Q_8, namely, the 4 1-dimensional representations which become trivial on restriction to the center Z of Q_8 and descend to the 4 1-dimensional representations of $Q_8/Z \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, and the 2-dimensional faithful representation. We denote these representations by 1_{Q_8}, $\rho_{(0,1)}$, $\rho_{(1,0)}$, $\rho_{(1,1)}$ and ρ_Q respectively. Similarly, we let 1_{C_3}, ρ_ζ and ρ_{ζ^2} be the three 1-dimensional complex irreducible representations of C_3. It can be easily seen that

$$R(Q_8,\mathbb{R}) = \mathbb{Z}\cdot[1_{Q_8}]\oplus\mathbb{Z}\cdot[\rho_{(0,1)}]\oplus\mathbb{Z}\cdot[\rho_{(1,0)}]\oplus\mathbb{Z}\cdot[\rho_{(1,1)}],$$

$$R(Q_8,\mathbb{C}) = 0,$$

$$R(Q_8,\mathbb{H}) = \mathbb{Z}\cdot[\rho_Q],$$

$$R(C_3,\mathbb{R}) = \mathbb{Z}\cdot[1_{C_3}],$$

$$R(C_3,\mathbb{C}) = \mathbb{Z}\cdot[\rho_\zeta]\oplus\mathbb{Z}\cdot[\rho_{\zeta^2}],$$

$$R(C_3,\mathbb{H}) = 0.$$

It follows that

$$R(G,\mathbb{R}) = \bigoplus_{x\in\{1_{Q_8},\rho_{(0,1)},\rho_{(1,0)},\rho_{(1,1)}\}}\mathbb{Z}\cdot[x\widehat{\otimes}1_{C_3}]\cong\mathbb{Z}^4,$$

$$R(G,\mathbb{C}) = \bigoplus_{\substack{x\in\{1_{Q_8},\rho_{(0,1)},\rho_{(1,0)},\rho_{(1,1)},\rho_Q\}\\ y\in\{\rho_\zeta,\rho_{\zeta^2}\}}}\mathbb{Z}\cdot[x\widehat{\otimes}y]\cong\mathbb{Z}^{10},$$

$$R(G,\mathbb{H}) = \mathbb{Z}\cdot[\rho_Q\widehat{\otimes}1_{C_3}]\cong\mathbb{Z},$$

$$RR(G,\mathbb{R}) = \bigoplus_{x\in\{1_{Q_8},\rho_{(0,1)},\rho_{(1,0)},\rho_{(1,1)}\}}\mathbb{Z}\cdot[x\widehat{\otimes}1_{C_3}]\cong\mathbb{Z}^4,$$

$$RR(G,\mathbb{C}) = \bigoplus_{x\in\{1_{Q_8},\rho_{(0,1)},\rho_{(1,0)},\rho_{(1,1)},\rho_Q\}}\mathbb{Z}\cdot[x\widehat{\otimes}\rho_\zeta\oplus\overline{x}\widehat{\otimes}\rho_{\zeta^2}]\cong\mathbb{Z}^5,$$

$$RR(G,\mathbb{H}) = \mathbb{Z}\cdot[\rho_Q\widehat{\otimes}1_{C_3}\oplus\overline{\rho_Q}\widehat{\otimes}1_{C_3}]\cong\mathbb{Z},$$

$$RH(G,\mathbb{R}) = \mathbb{Z}\cdot[\rho_Q\widehat{\otimes}1_{C_3}]\cong\mathbb{Z},$$

$$RH(G,\mathbb{C}) = \bigoplus_{x\in\{1_{Q_8},\rho_{(0,1)},\rho_{(1,0)},\rho_{(1,1)},\rho_Q\}}\mathbb{Z}\cdot[x\widehat{\otimes}\rho_\zeta\oplus\overline{x}\widehat{\otimes}\rho_{\zeta^2}]\cong\mathbb{Z}^5,$$

$$RH(G,\mathbb{H}) = \bigoplus_{x\in\{1_{Q_8},\rho_{(0,1)},\rho_{(1,0)},\rho_{(1,1)}\}}\mathbb{Z}\cdot[x\widehat{\otimes}1_{C_3}\oplus\overline{x}\widehat{\otimes}1_{C_3}]\cong\mathbb{Z}^4.$$

Some representations above should be equipped with suitable Real or Quaternionic structures given in Propositions 2.2.8 and 2.2.14. For example, the Real structure of $\rho_Q \widehat{\otimes} 1_{C_3} \oplus \overline{\rho_Q} \widehat{\otimes} 1_{C_3}$ in $RR(G, \mathbb{H})$ is given by swapping the two coordinates.

2.3 The module structure of KR-theory of compact simply-connected Lie groups

The following structure theorem for KR-theory, due to Seymour, is crucial in his computation of $KR^*(\mathrm{pt})$-module structure of $KR^*(G)$.

Theorem 2.3.1 ([Se, Theorem 4.2]). *Suppose that $K^*(X)$ is a free abelian group and decomposed by the involution $\overline{\sigma_X^*}$ into the following summands*

$$K^*(X) = M_+ \oplus M_- \oplus T \oplus \overline{\sigma_X^*} T,$$

where $\overline{\sigma_X^*}$ is identity on M_+ and negation on M_-. Suppose further that there exist $h_1, \ldots, h_n \in KR^*(X)$ such that $c(h_1), \ldots, c(h_n)$ form a basis of the $K^*(+)$-module $K^*(+) \otimes (M_+ \oplus M_-)$. Then, as $KR^*(pt)$-modules,

$$KR^*(X) \cong F \oplus r(K^*(+) \otimes T),$$

where F is the free $KR^*(pt)$-module generated by $h_1, \ldots, h_n$.

Remark 2.3.2. If $T = 0$, then the conditions in Theorem 2.3.1 are equivalent to $K^*(X)$ being free abelian and $c : KR^*(X) \to K^*(X)$ being surjective. In this special case the theorem implies that the map $KR^*(X) \otimes_{KR^*(\mathrm{pt})} K^*(\mathrm{pt}) \to K^*(X)$ defined by

$a \otimes b \mapsto c(a) \cdot b$ is a ring isomorphism. This smacks of the definition of weakly equivariant formality (cf. Remark 2.1.8) and inspires us to define a similar notion for equivariant KR-theory (cf. Definition 2.5.2). We say a real space is *real formal* if it satisfies the conditions of Theorem 2.3.1.

For any Real space X, $KR^{-1}(X)$ is isomorphic to the abelian group of equivariant homotopy classes of maps from X to $U(\infty)$ which respect σ_X and $\sigma_{\mathbb{R}}$ on $U(\infty)$. Similarly, $KR^{-5}(X)$, which is isomorphic to $KH^{-1}(X)$ by Proposition 1.3.10, is isomorphic to the abelian group of equivariant homotopy classes of maps from X to $U(2\infty)$ which respect σ_X and $\sigma_{\mathbb{H}}$ on $U(2\infty)$ (cf. remarks in the last two paragraphs of Appendix of [Se]). We can define maps analogous to those in Definition 2.1.1 in the context of KR-theory.

Definition 2.3.3. Let $\delta_{\mathbb{R}} : RR(G) \to KR^{-1}(G)$ and $\delta_{\mathbb{H}} : RH(G) \to KR^{-5}(G)$ be group homomorphisms which send a Real (resp. Quaternionic) representation to the KR-theory element represented by its homotopy class.

Proposition 2.3.4. *If $\rho \in RR(G)$, then $\delta_{\mathbb{R}}(\rho)$ is represented by the complex of vector bundles in Proposition 2.1.2 equipped with the Real structure given by*

$$\begin{aligned} \iota : \ & G \times \mathbb{R} \times V \to G \times \mathbb{R} \times V, \\ & (g, t, v) \mapsto (\sigma_G(g), t, \overline{v}). \end{aligned}$$

If $\rho \in RH(G)$, then $\delta_{\mathbb{H}}(\rho)$ can be similarly represented, with the Real structure replaced by the Quaternionic structure.

From this point on until the end of this section, we further assume that G is connected and simply-connected unless otherwise specified. It is known that $R(G)$ is a polynomial ring over $\mathbb{Z}$ generated by fundamental representations, which are permuted by $\overline{\sigma_G^*}$ (cf. [Se, Lemma 5.5]). Let

$$R(G) \cong \mathbb{Z}[\varphi_1, \ldots, \varphi_r, \theta_1, \ldots, \theta_s, \gamma_1, \ldots, \gamma_t, \overline{\sigma_G^*}\gamma_1, \ldots, \overline{\sigma_G^*}\gamma_t],$$

where $\varphi_i \in RR(G, \mathbb{R})$, $\theta_j \in RH(G, \mathbb{R})$, $\gamma_k \in R(G, \mathbb{C})$. Then $K^*(G)$, as a free abelian group, is generated by square-free monomials in $\delta(\varphi_1), \ldots, \delta(\varphi_r)$, $\delta(\theta_1), \ldots, \delta(\theta_s)$, $\delta(\gamma_1), \ldots, \delta(\gamma_t)$, $\delta(\overline{\sigma_G^*}\gamma_1), \ldots, \delta(\overline{\sigma_G^*}\gamma_t)$. Using Theorem 2.3.1, Seymour obtained

Theorem 2.3.5 ([Se, Theorem 5.6]**).** 1. *Suppose that $\overline{\sigma_G^*}$ acts as identity on $R(G)$, i.e. any irreducible Real representation of G is either of real type or quaternionic type. Then as $KR^*(pt)$-modules,*

$$KR^*(G) \cong \wedge_{KR^*(pt)}(\delta_{\mathbb{R}}(\varphi_1), \ldots, \delta_{\mathbb{R}}(\varphi_r), \delta_{\mathbb{H}}(\theta_1), \ldots, \delta_{\mathbb{H}}(\theta_s)).$$

2. *More generally, $c(\delta_{\mathbb{R}}(\varphi_i)) = \delta(\varphi_i)$, $c(\delta_{\mathbb{H}}(\theta_j)) = \beta^2 \cdot \delta(\theta_j)$, and there exist $\lambda_1, \ldots, \lambda_t \in KR^0(G)$ such that $c(\lambda_k) = \beta^3 \cdot \delta(\gamma_k)\delta(\overline{\sigma_G^*}\gamma_k)$, and*

$$KR^*(G) \cong P \oplus T \cdot P$$

as $KR^*(pt)$-module, where

- *$P \cong \bigwedge_{KR^*(pt)}(\delta_{\mathbb{R}}(\varphi_1), \ldots, \delta_{\mathbb{R}}(\varphi_r), \delta_{\mathbb{H}}(\theta_1), \ldots, \delta_{\mathbb{H}}(\theta_s), \lambda_1, \ldots, \lambda_t)$,*
- *T is the additive abelian group generated by the set*

$$\{r(\beta^i \cdot \delta(\gamma_1)^{\varepsilon_1} \cdots \delta(\gamma_t)^{\varepsilon_t}\delta(\overline{\sigma_G^*}\gamma_1)^{\nu_1} \cdots \delta(\overline{\sigma_G^*}\gamma_t)^{\nu_t})\},$$

where $\varepsilon_1, \ldots, \varepsilon_t$, $\nu_1, \ldots, \nu_t$ are either 0 or 1, ε_k and ν_k are not equal to 1 at the same time for $1 \le k \le t$, and the first index k_0 where $\varepsilon_{k_0} = 1$ is less than the first index k_1 where $\nu_{k_1} = 1$.

Moreover,

(*a*) $\lambda_k^2 = 0$ *for all* $1 \le k \le t$,

(*b*) $\delta_{\mathbb{R}}(\varphi_i)^2$ *and* $\delta_{\mathbb{H}}(\theta_j)^2$ *are divisible by* η.

Definition 2.3.6. Let $\omega_t := \delta_{\epsilon_t, 1-\nu_t}$ and

$$r_{i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t} := r\left(\beta^i \cdot \delta(\gamma_1)^{\varepsilon_1} \cdots \delta(\gamma_t)^{\varepsilon_t} \delta(\overline{\sigma_G^*}(\gamma_1))^{\nu_1} \cdots \delta(\overline{\sigma_G^*}(\gamma_t))^{\nu_t}\right) \in T.$$

Corollary 2.3.7. *1. $KR^*(G)$ is generated by $\delta_{\mathbb{R}}(\varphi_1), \ldots, \delta_{\mathbb{R}}(\varphi_r)$, $\delta_{\mathbb{H}}(\theta_1), \ldots, \delta_{\mathbb{H}}(\theta_s)$, $\lambda_1, \ldots, \lambda_t$ and $r_{i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t} \in T$ as an algebra over $KR^*(pt)$.*

2.

$$r^2_{i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t} = \begin{cases} \eta^2 \lambda_1^{\omega_1} \cdots \lambda_t^{\omega_t}, & \text{if } r_{i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t} \text{ is of degree } -1 \text{ or } -5, \\ \pm\mu \lambda_1^{\omega_1} \cdots \lambda_t^{\omega_t}, & \text{if } r_{i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t} \text{ is of degree } -2 \text{ or } -6, \\ 0 & \text{otherwise.} \end{cases}$$

The sign depends on i, $\varepsilon_1, \ldots, \varepsilon_t$, $\nu_1, \ldots, \nu_t$ and can be determined using formulae from (2) of Proposition 1.3.8.

3. $r_{i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t}\eta = 0$, and $r_{i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t}\mu = 2r_{i+2,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t}$.

Proof. The Corollary follows easily from the various properties of the realification map and the complexification map in Proposition 1.3.8, and the fact that $c(\lambda_k) =$

$\beta^3 \cdot \delta(\gamma_k)\delta(\overline{\sigma_G^*}\gamma_k)$. For example,

$$
\begin{aligned}
r_{i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t}\eta &= r(\beta^i \cdot \delta(\gamma_1)^{\varepsilon_1} \cdots \delta(\gamma_t)^{\varepsilon_t}\delta(\overline{\sigma_G^*}(\gamma_1))^{\nu_1} \cdots \delta(\overline{\sigma_G^*}(\gamma_t))^{\nu_t} c(\eta)) \\
&= 0, \\
r_{i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t}\mu &= r(\beta^i \cdot \delta(\gamma_1)^{\varepsilon_1} \cdots \delta(\gamma_t)^{\varepsilon_t}\delta(\overline{\sigma_G^*}(\gamma_1))^{\nu_1} \cdots \delta(\overline{\sigma_G^*}(\gamma_t))^{\nu_t} c(\mu)) \\
&= r_{i+2,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t}.
\end{aligned}
$$

□

In fact Theorems 2.1.4 and 2.3.1 also yield the following description of module structure of KR-theory of a compact connected Real Lie group with torsion-free fundamental group with a restriction on the types of the Real representations.

Theorem 2.3.8. *Let G be a compact connected real Lie group with $\pi_1(G)$ torsion-free. Suppose that $R(G, \mathbb{C}) = 0$, i.e. $\overline{\sigma_G^*}$ acts as identity on $R(G)$. Then $KR^*(G)$ is isomorphic to $\wedge^*_{KR^*(pt)}(\mathrm{Im}(\widetilde{\delta_{\mathbb{R}}}), \mathrm{Im}(\widetilde{\delta_{\mathbb{H}}}))$ as $KR^*(pt)$-modules.*

As we see from Theorem 2.3.5 and Corollary 2.3.7, to get a full description of the ring structure of $KR^*(G)$, it remains to figure out $\delta_{\mathbb{R}}(\varphi_i)^2$ and $\delta_{\mathbb{H}}(\theta_j)^2$. We will, in the end, obtain formulae for the squares by way of computing the ring structure of $KR^*_G(G)$ and applying the forgetful map. In particular, we will show that $\delta_{\mathbb{R}}(\varphi_i)^2$ and $\delta_{\mathbb{H}}(\theta_j)^2$ in general are non-zero. So, unlike the complex K-theory, $KR^*(G)$ is not an exterior algebra in general. Nevertheless, $KR^*(G)$ is not far from being an exterior algebra, in the sense of the following

Corollary 2.3.9.
1. *$KR^*(pt)_2$, which is the ring obtained by inverting the prime 2 in $KR^*(pt)$, is isomorphic to $\mathbb{Z}\left[\frac{1}{2}, \mu\right]/(\mu^2 - 4) \cong \mathbb{Z}\left[\frac{1}{2}, \beta^2\right]/((\beta^2)^2 - 1)$.*

2. *Suppose that $R(G, \mathbb{C}) = 0$. $KR^*(G)_2$, which is the ring obtained by inverting the prime 2 in $KR^*(G)$, is isomorphic to, as $KR^*(pt)_2$-algebra*

$$\bigwedge\nolimits_{KR^*(pt)_2} (\delta_{\mathbb{R}}(\varphi_1), \ldots, \delta_{\mathbb{R}}(\varphi_r), \delta_{\mathbb{H}}(\theta_1), \ldots, \delta_{\mathbb{H}}(\theta_s))$$

Remark 2.3.10. As we will see in the rest of this thesis, the condition $R(G, \mathbb{C}) = 0$ yields nice descriptions of KR-theory (homology) of G. We find it worthwhile to record some equivalent conditions, which can be found in Section 4.6.1.

2.4 The coefficient ring $KR^*_G(\mathrm{pt})$

In this section, we assume that G is a compact Real Lie group, and will prove a result on the coefficient ring $KR^*_G(\mathrm{pt})$. In [AS], all graded pieces of $KR^*_G(\mathrm{pt})$ were worked out using Real Clifford G-modules. We record them in the following

Proposition 2.4.1. *$KR^{-q}_G(pt)$, as abelian groups, for $0 \le q \le 7$, are isomorphic to $RR(G)$, $RR(G)/\rho(R(G))$, $R(G)/j(RH(G))$, 0, $RH(G)$, $RH(G)/\eta(R(G))$, $R(G)/i(RR(G))$ and 0 respectively, where the maps i, j, ρ, η are as in Propositions 2.2.9 and 2.2.16.*

Remark 2.4.2. Note from the above proposition that $KR^0_G(\mathrm{pt}) \oplus KR^{-4}_G(\mathrm{pt}) \cong RR(G) \oplus RH(G)$. In this way we can view $RR(G) \oplus RH(G)$ as a graded ring where $RR(G)$ is of degree 0 and $RH(G)$ of degree -4.

Proposition 2.4.3. 1. *Suppose $R(G, \mathbb{C}) = 0$. Then the map*

$$f : (RR(G, \mathbb{R}) \oplus RH(G, \mathbb{R})) \otimes KR^*(pt) \to KR^*_G(pt),$$

$$\rho_1 \otimes x_1 \oplus \rho_2 \otimes x_2 \mapsto \rho_1 \cdot x_1 + \rho_2 \cdot x_2$$

is an isomorphism of graded rings.

2. *In general,*

$$f : (RR(G,\mathbb{R}) \oplus RH(G,\mathbb{R})) \otimes KR^*(pt) \oplus r(R(G,\mathbb{C}) \otimes K^*(+)) \to KR^*_G(pt),$$
$$\rho_1 \otimes x_1 \oplus \rho_2 \otimes x_2 \oplus r(\rho_3 \otimes \beta^i) \mapsto \rho_1 \cdot x_1 + \rho_2 \cdot x_2 + r(\rho_3 \cdot \beta^i)$$

   is an isomorphism of graded abelian groups.

3. *If ρ is an irreducible complex representation of complex type, then $\eta r(\beta^i \cdot \rho) = 0$ and $\mu r(\beta^i \cdot \rho) = 2r(\beta^{i+2} \cdot \rho)$.*

Proof. The Proposition follows by verifying the isomorphism in different degree pieces against the description in Proposition 2.4.1. For example, in degree 0,

$$\begin{aligned}
&RR(G,\mathbb{R}) \otimes KR^0(\mathrm{pt}) \oplus RH(G,\mathbb{R}) \otimes KR^{-4}(\mathrm{pt}) \oplus r(R(G,\mathbb{C}) \otimes K^0(+)) \\
=&RR(G,\mathbb{R}) \oplus RH(G,\mathbb{R}) \otimes \mathbb{Z}\mu \oplus RR(G,\mathbb{C}) \\
\cong&RR(G,\mathbb{R}) \oplus RR(G,\mathbb{H}) \oplus RR(G,\mathbb{C}) \\
&(\text{if } [V] \in RH(G,\mathbb{R}), \text{ then } [V]\cdot\mu = [V \oplus V] \in RR(G,\mathbb{H})) \\
=&RR(G) \\
=&KR^0_G(\mathrm{pt}).
\end{aligned}$$

(3) follows from Proposition 1.3.8. ◻

Remark 2.4.4. In [AS], $KR^*_G(X)$, where the G-action is trivial, is given as the following direct sum of abelian groups

$$RR(G,\mathbb{R}) \otimes KR^*(X) \oplus RR(G,\mathbb{C}) \otimes KC^*(X) \oplus RR(G,\mathbb{H}) \otimes KH^*(X),$$

where $KC^*(X)$ and $KH^*(X)$ are Grothendieck groups of the so-called 'Complex vector bundles' and 'Quaternionic vector bundles' of X. We find Proposition 2.4.3, which is motivated by this description, better because the ring structure of the coefficient ring is more apparent when cast in this light. The proposition is, as we will see in the next section, a consequence of a structure theorem of equivariant KR-theory (Theorem 2.5.5), and therefore still holds true if the point is replaced by any general space X with trivial G-action.

2.5 Equivariant KR-theory rings of compact simply-connected Lie groups

Throughout this section we assume that G is a compact, connected and simply-connected Real Lie group unless otherwise specified. We will prove the main result of this paper, Theorem 2.5.33, which gives the ring structure of $KR^*_G(G)$. Our strategy is outlined as follows.

1. We obtain a result on the structure of $KR^*_G(G)$ (Corollary 2.5.10) which is analogous to Theorem 2.3.5 and Proposition 2.4.3. We define $\delta^G_{\mathbb{R}}(\varphi_i)$, $\delta^G_{\mathbb{H}}(\theta_j)$, λ^G_k and $r^G_{\rho,i,\varepsilon_1,\dots,\varepsilon_t,\nu_1,\dots,\nu_t}$ (cf. Definition 2.5.8 and Corollary 2.5.11), which generate $KR^*_G(G)$ as a $KR^*_G(\mathrm{pt})$-algebra, as a result of Corollary 2.5.10. We show that $(\lambda^G_k)^2 = 0$ (cf. Proposition 2.5.13).
2. We compute the module structure of $KR^*_{(U(n),\sigma_{\mathbb{F}})}(U(n), \sigma_{\mathbb{F}})$ for $\mathbb{F} = \mathbb{R}$ and $\mathbb{H}$.

3. Let T be the maximal torus of diagonal matrices in $U(n)$ and, by abuse of notation, $\sigma_{\mathbb{R}}$ be the inversion map on T, $\sigma_{\mathbb{H}}$ be the involution on $U(n)/T$ (where $n = 2m$ is even) defined by $gT \mapsto J_m\overline{g}T$. We show that the restriction map

$$p_G^* : \ KR^*_{(U(n),\sigma_{\mathbb{R}})}(U(n), \sigma_{\mathbb{R}}) \to KR^*_{(T,\sigma_{\mathbb{R}})}(T, \sigma_{\mathbb{R}})$$

and the map

$$q_G^* : \ KR^*_{(U(2m),\sigma_{\mathbb{H}})}(U(2m), \sigma_{\mathbb{H}}) \to KR^*_{(U(2m),\sigma_{\mathbb{H}})}(U(2m)/T \times T, \sigma_{\mathbb{H}} \times \sigma_{\mathbb{R}})$$

induced by the Weyl covering map $q_G : U(2m)/T{\times}T \to U(2m)$, $(gT, t) \mapsto gtg^{-1}$, are injective.

4. Let σ_n be the class of the standard representation of $U(n)$. We pass the computation of the two squares $\delta^G_{\mathbb{R}}(\sigma_n)^2 \in KR^*_{(U(n),\sigma_{\mathbb{R}})}(U(n), \sigma_{\mathbb{R}})$ and $\delta^G_{\mathbb{H}}(\sigma_{2m})^2 \in KR^*_{(U(2m),\sigma_{\mathbb{H}})}(U(2m), \sigma_{\mathbb{H}})$ through the induced map p_G^* and q_G^* to their images and get equations (2.5) and (2.6) in Proposition 2.5.29.

5. Applying induced maps φ_i^* and θ_j^* to equations (2.5) and (2.6) yields equations (2.7) and (2.8) in Theorem 2.5.30 which, together with Proposition 2.5.13 and some relations among η, μ and $r^G_{\rho,i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t}$ deduced from Proposition 1.3.8, describe completely the ring structure of $KR^*_G(G)$ (cf. Theorem 2.5.33).

Remark 2.5.1. 1. Seymour first suggested the analogues of Steps 3, 4 and 5 in the ordinary KR-theory case in [Se] in an attempt to compute $\delta_{\mathbb{R}}(\varphi_i)^2$ and $\delta_{\mathbb{H}}(\theta_j)^2$, but failed to establish Step 3, which he assumed to be true to make conjectures about $\delta_{\mathbb{R}}(\varphi_i)^2$.

2. In equivariant complex K-theory, $K_G^*(G/T \times T) \cong K_T^*(T)$ for any compact Lie group G, and the two maps p_G^* (the restriction map induced by the inclusion $T \hookrightarrow G$) and q_G^* which is induced by the Weyl covering map are the same. If $\pi_1(G)$ is torsion-free, then these two maps are shown to be injective (cf. [BZ]. In fact it is even shown there that the maps inject onto the Weyl invariants of $K_T^*(T)$). In the case of equivariant KR-theory, things are more complicated. First of all, while in the case where $(G, \sigma_G) = (U(n), \sigma_{\mathbb{R}})$, it is true that $KR_G^*(G/T \times T) \cong KR_T^*(T)$, and p_G^* and q_G^* are the same, it is no longer true in the case where $(G, \sigma_G) = (U(2m), \sigma_{\mathbb{H}})$. In Step 3, we use q_G^* for the quaternionic type involution case because we find that it admits an easier description than p_G^* does. Second, we do not know whether p_G^* and q_G^* are injective for general compact Real Lie groups (equipped with any Lie group involution). For our purpose it is sufficient to show the injectivity results in Step 3.

2.5.1 A structure theorem

Definition 2.5.2. A G-space X is a *Real equivariantly formal* space if

1. G is a compact Real Lie group,
2. X is a weakly equivariantly formal G-space, and
3. the forgetful map $KR_G^*(X) \to KR^*(X)$ admits a section $s_R : KR^*(X) \to KR_G^*(X)$ which is a $KR^*(\mathrm{pt})$-module homomorphism.

Remark 2.5.3. If X is a weakly equivariantly formal G-space, then the forgetful map $K_G^*(X) \to K^*(X)$ admits a (not necessarily unique) section $s : K^*(X) \to K_G^*(X)$ which is a group homomorphism.

Definition 2.5.4. For a section $s : K^*(X) \to K_G^*(X)$ (resp. $s_R : KR^*(X) \to KR_G^*(X)$) and $a \in K^*(X)$ (resp. $a \in KR^*(X)$), we call $s(a)$ (resp. $s_R(a)$) a (*Real*) *equivariant lift* of a, with respect to s (resp. s_R).

We first prove a structure theorem of equivariant KR-theory of Real equivariantly formal spaces.

Theorem 2.5.5. *Let X be a Real equivariantly formal space. For any element $a \in K^*(X)$ (resp. $a \in KR^*(X)$), let $a_G \in K_G^*(X)$ (resp. $a_G \in KR_G^*(X)$) be a (Real) equivariant lift of a with respect to a group homomorphic section s (resp. s_R which is a $KR^*(\mathrm{pt})$-module homomorphism). Then the map*

$$f : (RR(G,\mathbb{R}) \oplus RH(G,\mathbb{R})) \otimes KR^*(X) \oplus r(R(G,\mathbb{C}) \otimes K^*(X)) \to KR_G^*(X),$$
$$\rho_1 \otimes a_1 \oplus r(\rho_2 \otimes a_2) \mapsto \rho_1 \cdot (a_1)_G \oplus r(\rho_2 \cdot (a_2)_G).$$

is a group isomorphism. In particular, if $R(G,\mathbb{C}) = 0$, then f is a $KR_G^*(\mathrm{pt})$-module isomorphism.

Proof. Consider the following $H(p,q)$-systems

$$HR^\alpha(p,q) := KR^{-\alpha}(X \times S^{q,0}, X \times S^{p,0}) \cong KR^{-\alpha+p}(X \times S^{q-p,0}),$$
$$H^\alpha(p,q) := K^{-\alpha}(X \times S^{q,0}, X \times S^{p,0}) \cong K^{-\alpha+p}(X \times S^{q-p,0}),$$
$$HR_G^\alpha(p,q) := KR_G^{-\alpha}(X \times S^{q,0}, X \times S^{p,0}) \cong KR_G^{-\alpha+p}(X \times S^{q-p,0}).$$

For the last $H(p,q)$-system, G acts on $S^{q-p,0}$ trivially. The spectral sequences induced by these $H(p,q)$-systems converge to $KR^*(X)$, $K^*(X)$ and $KR^*_G(X)$ respectively (for the assertion for the first two $H(p,q)$-systems, see the proofs of Theorem 3.1 and Lemma 4.1 of [Se]. That the third $H(p,q)$-system converges to $KR^*_G(X)$ follows from a straightforward generalization of the aforementioned proofs by adding equivariant structure throughout). Consider the two long exact sequences for the pair $(X \times B^{q-p,0}, X \times S^{q-p,0})$, with the top exact sequence involving equivariant KR-theory and the bottom one ordinary KR-theory, and the vertical maps being forgetful maps. By applying the five-lemma, we have that each element in the first two $H(p,q)$-systems has a (Real) equivariant lift. Define a group homomorphism

$$f(p,q) : \ (RR(G,\mathbb{R}) \oplus RH(G,\mathbb{R})) \otimes HR^\alpha(p,q) \oplus r(R(G,\mathbb{C}) \otimes H^\alpha(p,q)) \to HR^\alpha_G(p,q)$$

by

$$\rho_1 \otimes a_1 \oplus r(\rho_2 \otimes a_2) \mapsto \rho_1 \cdot (a_1)_G \oplus r(\rho_2 \cdot (a_2)_G).$$

As $RR(G,\mathbb{R})$, $RH(G,\mathbb{R})$ and $R(G,\mathbb{C})$ are free abelian groups, and tensoring free abelian groups and taking cohomology commute, f is the abutment of $f(p,q)$. On the $E_1^{p,q}$-page, $f(p,q)$ becomes

$$\begin{aligned} f_1^{p,q} : \ (RR(G,\mathbb{R}) \oplus RH(G,\mathbb{R})) \otimes KR^{-q}(X \times S^{1,0}) \oplus r(R(G,\mathbb{C}) \otimes K^{-q}(X \times S^{1,0})) \\ \to KR_G^{-q}(X \times S^{1,0}). \end{aligned}$$

Note that $K^{-q}(X \times S^{1,0}) \cong K^{-q}(X) \oplus K^{-q}(X)$, $KR^{-q}(X \times S^{1,0}) \cong K^{-q}(X)$, and $KR_G^{-q}(X \times$

$S^{1,0}) \cong K_G^{-q}(X)$. With the above identification,

$$r: \; R(G,\mathbb{C}) \otimes K^{-q}(X \times S^{1,0}) \to KR_G^{-q}(X \times S^{1,0}) \cong K_G^{-q}(X),$$
$$\rho_1 \otimes (a_1, 0) \oplus \rho_2 \otimes (0, a_2) \mapsto \rho_1 \cdot (a_1)_G + \overline{\sigma_G^*} \rho_2 \cdot (\overline{\sigma_G^*} a_2)_G.$$

So $r(R(G,\mathbb{C}) \otimes K^{-q}(X \times S^{1,0})) = R(G,\mathbb{C}) \otimes K^{-q}(X)$. $f_1^{p,q}$ is a group homomorphism from $(RR(G,\mathbb{R}) \oplus RH(G,\mathbb{R}) \oplus R(G,\mathbb{C})) \otimes K^{-q}(X) \cong R(G) \otimes K^{-q}(X)$ to $K_G^{-q}(X)$, which is an isomorphism by weak equivariant formality of X. It follows that f is also an isomorphism. If $R(G,\mathbb{C}) = 0$, then by (1) of Proposition 2.4.3 and (3) of Definition 2.5.2, f is indeed a $KR_G^*(\mathrm{pt})$-module isomorphism. □

Remark 2.5.6. The term 'Real equivariant formality' is suggested by the observation that, if X is a Real equivariantly formal G-space and $R(G,\mathbb{C}) = 0$, then the map

$$KR_G^*(X) \otimes_{RR(G,\mathbb{R}) \oplus RH(G,\mathbb{R})} \mathbb{Z} \to KR^*(X)$$

induced by the forgetful map is a ring isomorphism, which smacks of the ring isomorphism in the definition of weak equivariant formality.

Lemma 2.5.7. *$\delta_{\mathbb{R}}(\varphi_i)$, $\delta_{\mathbb{H}}(\theta_j)$, λ_k and $r_{i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t} \in KR^*(G)$ all have Real equivariant lifts in $KR_G^*(G)$. Hence G is a Real equivariantly formal space.*

Proof. A natural choice of a Real equivariant lift of $\delta_{\mathbb{R}}(\varphi_i)$ is represented by the complex of vector bundles in Proposition 2.1.2 equipped with both the Real structure and equivariant structure defined for $\delta_{\mathbb{R}}(\varphi_i)$ and $\delta_G(\varphi)$ respectively. These two structures are easily seen to be compatible. A Real equivariant lift of $\delta_{\mathbb{H}}(\theta_j)$ can be similarly defined. The class

$$r(\beta^i \cdot \delta_G(\gamma_1)^{\varepsilon_1} \cdots \delta_G(\gamma_t)^{\varepsilon_t} \delta_G(\overline{\sigma_G^*}\gamma_1)^{\nu_1} \cdots \delta_G(\overline{\sigma_G^*}\gamma_t)^{\nu_t})$$

obviously is a Real equivariant lift of $r_{i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t}$. By adding the natural equivariant structure throughout the construction of λ_k in the proof of Proposition 4.6 in [Se], one can obtain a Real equivariant lift of λ_k. $\square$

Definition 2.5.8. We fix a choice of equivariant lift of any element $a \in K^*(G)$ by defining $\delta_G(\rho)$ to be the equivariant lift of $\delta(\rho)$. Similarly, we fix a choice of Real equivariant lift of $a \in KR^*(G)$ by defining $\delta^G_{\mathbb{R}}(\varphi_i)$, $\delta^G_{\mathbb{H}}(\theta_j)$, λ^G_k, and $r^G_{i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t}$ in the proof of Lemma 2.5.7 to be the Real equivariant lift of $\delta_{\mathbb{R}}(\varphi_i)$, $\delta_{\mathbb{H}}(\theta_j)$, λ_k and $r_{i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t}$.

Remark 2.5.9. λ^G_k satisfies $c(\lambda^G_k) = \beta^3\delta_G(\gamma_k)\delta_G(\overline{\sigma_G^*}\gamma_k)$.

Corollary 2.5.10. *Let G be a compact, connected and simply-connected Real Lie group. The map*

$$f : (RR(G,\mathbb{R}) \oplus RH(G,\mathbb{R})) \otimes KR^*(G) \oplus r(R(G,\mathbb{C}) \otimes K^*(G)) \to KR^*_G(G),$$
$$\rho_1 \otimes a_1 \oplus r(\rho_2 \otimes a_2) \mapsto \rho_1 \cdot (a_1)_G \oplus r(\rho_2 \cdot (a_2)_G)$$

is a group isomorphism. Here $(a_i)_G$ is the (Real) equivariant lift defined as in Definition 2.5.8. In particular, if $R(G,\mathbb{C}) = 0$, then f is an isomorphism of $KR^*_G(\mathrm{pt})$-modules from $KR^*_G(\mathrm{pt}) \otimes K^*(G)$ to $KR^*_G(G)$.

Proof. The result follows from Theorem 2.5.5 and Lemma 2.5.7. In the special case where $R(G,\mathbb{C}) = 0$, $KR^*(G)$ is isomorphic to $KR^*(pt) \otimes K^*(G)$ as $KR^*(\mathrm{pt})$-modules by (1) of Theorem 2.3.5, and applying Theorem 2.5.5 and Proposition 2.4.3 give $KR^*_G(G) \cong RR(G) \otimes KR^*(G) \cong RR(G) \otimes KR^*(\mathrm{pt}) \otimes K^*(G) \cong KR^*_G(\mathrm{pt}) \otimes K^*(G)$. In this way f is a $KR^*_G(\mathrm{pt})$-module isomorphism from $KR^*_G(\mathrm{pt}) \otimes K^*(G)$ to $KR^*_G(G)$. $\square$

Corollary 2.5.11. *Let*

$$r^G_{\rho,i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t} := r(\beta^i \cdot \rho\delta_G(\gamma_1)^{\varepsilon_1} \cdots \delta_G(\gamma_t)^{\varepsilon_t}\delta_G(\overline{\sigma^*_G}\gamma_1)^{\nu_1} \cdots \delta_G(\overline{\sigma^*_G}\gamma_t)^{\nu_t}),$$

where $\rho \in R(G,\mathbb{C}) \oplus \mathbb{Z} \cdot \rho_{\mathrm{triv}}$ and $\varepsilon_1,\ldots,\varepsilon_t$, $\nu_1,\ldots,\nu_t$ are as in Theorem 2.3.1. Then $KR^*_G(G)$, as an algebra over $KR^*_G(pt)$, is generated by $\delta^G_{\mathbb{R}}(\varphi_1),\ldots,\delta^G_{\mathbb{R}}(\varphi_r)$, $\delta^G_{\mathbb{H}}(\theta_1),\ldots,\delta^G_{\mathbb{H}}(\theta_s)$, $\lambda_1,\ldots,\lambda_t$, and $r^G_{\rho,i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t}$.

Remark 2.5.12. If $\rho = \rho_{\mathrm{triv}}$, then $r^G_{\rho,i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t} = r^G_{i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t}$. If $\rho \in R(G,\mathbb{C})$, then $r^G_{\rho,i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t}$ comes from $r(R(G,\mathbb{C}) \otimes K^*(G))$ in the decomposition of Theorem 2.5.5.

Now we are in a position to compute $(\lambda^G_k)^2$ by imitating the proof of Proposition 4.7 in [Se].

Proposition 2.5.13. $(\lambda^G_k)^2 = 0$.

Proof. Consider the Real Lie group $(U(n) \times U(n), \sigma_{\mathbb{C}})$, where $\sigma_{\mathbb{C}}(g_1, g_2) = (\overline{g_2}, \overline{g_1})$. Let $p_j : U(n) \times U(n) \to U(n)$ be the projection onto the j-th factor, and $u_i = p_1^*(\wedge^i\sigma_n)$, $v_i = p_2^*(\wedge^i\sigma_n)$. Thus $\overline{\sigma^*_{\mathbb{C}}}u_i = v_i$. A decomposition of $K^*(U(n) \times U(n))$ by the induced involution $\overline{\sigma^*_{\mathbb{C}}}$ is given by $M \oplus T \oplus \overline{\sigma^*_{\mathbb{C}}}T$, where M is the subalgebra generated by $\delta(u_1)\delta(v_1),\ldots,\delta(u_n)\delta(v_n)$. By Proposition 2.3.1, there exist $h_1,\ldots,h_n \in KR^0(U(n) \times U(n), \sigma_{\mathbb{C}})$ such that $c(h_i) = \beta^3\delta(u_i)\delta(v_i)$, and $KR^*(U(n) \times U(n), \sigma_{\mathbb{C}}) \cong F \oplus r(K^*(+) \otimes T)$, where F is the $KR^*(\mathrm{pt})$-module freely generated by monomials in $h_1, \ldots, h_n$. By Corollary 2.5.10,

$$KR^*_{(U(n)\times U(n),\sigma_{\mathbb{C}})}(U(n)\times U(n),\sigma_{\mathbb{C}}) \cong RR(U(n)\times U(n),\sigma_{\mathbb{C}},\mathbb{R})\otimes(F\oplus r(K^*(+)\otimes T))$$
$$\oplus\, r(R(U(n)\times U(n),\sigma_{\mathbb{C}},\mathbb{C})\otimes K^*(U(n)\times U(n))).$$

Let h_i^G be the equivariant lift of h_i as defined in Definition 2.5.8. So $c(h_i^G) = \beta^3\delta_G(u_i)\delta_G(v_i)$ and $c((h_i^G)^2) = 0$. Consequently $(h_i^G)^2 = \eta k_i$ for some $k_i \in KR^{-7}_{(U(n)\times U(n),\sigma_{\mathbb{C}})}(U(n)\times U(n),\sigma_{\mathbb{C}})$ (cf. Gysin sequence (3.4) in [At3] and its equivariant analogue). Since $\eta\cdot r(\cdot) = 0$, we may assume that k_i is from the component $RR(U(n)\times U(n),\sigma_{\mathbb{C}},\mathbb{R})\otimes F$. But the degree -7 piece of the later is 0. So $(h_i^G)^2 = 0$.

Consider the map

$$\gamma_k\times\overline{\sigma^*_{\mathbb{C}}}\gamma_k :\ (G,\sigma_G)\to(U(n)\times U(n),\sigma_{\mathbb{C}}),$$
$$g\mapsto(\gamma_k(g),\overline{\gamma_k(\sigma_G(g))}).$$

It can be easily seen that $(\gamma_k\times\overline{\sigma^*_{\mathbb{C}}}\gamma_k)^*(h_1^G) = \lambda_k^G$. So $(\lambda_k^G)^2 = 0$. □

2.5.2 The module structure of $KR^*_{(U(n),\sigma_{\mathbb{F}})}(U(n),\sigma_{\mathbb{F}})$

Definition 2.5.14. Let σ_n be (the class of) the standard representation of $U(n)$.

Proposition 2.5.15. *$\sigma_n, \wedge^2\sigma_n, \ldots, \wedge^n\sigma_n \in RR(U(n),\sigma_{\mathbb{R}},\mathbb{R})$, $\wedge^{2i}\sigma_{2m}\in RR(U(2m),\sigma_{\mathbb{H}},\mathbb{R})$ and $\wedge^{2i+1}\sigma_{2m}\in RH(U(2m),\sigma_{\mathbb{H}},\mathbb{R})$. Also, both $R(U(n),\sigma_{\mathbb{R}},\mathbb{C})$ and $R(U(2m),\sigma_{\mathbb{H}},\mathbb{C})$ are* 0.

Proof. For the involution $\sigma_{\mathbb{R}}$ and $\wedge^i\sigma_n$, define the bilinear form

$$B_{\mathbb{R}}: \wedge^i\sigma_n \times \sigma_{\mathbb{R}}^* \wedge^i \sigma_n \to \mathbb{C},$$
$$(v_1 \wedge \cdots \wedge v_i, w_1 \wedge \cdots \wedge w_i) \mapsto \det(\langle v_j, \overline{w_k}\rangle).$$

Obviously the form is $U(n)$-invariant, symmetric and non-degenerate. By Proposition 2.2.10, each of $\wedge^i\sigma_n$, $1 \leq i \leq n$ is a Real representation of real type. Similarly, define, for the involution $\sigma_{\mathbb{H}}$ and $\wedge^i\sigma_{2m}$, a bilinear form

$$B_{\mathbb{H}}: \wedge^i\sigma_{2m} \times \sigma_{\mathbb{H}}^*\wedge^i\sigma_{2m} \to \mathbb{C},$$
$$(v_1 \wedge \cdots \wedge v_i, w_1 \wedge \cdots \wedge w_i) \mapsto \det(\langle J_m v_j, \overline{w_k}\rangle).$$

It is $U(n)$-invariant because

$$\begin{aligned} B_{\mathbb{H}}(gv, gw) &= \det(\langle J_m g v_j, \overline{J_m \overline{g} J_m^{-1} w_k}\rangle) \\ &= \det(\langle J_m g v_j, J_m g J_m^{-1} \overline{w_k}\rangle) \\ &= \det(\langle v_j, J_m^{-1}\overline{w_k}\rangle) \\ &= \det(\langle J_m v_j, \overline{w_k}\rangle). \end{aligned}$$

Moreover

$$\begin{aligned} B_{\mathbb{H}}(v, w) &= \det(\langle J_m v_j, \overline{w_k}\rangle) \\ &= \det(\langle -v_j, J_m \overline{w_k}\rangle) \\ &= \det(-\overline{\langle J_m \overline{w_k}, v_j\rangle}) \\ &= \det(-\langle J_m w_k, \overline{v_j}\rangle) \\ &= (-1)^i B_{\mathbb{H}}(w, v). \end{aligned}$$

So by Propositions 2.2.8, 2.2.10 and 2.2.14, $\wedge^i \sigma_{2m}$ is a Real representation of real type when i is even and a Quaternionic representation of real type when i is odd. There are no complex representations of complex type because $\wedge^i \sigma_n \cong \sigma_{\mathbb{F}}^* \overline{\wedge^i \sigma_n}$ for $\mathbb{F} = \mathbb{R}$ and $\mathbb{H}$. □

Lemma 2.5.16. *$KR^*_{(U(n),\sigma_{\mathbb{F}})}(U(n),\sigma_{\mathbb{F}})$ is isomorphic to $\Omega^*_{KR^*_{(U(n),\sigma_{\mathbb{F}})}(pt)/KR^*(pt)}$ as $KR^*_{(U(n),\sigma_{\mathbb{F}})}(pt)$-modules, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{H}$.*

Proof. Theorem 2.5.5 implies that, $KR^*_{(U(n),\sigma_{\mathbb{R}})}(U(n), \sigma_{\mathbb{R}}) \cong RR(U(n), \sigma_{\mathbb{R}}) \otimes KR^*(U(n), \sigma_{\mathbb{R}})$ and $KR^*_{(U(n),\sigma_{\mathbb{H}})}(U(n), \sigma_{\mathbb{H}}) \cong (RR(U(n), \sigma_{\mathbb{H}}, \mathbb{R}) \oplus RH(U(n), \sigma_{\mathbb{H}}, \mathbb{R})) \otimes KR^*(U(n), \sigma_{\mathbb{H}})$. Moreover, by Theorem 2.3.8 and Proposition 2.5.15,

$$KR^*(U(n), \sigma_{\mathbb{R}}) \cong \bigwedge\nolimits_{KR^*(\mathrm{pt})}(\delta_{\mathbb{R}}(\sigma_n), \ldots, \delta_{\mathbb{R}}(\wedge^n \sigma_n))$$

and

$$KR^*(U(2m), \sigma_{\mathbb{H}}) \cong \bigwedge\nolimits_{KR^*(\mathrm{pt})}(\delta_{\mathbb{H}}(\sigma_{2m}), \delta_{\mathbb{R}}(\wedge^2 \sigma_{2m}), \ldots, \delta_{\mathbb{R}}(\wedge^{2m} \sigma_{2m})).$$

Putting all these together and applying Theorem 2.4.3, we get the desired conclusion. □

Remark 2.5.17. As ungraded $KR^*(\mathrm{pt})$-modules, both $KR^*_{(U(2m),\sigma_{\mathbb{R}})}(U(2m), \sigma_{\mathbb{R}})$ and $KR^*_{(U(2m),\sigma_{\mathbb{H}})}(U(2m), \sigma_{\mathbb{H}})$ are isomorphic to $K^*_{U(2m)}(U(2m)) \otimes KR^*(\mathrm{pt})$.

2.5.3 Injectivity results

This step involves proving that the restriction map p_G^* to the equivariant KR-theory of the maximal torus and the map q_G^* induced by the Weyl covering map are injective.

Lemma 2.5.18. *Let G be a compact Lie group and X a G-space. Let $i_1^* : K_G^*(X) \to K_T^*(X)$ be the map which restricts the G-action to T-action. Then*

$$i_1^* \otimes \mathrm{Id}_R : \; K_G^*(X) \otimes R \to K_T^*(X) \otimes R$$

is injective for any ring R.

Proof. By [At4, Proposition 4.9], i_1^* is split injective. So is $i_1^* \otimes \mathrm{Id}_R$ for any ring R. □

Lemma 2.5.19. *Let $i_2^* : K_T^*(G) \to K_T^*(T) \cong R(T) \otimes K^*(T)$ be the map induced by the inclusion $T \hookrightarrow G$. Then*

$$i_2^* \delta_T(\rho) = \sum_{j=1}^{\dim \rho} e^{\tau_j} \otimes \delta(\tau_j) \in K_T^{-1}(T),$$

where τ_j are the weights of ρ.

Proof. Let V be the vector space underlying the representation ρ. $\delta_T(\rho)$ is represented by the complex of T-equivariant vector bundles

$$0 \longrightarrow G \times \mathbb{R} \times V \longrightarrow G \times \mathbb{R} \times V \longrightarrow 0,$$

$$(g, t, v) \mapsto (g, t, -t\rho(g)v) \qquad \text{if } \; t \geq 0,$$

$$(g, t, v) \mapsto (g, t, tv) \qquad \text{if } \; t \leq 0$$

which, on restricting to

$$0 \longrightarrow T \times \mathbb{R} \times V \longrightarrow T \times \mathbb{R} \times V \longrightarrow 0$$

is decomposed into a direct sum of complexes of 1-dimensional T-equivariant vector bundles, each of which corresponds to a weight of ρ. □

Lemma 2.5.20. *Let G be a simply-connected, connected compact Lie group and $\rho_1, \ldots, \rho_l$ be its fundamental representations. Then*

$$i_2^*\left(\prod_{i=1}^{l} \delta_T(\rho_i)\right) = d_G \otimes \prod_{i=1}^{l} \delta(\varpi_i), \tag{2.4}$$

where ϖ_i the i-th fundamental weight and $d_G = \sum_{w \in W} sgn(w) e^{w \cdot \sum_{i=1}^{l} \varpi_i} \in R(T)$ is the Weyl denominator.

Proof. Equation (2.4) follows from Lemma 2.5.19 and Lemma 3 of [At2]. □

Lemma 2.5.21. *Let M be an R-module freely generated by $m_1, \ldots, m_l$, and N an R-module. If $f : M \to N$ is an R-module homomorphism, and $rf(m_1) \wedge \cdots \wedge f(m_l) \in \bigwedge_R^l N$ is nonzero for all $r \in R \setminus \{0\}$, then*

$$\bigwedge^* f : \bigwedge_R^* M \to \bigwedge_R^* N$$

is injective.

Proof. It suffices to show that $\bigwedge^k f$ is injective for $1 \leq k \leq l$. Suppose $I \subseteq \{1, \ldots, l\}$, $|I| = k$, $m_I := \bigwedge_{i \in I} m_i$ and $f(m_I) := \bigwedge_{i \in I} f(m_i)$. If $\sum_{|I|=k} r_I m_I \in \ker(\bigwedge^k f)$, then for any J with $|J| = k$,

$$0 = \sum_{|I|=k} r_I f(m_I) \wedge f(m_{J^c}) = r_J f(m_1) \wedge \cdots \wedge f(m_l).$$

Hence $\sum_{|I|=k} r_I m_I = 0$ and the conclusion follows. □

Lemma 2.5.22. *Let G be a simply-connected, connected and compact Lie group. Then the map*

$$i_2^* \otimes \mathrm{Id}_R : K_T^*(G) \otimes R \to K_T^*(T) \otimes R$$

is injective for any ring R.

Proof. Note that

$$K_G^*(G) \otimes_{R(G)} R(T) \to K_T^*(G),$$
$$a \otimes \rho \mapsto i_1^*(a) \cdot \rho.$$

is an $R(T)$-algebra isomorphism (cf. [HLS, Theorem 4.4]). Using Theorem 2.1.7, we have that $K_T^*(G)$ is isomorphic, as an $R(T)$-algebra, to $\bigwedge^*_{R(T)} M$, where M is the $R(T)$-module freely generated by $\delta_T(\rho_1), \ldots, \delta_T(\rho_l)$. We also observe that $K_T^*(T)$ is isomorphic, as an $R(T)$-algebra, to $\bigwedge^*_{R(T)} N$, where N is the $R(T)$-module freely generated by $\delta(\varpi_1), \ldots, \delta(\varpi_l)$. Note that the hypotheses of Lemma 2.5.21 are satisfied by $f = i_2^* \otimes \mathrm{Id}_{\mathbb{Z}_m}$ for any $m \geq 2$, as $r \prod_{i=1}^{l} i_2^* \delta_T(\rho_i) = r i_2^* \prod_{i=1}^{l} \delta_T(\rho_i) = r d_G \otimes \prod_{i=1}^{l} \delta(\varpi_i)$ (by Lemma 2.5.20) is indeed nonzero for any nonzero r in $\mathbb{Z}_m$ (the coefficients of d_G are either 1 or -1, so after reduction mod m rd_G is still nonzero). Now that $i_2^* \otimes \mathbb{Z}_m$ is injective, so is $i_2^* \otimes \mathrm{Id}_R$ for any ring R. $\square$

Proposition 2.5.23. 1. *Let G be a compact, connected and simply-connected Real Lie group such that $RR(G, \mathbb{C}) = RR(G, \mathbb{H}) = 0$ and there exists a maximal torus T on which the involution acts by inversion. Then the restriction map*

$$KR_G^*(G) \to KR_T^*(T)$$

is injective.

2. *The map $p_G^* : KR^*_{(U(n),\sigma_{\mathbb{R}})}(U(n), \sigma_{\mathbb{R}}) \to KR^*_{(T,\sigma_{\mathbb{R}})}(T, \sigma_{\mathbb{R}})$ is injective.*

Proof. By Corollary 2.5.10, $KR^*_G(G) \cong K^*_G(G) \otimes KR^*(\mathrm{pt})$ and $KR^*_T(T) \cong K^*_T(T) \otimes KR^*(\mathrm{pt})$ as $KR^*(\mathrm{pt})$-modules. Using this identification, we can as well identify the restriction map with $i^* \otimes \mathrm{Id}_{KR^*(\mathrm{pt})}$, where $i^* := i_2^* \circ i_1^*$. Part (1) then follows from Lemmas 2.5.18 and 2.5.22. Part (2) is immediate once we apply Lemma 2.5.18 and note that the proofs of Lemmas 2.5.20 and 2.5.22 can be adapted to the case $G = U(n)$ by letting $\sigma_n, \dots, \wedge^n \sigma_n$ play the role of the fundamental representations and their highest weights, the fundamental weights. $\square$

Lemma 2.5.24. *$KR^*_{(U(2m),\sigma_{\mathbb{H}})}(U(2m)/T, \sigma_{\mathbb{H}}) \cong \mathbb{Z}[e_1^{\mathbb{H}}, \dots, e_{2m}^{\mathbb{H}}, (e_1^{\mathbb{H}} e_2^{\mathbb{H}} \cdots e_{2m}^{\mathbb{H}})^{-1}] \otimes KR^*(\mathrm{pt})$ as rings, where $e_i^{\mathbb{H}}$ lives in the degree -4 piece.*

Proof. It is known that $K^*(U(2m)/T) \cong \mathbb{Z}[\alpha_1, \dots, \alpha_{2m}]/(s_i - \binom{2m}{i} | 1 \leq i \leq 2m)$, where $\alpha_i = [U(2m) \times_T \mathbb{C}_{e_i}]$ and s_i is the i-th elementary symmetric polynomial (cf. [At, Proposition 2.7.13]). The induced map $\overline{\sigma_{\mathbb{H}}^*}$ acts as identity on $K^*(U(2m)/T)$. The involution $\sigma_{\mathbb{H}}$ on the base lifts to a Quaternionic structure on the associated complex line bundle, so there exist $\alpha_1^{\mathbb{H}}, \dots, \alpha_{2m}^{\mathbb{H}} \in KR^{-4}(U(2m)/T)$, such that their complexifications are $\beta^2\alpha_1, \dots, \beta^2\alpha_{2m} \in K^*(U(2m)/T)$. By Theorem 2.3.1, $KR^*(U(2m)/T, \sigma_{\mathbb{H}})$ is a $KR^*(\mathrm{pt})$-module generated by polynomials in $\alpha_1^{\mathbb{H}}, \dots, \alpha_{2m}^{\mathbb{H}} \in KR^{-4}(U(2m)/T, \sigma_{\mathbb{H}})$. In fact it is not hard to see that $KR^*(U(2m)/T, \sigma_{\mathbb{H}})$ is isomorphic to

$$\mathbb{Z}[\alpha_1^{\mathbb{H}}, \dots, \alpha_{2m}^{\mathbb{H}}] \otimes KR^*(\mathrm{pt}) \Bigg/ \left(s_{2k} - \binom{2m}{2k}, s_{2k-1} - \frac{1}{2}\mu \binom{2m}{2k-1} \,\middle|\, 1 \leq k \leq m \right).$$

Also obvious is that each of $\alpha_i^{\mathbb{H}}$ has an equivariant lift $e_i^{\mathbb{H}} \in KR^{-4}_{(U(2m),\sigma_{\mathbb{H}})}(U(2m)/T, \sigma_{\mathbb{H}})$.

Now that all the hypotheses in Theorem 2.5.5 are satisfied, we can apply it, together with the fact that $R(U(2m), \sigma_{\mathbb{H}}, \mathbb{C}) = 0$ (cf. Proposition 2.5.15) to see that $KR^*_{(U(2m),\sigma_{\mathbb{H}})}(U(2m)/T, \sigma_{\mathbb{H}})$ is isomorphic to

$$(RR(U(2m), \sigma_{\mathbb{H}}, \mathbb{R}) \oplus RH(U(2m), \sigma_{\mathbb{H}}, \mathbb{R})) \otimes KR^*(U(2m)/T, \sigma_{\mathbb{H}})$$

as $RR(U(2m), \sigma_{\mathbb{H}}, \mathbb{R}) \oplus RH(U(2m), \sigma_{\mathbb{H}}, \mathbb{R})$-modules (actually as rings). Noting that

$$RR(U(2m), \sigma_{\mathbb{H}}, \mathbb{R}) \oplus RH(U(2m), \sigma_{\mathbb{H}}, \mathbb{R}) \cong \mathbb{Z}[s_1, \ldots, s_{2m}, s_{2m}^{-1}]$$

we establish the Lemma. $\square$

Proposition 2.5.25. $KR^*_{(U(2m),\sigma_{\mathbb{H}})}(U(2m)/T \times T, \sigma_{\mathbb{H}} \times \sigma_{\mathbb{R}})$ *is isomorphic to*

$$\mathbb{Z}[e_1^{\mathbb{H}}, \ldots, e_{2m}^{\mathbb{H}}, (e_1^{\mathbb{H}} \cdots e_{2m}^{\mathbb{H}})^{-1}] \otimes KR^*(T, \sigma_{\mathbb{R}})$$

as graded rings.

Proof. First, by [P, Theorem 1], Proposition 2.4.3 and Lemma 2.5.24, $KR^*_{(U(2m),\sigma_{\mathbb{H}})}(U(2m)/T, \sigma_{\mathbb{H}})$ is a free $KR^*_{(U(2m),\sigma_{\mathbb{H}})}(\mathrm{pt})$-module. The same is also true of $KR^*_{(U(2m),\sigma_{\mathbb{H}})}(T, \sigma_{\mathbb{R}})$ since, by Theorem 2.5.5, it is isomorphic to $RR(U(2m), \sigma_{\mathbb{H}}) \otimes KR^*(T, \sigma_{\mathbb{R}})$, which in turn is isomorphic to $KR^*_{(U(2m),\sigma_{\mathbb{H}})}(\mathrm{pt}) \otimes K^*(T)$. The proposition follows from a version of Künneth formula for equivariant KR-theory. $\square$

Remark 2.5.26. $KR^*_{(U(2m),\sigma_{\mathbb{H}})}(U(2m)/T \times T, \sigma_{\mathbb{H}} \times \sigma_{\mathbb{R}})$ is isomorphic to $K^*_T(T) \otimes KR^*(\mathrm{pt})$ and $KR^*_{(T,\sigma_{\mathbb{R}})}(T, \sigma_{\mathbb{R}})$ as ungraded $KR^*(\mathrm{pt})$-modules.

Proposition 2.5.27. *The map q_G^* is injective.*

Proof. By Remarks 2.5.17 and 2.5.26, $KR^*_{(U(2m),\sigma_{\mathbb{H}})}(U(2m),\sigma_{\mathbb{H}})$ and $KR^*_{(U(2m),\sigma_{\mathbb{H}})}(U(2m)/T\times T,\sigma_{\mathbb{H}}\times\sigma_{\mathbb{R}})$ are isomorphic to $KR^*_{(U(2m),\sigma_{\mathbb{R}})}(U(2m),\sigma_{\mathbb{R}})$ and $KR^*_{(T,\sigma_{\mathbb{R}})}(T,\sigma_{\mathbb{R}})$ respectively, as ungraded $KR^*(\mathrm{pt})$-modules. It is not hard to see that q_G^* can be identified with p_G^* under these isomorphisms. Now the result follows from Proposition 2.5.23. □

2.5.4 Squares of algebra generators of real and quaternionic types

Lemma 2.5.28. *$KR^*(T,\sigma_{\mathbb{R}})$ is isomorphic to the exterior algebra over $KR^*(\mathrm{pt})$ generated by $\delta_{\mathbb{R}}(e_1),\dots,\delta_{\mathbb{R}}(e_n)$, as $KR^*_{(T,\sigma_{\mathbb{R}})}(\mathrm{pt})$-modules. Here e_i is the 1-dimensional complex representation with weight being the i-th standard basis vector of the weight lattice. Moreover, $\delta_{\mathbb{R}}(e_i)^2=\eta\delta_{\mathbb{R}}(e_i)$.*

Proof. Since $R(T,\sigma_{\mathbb{R}},\mathbb{C})=0$, the module structure follows from Theorem 2.3.8. For the second part of the Lemma, see the appendix of [Se]. □

Proposition 2.5.29. *In $KR^*_{(U(n),\sigma_{\mathbb{R}})}(U(n),\sigma_{\mathbb{R}})$*

$$\delta^G_{\mathbb{R}}(\sigma_n)^2=\eta(\sigma_n\cdot\delta^G_{\mathbb{R}}(\sigma_n)-\delta^G_{\mathbb{R}}(\wedge^2\sigma_n)). \tag{2.5}$$

In $KR^*_{(U(2m),\sigma_{\mathbb{H}})}(U(2m),\sigma_{\mathbb{H}})$,

$$\delta^G_{\mathbb{H}}(\sigma_{2m})^2=\eta(\sigma_{2m}\cdot\delta^G_{\mathbb{H}}(\sigma_{2m})-\delta^G_{\mathbb{R}}(\wedge^2\sigma_{2m})). \tag{2.6}$$

Proof. Now that we have shown that p_G^* and q_G^* are injective by Propositions 2.5.23 and 2.5.27, we can compute $\delta_{\mathbb{R}}^G(\sigma_n)^2$ and $\delta_{\mathbb{H}}^G(\sigma_{2m})^2$ by passing the computation through p_G^* and q_G^* to their images. We prove the case $\mathbb{F} = \mathbb{R}$. The proof of the case $\mathbb{F} = \mathbb{H}$ is similar so we leave it to the reader. Note that

$$
\begin{aligned}
p_G^*(\delta_{\mathbb{R}}^G(\sigma_n)^2) &= p_G^*(\delta_{\mathbb{R}}^G(\sigma_n))^2 \\
&= \left(\sum_{i=1}^n e_i \otimes \delta_{\mathbb{R}}(e_i)\right)^2 \\
&= \sum_{i=1}^n e_i^2 \otimes \eta\delta_{\mathbb{R}}(e_i) + \sum_{i\neq j} e_i e_j \otimes \delta_{\mathbb{R}}(e_i)\delta_{\mathbb{R}}(e_j) \\
&\overset{\text{Lemma 2.5.28}}{=} \sum_{i=1}^n e_i^2 \otimes \eta\delta_{\mathbb{R}}(e_i) + \sum_{i<j} e_i e_j \otimes (\delta_{\mathbb{R}}(e_i)\delta_{\mathbb{R}}(e_j) + \delta_{\mathbb{R}}(e_j)\delta_{\mathbb{R}}(e_i)) \\
&= \sum_{i=1}^n e_i^2 \otimes \eta\delta_{\mathbb{R}}(e_i).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
p^*(\sigma_n \cdot \delta_{\mathbb{R}}^G(\sigma_n)) &= \left(\sum_{i=1}^n e_i \otimes 1\right)\left(\sum_{i=1}^n e_i \otimes \delta_{\mathbb{R}}(e_i)\right) \\
&= \sum_{1\leq i,j\leq n} e_i e_j \otimes \delta_{\mathbb{R}}(e_i) \\
&= \sum_{i=1}^n e_i^2 \otimes \delta_{\mathbb{R}}(e_i) + \sum_{i\neq j} e_j e_i \otimes \delta_{\mathbb{R}}(e_i) \\
&= \sum_{i=1}^n e_i^2 \otimes \delta_{\mathbb{R}}(e_i) + \sum_{1\leq i<j\leq n} e_j e_i \otimes (\delta_{\mathbb{R}}(e_i) + \delta_{\mathbb{R}}(e_j)) \\
&= \sum_{i=1}^n e_i^2 \otimes \delta_{\mathbb{R}}(e_i) + p_G^*(\delta_{\mathbb{R}}^G(\wedge^2\sigma_n)).
\end{aligned}
$$

From the above equations we obtain

$$
p_G^*(\delta_{\mathbb{R}}^G(\sigma_n)^2) = \eta p_G^*(\sigma_n \cdot \delta_{\mathbb{R}}^G(\sigma_n) - \delta_{\mathbb{R}}^G(\wedge^2\sigma_n)).
$$

By Proposition 2.5.23,

$$\delta^G_{\mathbb{R}}(\sigma_n)^2 = \eta(\sigma_n \cdot \delta^G_{\mathbb{R}}(\sigma_n) - \delta^G_{\mathbb{R}}(\wedge^2\sigma_n)). \qquad \square$$

Theorem 2.5.30. *Let G be a Real compact Lie group. Then*

$$\delta^G_{\mathbb{R}}(\varphi_i)^2 = \eta(\varphi_i \cdot \delta^G_{\mathbb{R}}(\varphi_i) - \delta^G_{\mathbb{R}}(\wedge^2\varphi_i)), \tag{2.7}$$

$$\delta^G_{\mathbb{H}}(\theta_j)^2 = \eta(\theta_j \cdot \delta^G_{\mathbb{H}}(\theta_j) - \delta^G_{\mathbb{R}}(\wedge^2\theta_j)). \tag{2.8}$$

Proof. The induced map $\varphi_i^* : KR^*_{(U(n),\sigma_{\mathbb{R}})}(U(n), \sigma_{\mathbb{R}}) \to KR^*_{(G,\sigma_G)}(G, \sigma_G)$ sends σ_n to φ_i, and $\delta^G_{\mathbb{R}}(\sigma_n)$ to $\delta^G_{\mathbb{R}}(\varphi_i)$. Likewise, the induced map $\theta_j^* : KR^*_{(U(2m),\sigma_{\mathbb{H}})}(U(2m), \sigma_{\mathbb{H}})$ sends σ_{2m} to θ_j, and $\delta^G_{\mathbb{H}}(\sigma_{2m})$ to $\delta^G_{\mathbb{H}}(\theta_j)$. The result now follows from Proposition 2.5.29. $\square$

To further express $\delta^G_{\mathbb{R}}(\wedge^2\varphi_i)$ and $\delta^G_{\mathbb{R}}(\wedge^2\theta_j)$ in terms of the module generators associated with the fundamental representations, we may use the following derivation property of $\delta^G_{\mathbb{R}}$ and $\delta^G_{\mathbb{H}}$.

Proposition 2.5.31. *$\delta^G_{\mathbb{R}} \oplus \delta^G_{\mathbb{H}}$ is a derivation of the graded ring $RR(G) \oplus RH(G)$ (with $RR(G)$ of degree 0 and $RH(G)$ of degree -4) taking values in the graded module $KR^{-1}_G(G) \oplus KR^{-5}_G(G)$.*

Proof. We refer the reader to the proof of Proposition 3.1 of [BZ] with the definition of $\delta_G(\rho)$ given there (which is incorrect) replaced by the one in Definition 2.1.5. One just need to simply check that the homotopy ρ_s in the proof for $t \geq 0$ intertwines with both $\sigma_{\mathbb{R}}$ and $\sigma_{\mathbb{H}}$. $\square$

Corollary 2.5.32. *In* $KR^*_{(U(n),\sigma_{\mathbb{R}})}(U(n),\sigma_{\mathbb{R}})$,

$$\delta^G_{\mathbb{R}}(\wedge^k\sigma_n)^2 = \eta\sum_{i=1}^{2k}\wedge^{2k-i}\sigma_n\cdot\delta^G_{\mathbb{R}}(\wedge^i\sigma_n).$$

In $KR^*_{(U(2m),\sigma_{\mathbb{H}})}(U(2m),\sigma_{\mathbb{H}})$,

$$\delta^G_{\mathbb{H}}(\wedge^{2k-1}\sigma_{2m})^2 = \eta\sum_{j=1}^{2k-1}\left(\wedge^{4k-2j-1}\sigma_{2m}\cdot\delta^G_{\mathbb{H}}(\wedge^{2j-1}\sigma_{2m}) + \wedge^{4k-2j-2}\sigma_{2m}\cdot\delta^G_{\mathbb{R}}(\wedge^{2j}\sigma_{2m})\right),$$

$$\delta^G_{\mathbb{R}}(\wedge^{2k}\sigma_{2m})^2 = \eta\sum_{j=1}^{2k}\left(\wedge^{4k-2j+1}\sigma_{2m}\cdot\delta^G_{\mathbb{H}}(\wedge^{2j-1}\sigma_{2m}) + \wedge^{4k-2j}\sigma_{2m}\cdot\delta^G_{\mathbb{R}}(\wedge^{2j}\sigma_{2m})\right).$$

Proof. By the definition,

$$(\wedge^k\sigma_n)^*(\delta^G_{\mathbb{R}}(\wedge^2\sigma_{\binom{n}{k}})) = \delta^G_{\mathbb{R}}(\wedge^2(\wedge^k\sigma_n)).$$

By Exercise 15.32 of [FH],

$$\wedge^2(\wedge^k\sigma_n) = \bigoplus_i \Gamma_{\varpi_{k-2i+1}+\varpi_{k+2i-1}}.$$

By Giambelli's formula,

$$\Gamma_{\varpi_{k-2i+1}+\varpi_{k+2i-1}} = \wedge^{k+2i-1}\sigma_n\cdot\wedge^{k-2i+1}\sigma_n - \wedge^{k+2i}\sigma_n\cdot\wedge^{k-2i}\sigma_n.$$

By Proposition 2.5.31

$$\begin{aligned}\delta^G_{\mathbb{R}}(\Gamma_{\varpi_{k-2i+1}+\varpi_{k+2i-1}}) = {} & \wedge^{k+2i-1}\sigma_n\cdot\delta^G_{\mathbb{R}}(\wedge^{k-2i+1}\sigma_n) + \wedge^{k-2i+1}\sigma_n\cdot\delta^G_{\mathbb{R}}(\wedge^{k+2i-1}\sigma_n)\\ & - \wedge^{k+2i}\sigma_n\cdot\delta^G_{\mathbb{R}}(\wedge^{k-2i}\sigma_n) - \wedge^{k-2i}\sigma_n\cdot\delta^G_{\mathbb{R}}(\wedge^{k+2i}\sigma_n).\end{aligned}$$

Now the first equation is immediate. The second and third equations can be derived similarly. □

Putting together the previous results yields the following full description of the ring structure of $KR^*_G(G)$.

Theorem 2.5.33. *Let G be a simply-connected, connected and compact Real Lie group. Viewing G as a Real G-space with adjoint action, we have*

1. (*Corollary* 2.5.10) *The map*

$$f : (RR(G,\mathbb{R}) \oplus RH(G,\mathbb{R})) \otimes KR^*(G) \oplus r(R(G,\mathbb{C}) \otimes K^*(G)) \to KR^*_G(G)$$
$$\rho_1 \otimes a_1 \oplus r(\rho_2 \otimes a_2) \mapsto \rho_1 \cdot (a_1)_G \oplus r(\rho_2 \cdot (a_2)_G)$$

   is a group isomorphism. In particular, if $R(G,\mathbb{C}) = 0$, then f is an isomorphism of $KR^*_G(\mathrm{pt})$-modules.

2. (*Corollary* 2.5.11) *$KR_G(G)$ is generated by $\delta^G_{\mathbb{R}}(\varphi_1), \ldots, \delta^G_{\mathbb{R}}(\varphi_r)$, $\delta^G_{\mathbb{H}}(\theta_1), \ldots, \delta^G_{\mathbb{H}}(\theta_s)$, $\lambda^G_1, \ldots, \lambda^G_t$ and $r^G_{\rho,i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t}$ as an algebra over $KR^*_G(\mathrm{pt})$. Moreover,*

   (a) (*Proposition 2.5.13*) *$(\lambda^G_k)^2 = 0$ for all $1 \le k \le t$.*

   (b)

$$(r^G_{\rho,i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t})^2 = \begin{cases} \eta^2(\rho \cdot \overline{\sigma^*_G}\rho)(\lambda^G_1)^{\omega_1}\cdots(\lambda^G_t)^{\omega_t} & \text{if } r^G_{\rho,i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t} \\ & \text{is of degree } -1 \text{ or } -5, \\ \pm\mu(\rho \cdot \overline{\sigma^*_G}\rho)(\lambda^G_1)^{\omega_1}\cdots(\lambda^G_t)^{\omega_t} & \text{if } r^G_{\rho,i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t} \\ & \text{is of degree } -2 \text{ or } -6, \\ 0 & \text{otherwise.} \end{cases}$$

   The sign can be determined using formulae in (2) *of Proposition* 1.3.8.

(c) $r^G_{\rho,i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t}\eta = 0$*, and* $r^G_{\rho,i,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t}\mu = 2r^G_{\rho,i+2,\varepsilon_1,\ldots,\varepsilon_t,\nu_1,\ldots,\nu_t}$.

(d) (*Proposition* 2.5.30) $\delta^G_{\mathbb{R}}(\varphi_i)^2 = \eta(\varphi_i \cdot \delta^G_{\mathbb{R}}(\varphi_i) - \delta^G_{\mathbb{R}}(\wedge^2\varphi_i))$, $\delta^G_{\mathbb{H}}(\theta_j)^2 = \eta(\theta_j \cdot \delta^G_{\mathbb{H}}(\theta_j) - \delta^G_{\mathbb{R}}(\wedge^2\theta_j))$. *One can express* $\delta^G_{\mathbb{R}}(\wedge^2\varphi_i)$ *and* $\delta^G_{\mathbb{H}}(\wedge^2\theta_j)$ *in terms of the algebra generators using the derivation property of* $\delta^G_{\mathbb{R}}$ *and* $\delta^G_{\mathbb{H}}$ (*cf. Proposition* 2.5.31).

Proof. Only (2a) and (2c) need explanation, but they are just equivariant analogues of Corollary 2.3.7 and follow from (2) of Proposition 1.3.8 and Remark 2.5.9. □

Remark 2.5.34. 1. $KR^*_G(\mathrm{pt})_2$, which is the ring obtained by inverting the prime 2 in $KR^*_G(\mathrm{pt})$, is isomorphic to

$$(RR(G,\sigma_G,\mathbb{R}) \oplus RH(G,\sigma_G,\mathbb{R})) \otimes \mathbb{Z}\left[\tfrac{1}{2},\beta^2\right]/((\beta^2)^2-1) \oplus r(R(G,\sigma_G,\mathbb{C}) \otimes \mathbb{Z}\left[\tfrac{1}{2},\beta\right]/(\beta^4-1)).$$

2. If $R(G,\mathbb{C}) = 0$, then $KR^*_G(G)_2$, which is the ring obtained by inverting the prime 2 in $KR^*_G(G)$, is isomorphic to, as $KR^*_G(\mathrm{pt})_2$-algebra,

$$\bigwedge\nolimits_{KR^*_G(\mathrm{pt})_2}\left(\delta^G_{\mathbb{R}}(\varphi_1), \ldots, \delta^G_{\mathbb{R}}(\varphi_r), \delta^G_{\mathbb{H}}(\theta_1), \ldots, \delta^G_{\mathbb{H}}(\theta_s)\right).$$

2.6 Applications and examples

Applying the forgetful map $KR^*_G(G) \to KR^*(G)$ to Proposition 2.5.31 and Theorem 2.5.33, we solve the problem of finding a description of the ring structure of $KR^*(G)$ which was left open by Seymour in [Se].

Theorem 2.6.1. *Let G be a simply-connected, connected and compact Real Lie group. Then*

$$\delta_{\mathbb{R}}(\varphi_i)^2 = \eta\big(\dim(\varphi_i)\cdot\delta_{\mathbb{R}}(\varphi_i) - \delta_{\mathbb{R}}(\wedge^2\varphi_i)\big),$$
$$\delta_{\mathbb{H}}(\theta_j)^2 = \eta\delta_{\mathbb{R}}(\wedge^2\theta_j).$$

One can express $\delta_{\mathbb{R}}(\wedge^2\varphi_i)$ and $\delta_{\mathbb{H}}(\wedge^2\theta_j)$ in terms of the generators in Proposition 2.3.5 *using the derivation property of $\delta_{\mathbb{R}}$ and $\delta_{\mathbb{H}}$ got by applying the forgetful map to Proposition* 2.5.31*. The above equations, together with Theorem* 2.3.5 *and Corollary* 2.3.7*, constitute a complete description of the ring structure of $KR^*(G)$.*

Remark 2.6.2. Seymour's conjecture concerning $\delta_{\mathbb{R}}(\sigma_n)^2$ is true. However, his conjecture that if $x \in KR^{-5}(X)$, then $x^2 = 0$ is false, as evidenced by the ring structure of $KR^*(U(2m), \sigma_{\mathbb{H}})$.

Example 2.6.3. Let G be a simply-connected, connected and compact Real Lie group with no fundamental representations of complex type. Equip G with both the trivial G-action and the adjoint action. Both $K^*_{G_{\mathrm{triv}}}(G)$ and $K^*_{G_{\mathrm{Ad}}}(G)$ are isomorphic to $\Omega^*_{R(G)/\mathbb{Z}}$ as rings. On the other hand, though both $KR^*_{G_{\mathrm{triv}}}(G)$ and $KR^*_{G_{\mathrm{Ad}}}(G)$ are isomorphic to $\Omega^*_{KR^*_G(\mathrm{pt})/KR^*(\mathrm{pt})}$ as $KR^*_G(\mathrm{pt})$-modules, they are not isomorphic as rings, as one can tell from the squares of the generators of both rings. For instance, in $KR^*_{G_{\mathrm{triv}}}(G)$,

$$\delta^G_{\mathbb{R}}(\varphi_i)^2 = \eta\left(\dim(\varphi_i)\delta^G_{\mathbb{R}}(\varphi_i) - \delta^G_{\mathbb{R}}(\wedge^2\varphi_i)\right)$$

whereas in $KR^*_{G_{\mathrm{Ad}}}(G)$,

$$\delta^G_{\mathbb{R}}(\varphi_i)^2 = \eta\left(\varphi_i\cdot\delta^G_{\mathbb{R}}(\varphi_i) - \delta^G_{\mathbb{R}}(\wedge^2\varphi_i)\right).$$

In this example KR-theory can tell apart two different group actions, while K-theory cannot.

Example 2.6.4. Let $(G, \sigma_G) = (Sp(2m), \mathrm{Id})$. Then $R(Sp(2m)) \cong \mathbb{Z}[\sigma_{2m}^1, \sigma_{2m}^2, \ldots, \sigma_{2m}^m]$, where σ_{2m}^i is the class of the irreducible representation with highest weight $L_1+L_2+\cdots+L_i$. Note that $\sigma_{2m}^{2k-1} \in RH(Sp(2m), \mathrm{Id}, \mathbb{R})$, $\sigma_{2m}^{2k} \in RR(Sp(2m), \mathrm{Id}, \mathbb{R})$. Moreover, $\sigma_{2m}^i + \wedge^{i-2}\sigma_{2m} = \wedge^i\sigma_{2m}$ for $1 \le i \le m$. The equivariant KR-theory $KR^*_{(Sp(2m),\mathrm{Id})}(Sp(2m), \mathrm{Id})$ is isomorphic to, as $KR^*_{(Sp(2m),\mathrm{Id})}(\mathrm{pt})$-modules, the exterior algebra over $KR^*_{(Sp(2m),\mathrm{Id})}(\mathrm{pt})$ generated by $\delta^G_{\mathbb{H}}(\sigma_{2m}^{2k-1})$ and $\delta^G_{\mathbb{R}}(\sigma_{2m}^{2k})$ for $1 \le k \le m$ by Theorem 2.5.33. The restriction map

$$i^* : KR^*_{(U(2m),\sigma_{\mathbb{H}})}(U(2m), \sigma_{\mathbb{H}}) \to KR^*_{(Sp(2m),\mathrm{Id})}(Sp(2m), \mathrm{Id})$$

sends $\delta^G_{\mathbb{R}}(\wedge^{2k}\sigma_{2m} - \wedge^{2k-2}\sigma_{2m})$ to $\delta^G_{\mathbb{R}}(\sigma_{2m}^{2k})$ and $\delta^G_{\mathbb{H}}(\wedge^{2k+1}\sigma_{2m} - \wedge^{2k-1}\sigma_{2m})$ to $\delta^G_{\mathbb{H}}(\sigma_{2m}^{2k+1})$. Applying i^* to the relevant equations in Corollary 2.5.32, we get

$$\begin{aligned}
\delta^G_{\mathbb{R}}(\sigma_{2m}^{2k})^2 &= \eta \sum_{i=1}^{2k} \left(\sigma_{2m}^{4k-2i} \cdot \delta^G_{\mathbb{R}}(\sigma_{2m}^{2i}) + \sigma_{2m}^{4k-2i+1} \cdot \delta^G_{\mathbb{H}}(\sigma_{2m}^{2i-1})\right), \\
\delta^G_{\mathbb{H}}(\sigma_{2m}^{2k-1})^2 &= \eta \sum_{i=1}^{2k-1} \left(\sigma_{2m}^{4k-2-2i} \cdot \delta^G_{\mathbb{R}}(\sigma_{2m}^{2i}) + \sigma_{2m}^{4k-1-2i} \cdot \delta^G_{\mathbb{H}}(\sigma_{2m}^{2i-1})\right).
\end{aligned}$$

Example 2.6.5. Let $(G, \sigma_G) = (G_2, \mathrm{Id})$. Then $RR(G_2) \cong \mathbb{Z}[\sigma_1, \sigma_2]$, where σ_1 and σ_2 are the classes of irreducible representations of dimensions 7 and 14, respectively. Note that both σ_1 and σ_2 are in $RR(G_2, \mathrm{Id}, \mathbb{R})$, and that $\wedge^2\sigma_1 = \sigma_1 + \sigma_2$, $\wedge^2\sigma_2 = \sigma_1^3 - \sigma_1^2 - 2\sigma_1\sigma_2 - \sigma_1$. The equivariant KR-theory $KR^*_{(G_2,\mathrm{Id})}(G_2, \mathrm{Id})$ is isomorphic to, as $KR^*_{(G_2,\mathrm{Id})}(\mathrm{pt})$-modules, the exterior algebra over $KR^*_{(G_2,\mathrm{Id})}(\mathrm{pt})$ generated by $\delta^G_{\mathbb{R}}(\sigma_1)$ and $\delta^G_{\mathbb{R}}(\sigma_2)$, by Theorem 2.5.5. Using Theorem 2.5.30 and Proposition 2.5.31, we have

$$\delta^G_{\mathbb{R}}(\sigma_1)^2 = \eta((\sigma_1 - 1) \cdot \delta^G_{\mathbb{R}}(\sigma_1) + \delta^G_{\mathbb{R}}(\sigma_2)),$$

$$\delta^G_{\mathbb{R}}(\sigma_2)^2 = \eta((\sigma_1^2 - 1) \cdot \delta^G_{\mathbb{R}}(\sigma_1) + \sigma_2 \cdot \delta^G_{\mathbb{R}}(\sigma_2)).$$

CHAPTER 3

THE ANTI-INVOLUTION CASE

In this Chapter, we assume that G is compact, connected and simply-connected, as before. We consider G as the Real space G^- equipped with instead with an anti-involution a_G, with (G, σ_G) acting on it by conjugation. We will show that G^- is also a Real equivariant formal space, in the sense of Definition 2.5.2. Nevertheless the real and quaternionic type generators of $KR^*_G(G^-)$ are of different degrees from those of $KR^*_G(G)$ we consider in the previous Chapter. Moreover the squares of those generators are zero (see (1) of Theorem 3.2.8), in stark contrast to the case of $KR^*_G(G)$ (see Theorem 2.5.30). We henceforth obtain a generalization of Brylinski-Zhang's result (see (2) of Theorem 3.2.8).

3.1 The algebra generators of $KR^*_G(G^-)$

Lemma 3.1.1. *Let X be a finite CW-complex equipped with an involution. We have that*

$$KR^1(X) \cong [X, (U(\infty), a_{\mathbb{R}})]_{\mathbb{R}}$$

$$KR^{-3}(X) \cong [X, (U(2\infty), a_{\mathbb{H}})]_{\mathbb{R}}$$

where $[X, Y]_{\mathbb{R}}$ means the space of Real homotopy classes of Real maps from X to Y.

Proof. See Proposition 1.3.5. □

Definition 3.1.2. Let $\delta_{\mathbb{R}}^{\mathrm{inv}} : RR(G) \to KR^1(G^-)$ send ρ to the Real homotopy class of it, viewed as the Real map $G^- \to (U(\infty), a_{\mathbb{R}})$. Define $\delta_{\mathbb{H}}^{\mathrm{inv}} : RH(G) \to KR^{-3}(G^-)$ similarly.

Proposition 3.1.3. *If ρ be in $RR(G)$ with (V, σ_V) being the underlying finite dimensional Real vector space of the Real unitary representation, then $\delta_{\mathbb{R}}(\rho)$ is represented by the following complex of Real vector bundles*

$$0 \longrightarrow G \times \mathbb{R} \times \mathbb{C} \times (V \oplus V) \longrightarrow G \times \mathbb{R} \times \mathbb{C} \times (V \oplus V) \longrightarrow 0$$

$$(g, t, z, v_1, v_2) \mapsto \begin{cases} \left(g, t, z, \begin{pmatrix} -t\rho(g) & \bar{z}I_V \\ zI_V & t\rho(g)^* \end{pmatrix} \begin{pmatrix} iv_1 \\ iv_2 \end{pmatrix}\right) & \text{if } t \geq 0 \\ \left(g, t, z, \begin{pmatrix} tI_V & \bar{z}I_V \\ zI_V & -tI_V \end{pmatrix} \begin{pmatrix} iv_1 \\ iv_2 \end{pmatrix}\right) & \text{if } t \leq 0 \end{cases}$$

where the Real structure on $G \times \mathbb{R} \times \mathbb{C} \times (V \oplus V)$ is given by

$$(g, t, z, v_1, v_2) \mapsto (\sigma_G(g)^{-1}, t, -z, \sigma_V(v_2), \sigma_V(v_1))$$

Similarly, if $\rho \in RH(G)$ with (V, J_V) being the underlying finite dimensional Quaternionic vector space of the Quaternionic unitary representation, then $\delta_{\mathbb{H}}(\rho)$ is represented by the same complex of Real vector bundles except that the Real structure on $G \times \mathbb{R} \times \mathbb{C} \times (V \oplus V)$ is given by

$$(g, t, z, v_1, v_2) \mapsto (\sigma_G(g)^{-1}, t, z, -J_V(v_2), J_V(v_1))$$

Proof. It is straightforward to verify that the given Real structures indeed commute with the middle maps of the complex of vector bundles, and that they are canonical. The complex of vector bundles, with the Real structures forgotten, is the tensor product of the following two complexes

$$0 \longrightarrow G \times \mathbb{C} \times V \longrightarrow G \times \mathbb{C} \times V \longrightarrow 0$$
$$(g, z, v) \mapsto (g, z, izv)$$

$$0 \longrightarrow G \times \mathbb{R} \times V \longrightarrow G \times \mathbb{R} \times V \longrightarrow 0$$
$$(g, t, v) \mapsto \begin{cases} (g, t, -it\rho(g)v) & \text{if } t \geq 0 \\ (g, t, itv) & \text{if } t \leq 0 \end{cases}$$

(cf. [ABS, Proposition 10.4]) which represent the Bott class $\beta \in K^{-2}(G)$ and $\delta(\rho) \in K^{-1}(G)$ as defined in [BZ] respectively (the middle maps of the above two complexes differ from the ones conventionally used to define β and $\delta(\rho)$ by multiplication by i, which is homotopy equivalent to the constant map). Besides, the KR-theory classes represented by the complexes of Real vector bundles live in degree 1 and -3 pieces respectively because of the type of the involution of the middle maps restricted to $\mathbb{R} \times \mathbb{C}$. In sum, the two complexes of Real vector bundles represent canonical Real lifts of $\delta(\rho)$. Therefore they must represent $\delta^{\mathrm{inv}}_{\mathbb{R}}(\rho)$ (resp. $\delta^{\mathrm{inv}}_{\mathbb{H}}(\rho)$). □

Definition 3.1.4. Let $\delta^{G,\mathrm{inv}}_{\mathbb{R}} : RR(G) \to KR^1_G(G^-)$ send ρ to the complex of Real vector bundles as in Proposition 3.1.3 equipped with the equivariant structure given by

$$(g, t, z, v_1, v_2) \mapsto (\sigma_G(g)^{-1}, t, z, \rho(g)v_1, \rho(g)v_2)$$

Define $\delta^{G,\mathrm{inv}}_{\mathbb{H}} : RH(G) \to KR^{-3}_G(G^-)$ similarly.

Proposition 3.1.5. *Identifying $RR(G)$ with $KR_G^0(pt)$ and $RH(G)$ with $KR_G^{-4}(pt)$ (cf. [AS, Sect. 8]), $\delta_{\mathbb{R}}^{G,inv} \oplus \delta_{\mathbb{H}}^{G,inv}$ is a derivation of the graded ring $KR_G^0(pt) \oplus KR_G^{-4}(pt)$ taking values in $KR_G^1(G^-) \oplus KR_G^{-3}(G^-)$.*

Proof. The proof can be easily adapted from the one of [BZ, Proposition 3.1] by straightforwardly modifying the homotopy ρ_s and replacing the definition of the map δ_G given there (which is incorrect) with the one in Definition 2.1.5. The modified homotopy can be easily seen to intertwines with both Real structures of the complex of Real vector bundles as in Proposition 3.1.3. □

Proposition 3.1.6. $\delta_G(\overline{a_G^*}\rho) = -\delta_G(\overline{\sigma_G^*}\rho)$

Proof. Viewing $\overline{\sigma_G^*}\rho$ and $\overline{a_G^*}\rho$ as maps from G to $U(\infty)$, $\overline{\sigma_G^*}\rho \cdot \overline{a_G^*}\rho$ is the constant map with image being the identity. It follows that

$$0 = \delta_G(\overline{\sigma_G^*}\rho \cdot \overline{a_G^*}\rho) = \delta_G(\overline{\sigma_G^*}\rho) + \delta_G(\overline{a_G^*}\rho)$$

The last equality is the equivariant analogue of [At, Lemma 2.4.6]. □

The fundamental representations of G are permuted by $\overline{\sigma_G^*}$ (cf. [Se, Lemma 5.5]). Following the notations in Chapter 2, we let $\varphi_1, \cdots, \varphi_r$, $\theta_1, \cdots, \theta_s, \gamma_1, \cdots, \gamma_t, \overline{\sigma_G^*}\gamma_1, \cdots, \overline{\sigma_G^*}\gamma_t$ be the fundamental representations of G, where $\varphi_i \in RR(G, \mathbb{R})$, $\theta_j \in RH(G, \mathbb{R})$ and $\gamma_k \in R(G, \mathbb{C})$.

Definition 3.1.7. Let $\lambda_k^{G,\text{inv}}$ be the element in $KR_G^0(G^-)$ constructed by adding the natural equivariant structure throughout the construction of λ_k in the proof of [Se, Proposition 4.6] such that $c(\lambda_k^{G,\text{inv}}) = \beta^3\delta_G(\gamma_k)\delta_G(\overline{a_G^*}\gamma_k) = -\beta^3\delta_G(\gamma_k)\delta_G(\overline{\sigma_G^*}\gamma_k)$.

Applying [Se, Theorem 4.2], one can get the $KR^*(\mathrm{pt})$-module structure of $KR^*(G, a_G)$ (compare with [Se, Theorem 5.6]), from which one can further obtain the $KR^*_G(\mathrm{pt})$-module structure of $KR^*_G(G^-)$, by observing that G^- is a Real equivariantly formal (G, σ_G)-space (cf. Definition 2.5.2) and applying the structure theorem for Real equivariantly formal spaces (cf. Theorem 2.5.5). We shall state the following description of $KR^*_G(G^-)$ without proof. We refer the reader to Corollaries 2.5.10, 2.5.11, Proposition 2.5.13 and Theorem 2.5.33 for comparison.

Theorem 3.1.8. *1. The map*

$$f : (RR(G,\mathbb{R}) \oplus RH(G,\mathbb{R})) \otimes KR^*(G^-) \oplus r(R(G,\mathbb{C}) \otimes K^*(G)) \to KR^*_G(G^-)$$

$$\rho_1 \otimes x_1 \oplus r(\rho_2 \otimes x_2) \mapsto \rho_1 \cdot (x_1)_G \oplus r(\rho_2 \cdot (x_2)_G)$$

is a group isomorphism, where $x_G \in KR^*_G(G^-)$ is a Real equivariant lift of $x \in KR^*(G^-)$. If $R(G,\mathbb{C}) = 0$, then f is an isomorphism of $KR^*_G(pt)$-modules.

2. *$KR^*_G(G^-)$ is generated as an algebra over $KR^*_G(pt)$ (for descriptions of the coefficient ring see Section 2.4.3) by $\delta^{G,inv}_{\mathbb{R}}(\varphi_1), \cdots, \delta^{G,inv}_{\mathbb{R}}(\varphi_r)$, $\delta^{G,inv}_{\mathbb{H}}(\theta_1), \cdots, \delta^{G,inv}_{\mathbb{H}}(\theta_s)$, $\lambda^{G,inv}_1, \cdots, \lambda^{G,inv}_t$ and*

$$\{r^{G,inv}_{\rho,i,\varepsilon_1,\cdots,\varepsilon_t,\nu_1,\cdots,\nu_t} := r(\beta^i \cdot \rho\delta_G(\gamma_1)^{\varepsilon_1} \cdots \delta_G(\gamma_t)^{\varepsilon_t} \delta_G(\overline{a^*_G}\gamma_1)^{\nu_1} \cdots \delta_G(\overline{a^*_G}\gamma_t)^{\nu_t}\}$$

where $\rho \in R(G,\mathbb{C}) \oplus \mathbb{Z} \cdot \rho_{triv}$, $\varepsilon_1, \cdots, \varepsilon_t$, $\nu_1, \cdots, \nu_t$ are either 0 or 1, ε_k and ν_k are not equal to 1 at the same time for $1 \leq k \leq t$, and the first index k_0 where $\varepsilon_{k_0} = 1$ is less than the first index k_1 where $\nu_{k_1} = 1$. Moreover,

   (a) $(\lambda^{G,inv}_k)^2 = 0$.

(b) *Let* $\omega_t := \delta_{\varepsilon_t, 1-\nu_t}$. *Then*

$$(r^G_{\rho,i,\varepsilon_1,\cdots,\varepsilon_t,\nu_1,\cdots,\nu_t})^2 = \begin{cases} \eta^2(\rho\cdot\overline{\sigma^*_G}\rho)(\lambda^G_1)^{\omega_1}\cdots(\lambda^G_t)^{\omega_t} & \text{if } r^G_{\rho,i,\varepsilon_1,\cdots,\varepsilon_t,\nu_1,\cdots,\nu_t} \\ & \text{is of degree } -1 \text{ or } -5 \\ \pm\mu(\rho\cdot\overline{\sigma^*_G}\rho)(\lambda^G_1)^{\omega_1}\cdots(\lambda^G_t)^{\omega_t} & \text{if } r^G_{\rho,i,\varepsilon_1,\cdots,\varepsilon_t,\nu_1,\cdots,\nu_t} \\ & \text{is of degree } -2 \text{ or } -6 \\ 0 & \text{otherwise} \end{cases}$$

The sign can be determined using formulae in Proposition 1.3.8(2).

(c) $r^G_{\rho,i,\varepsilon_1,\cdots,\varepsilon_t,\nu_1,\cdots,\nu_t}\eta = 0$, *and* $r^G_{\rho,i,\varepsilon_1,\cdots,\varepsilon_t,\nu_1,\cdots,\nu_t}\mu = 2r^G_{\rho,i+2,\varepsilon_1,\cdots,\varepsilon_t,\nu_1,\cdots,\nu_t}$.

Corollary 3.1.9. *In particular, if* $R(G,\mathbb{C}) = 0$, *then*

$$\begin{aligned} KR^*_G(G^-) &= \bigwedge\nolimits_{KR^*_G(pt)}(\delta^{G,inv}_{\mathbb{R}}(\varphi_1),\cdots,\delta^{G,inv}_{\mathbb{R}}(\varphi_r),\delta^{G,inv}_{\mathbb{H}}(\theta_1),\cdots,\delta^{G,inv}_{\mathbb{H}}(\theta_s)) \\ &\cong \Omega_{KR^*_G(pt)/KR^*(pt)} \end{aligned}$$

as $KR^*_G(pt)$*-modules.*

As we can see, the module structure of $KR^*_G(G^-)$ is very similar to that of $KR^*_G(G)$, except that the degrees of the generators are different. Now it remains to find $\delta^{G,\mathrm{inv}}_{\mathbb{R}}(\varphi_i)^2$ and $\delta^{G,\mathrm{inv}}_{\mathbb{H}}(\theta_j)^2$ so as to complete the description of the ring structure of $KR^*_G(G^-)$. As it turns out, these squares are all zero, in stark contrast to the involutive automorphism case.

3.2 Squares of the real and quaternionic type generators

This section is devoted to proving that the squares of the real and quaternionic generators are zero, following the strategy outlined in Section 2.5.

Applying Brylinski-Zhang's result on the equivariant K-theory of compact connected Lie group G with $\pi_1(G)$ torsion-free and the structure theorem for Real equivariantly formal space (cf. Theorem 2.5.5), we have

Proposition 3.2.1. *For $\mathbb{F} = \mathbb{R}$ or $\mathbb{H}$, we have the following $KR^*_{(U(n),\sigma_{\mathbb{F}})}(pt)$-module isomorphism*

$$KR^*_{(U(n),\sigma_{\mathbb{F}})}(U(n), a_{\mathbb{F}}) \cong \Omega_{KR^*_{(U(n),\sigma_{\mathbb{F}})}(pt)/KR^*(pt)}$$

The set $\{\delta^{G,inv}_{\mathbb{R}}(\sigma_n), \delta^{G,inv}_{\mathbb{R}}(\bigwedge^2 \sigma_n), \cdots, \delta^{G,inv}_{\mathbb{R}}(\bigwedge^n \sigma_n)\}$ is a set of primitive generators for the case $\mathbb{F} = \mathbb{R}$, while $\{\delta^{G,inv}_{\mathbb{H}}(\sigma_{2m}), \delta^{G,inv}_{\mathbb{R}}(\bigwedge^2 \sigma_{2m}), \cdots, \delta^{G,inv}_{\mathbb{R}}(\bigwedge^{2m} \sigma_{2m})\}$ is a set of primitive generators for the case $\mathbb{F} = \mathbb{H}$.

Corollary 3.2.2. *We have the following isomorphism*

$$KR^*_{(U(n),\sigma_{\mathbb{F}})}(U(n), a_{\mathbb{F}}) \cong \Omega_{R(U(n))/\mathbb{Z}} \otimes KR^*(pt)$$

as ungraded $KR^*(pt)$-modules.

Definition 3.2.3. Let

$$p^*_{G,\text{inv}} : KR^*_{(U(n),\sigma_{\mathbb{R}})}(U(n), a_{\mathbb{R}}) \to KR^*_{(T,\sigma_{\mathbb{R}})}(T, \text{Id})$$

be the restriction map and the map

$$q^*_{G,\text{inv}} : KR^*_{(U(2m),\sigma_{\mathbb{H}})}(U(2m), a_{\mathbb{H}}) \to KR^*_{(U(2m),\sigma_{\mathbb{H}})}(U(2m)/T \times T, \sigma_{\mathbb{H}} \times \text{Id})$$

induced by the Weyl covering map

$$q_G : U(2m)/T \times T \to U(2m)$$
$$(gT, t) \mapsto gtg^{-1}$$

Proposition 3.2.4. *Identifying $KR^*_{(T,\sigma_{\mathbb{R}})}(T, Id)$ with $RR(T, \sigma_{\mathbb{R}}) \otimes KR^*(T, Id)$, we have*

$$p^*_{G,inv}(\delta^{G,inv}_{\mathbb{R}}(\bigwedge^k \sigma_n)) = \sum_{1 \le j_1 < \cdots < j_k \le n} e_{j_1} \cdots e_{j_k} \otimes \delta^{inv}_{\mathbb{R}}(e_{j_1} + \cdots + e_{j_k})$$

*where e_i is the 1-dimensional Real representation of $(T, \sigma_{\mathbb{R}})$ with weight being the i-th standard basis vector of the weight lattice. Similarly, identifying $KR_{(U(2m),\sigma_{\mathbb{H}})}(U(2m)/T \times T, \sigma_{\mathbb{H}} \times Id)$ with $\mathbb{Z}[e^{\mathbb{H}}_1, \cdots, e^{\mathbb{H}}_{2m}, (e^{\mathbb{H}}_1 \cdots e^{\mathbb{H}}_{2m})^{-1}] \otimes KR^*(T, Id)$ (cf. [**?**, Proposition 4.25]), where $e^{\mathbb{H}}_i$ is the degree -4 class in $KR_{(U(2m),\sigma_{\mathbb{H}})}(U(2m)/T, \sigma_{\mathbb{H}})$ represented by the Quaternionic line bundle $U(2m) \times_T \mathbb{C}_{e_i}$, we have, for $\mathbb{F} = \mathbb{R}$ or $\mathbb{H}$ (depending on the parity of k),*

$$q^*_{G,inv}(\delta^{G,inv}_{\mathbb{F}}(\bigwedge^k \sigma_{2m})) = \sum_{1 \le j_1 < \cdots < j_k \le 2m} e^{\mathbb{H}}_{j_1} \cdots e^{\mathbb{H}}_{j_k} \otimes \delta^{inv}_{\mathbb{R}}(e_{j_1} + \cdots + e_{j_k})$$

Proof. The proof is similar to Lemma 2.5.19. The Proposition follows from the fact that the complex of $U(n)$-equivariant Real vector bundles representing $\delta^{G,\mathrm{inv}}_{\mathbb{F}}(\bigwedge^k \sigma_n)$, as in Proposition 3.1.3, is decomposed into a direct sum of complexes of T-equivariant Real vector bundles, each of which corresponds to a weight of $\bigwedge^k \sigma_n$. □

Proposition 3.2.5. *Both $p^*_{G,inv}$ and $q^*_{G,inv}$ are injective.*

Proof. By Lemma 2.5.19 and Proposition 2.5.25, and Corollary 3.2.2 and Proposition 3.2.4, we can identify both $p^*_{G,\mathrm{inv}}$ and $q^*_{G,\mathrm{inv}}$ with the map

$$i^* \otimes \mathrm{Id}_{KR^*(\mathrm{pt})} : K^*_{U(n)}(U(n)) \otimes KR^*(\mathrm{pt}) \to K^*_T(T) \otimes KR^*(\mathrm{pt})$$

where the restriction map i^* can factor through $K_T^*(U(n))$ as

$$K_{U(n)}^*(U(n)) \xrightarrow{i_1^*} K_T^*(U(n)) \xrightarrow{i_2^*} K_T^*(T)$$

$i_1^* \otimes \mathrm{Id}_{KR^*(\mathrm{pt})}$ is injective because i_1^* is split injective by [At3, Proposition 4.9]. By adapting Lemma 2.5.20 to the case $G = U(n)$, we have

$$i_2^*\left(\prod_{i=1}^{n} \delta_T(\bigwedge^i \sigma_n)\right) = d_{U(n)} \otimes \prod_{i=1}^{n} \delta(e_i)$$

where $d_{U(n)}$ is the Weyl denominator for $U(n)$. By Lemma 2.5.21 and the fact that $rd_{U(n)} \otimes \prod_{i=1}^{n} \delta(e_i) \neq 0$ for all $r \in KR^*(\mathrm{pt}) \setminus \{0\}$, $i_2^* \otimes \mathrm{Id}_{KR^*(\mathrm{pt})}$ is injective as well. Thus $i^* \otimes \mathrm{Id}_{KR^*(\mathrm{pt})}$, as well as $p_{G,\mathrm{inv}}^*$ and $q_{G,\mathrm{inv}}^*$, are injective. □

Lemma 3.2.6. *Let e be the standard representation of S^1. Then $\delta_{\mathbb{R}}^{inv}(e)^2 = 0$ in $KR^*(S^1, Id)$.*

Proof. Note that $\delta_{\mathbb{R}}^{\mathrm{inv}}(e) \in KR^{-7}(S^1, \mathrm{Id})$. So $\delta_{\mathbb{R}}^{\mathrm{inv}}(e)^2 \in KR^{-6}(S^1, \mathrm{Id}) \cong KR^{-7}(\mathrm{pt}) = 0$. □

Proposition 3.2.7. *For $\mathbb{F} = \mathbb{R}$ or $\mathbb{H}$, $\delta_{\mathbb{F}}^{G,inv}(\sigma_n)^2 = 0$ in $KR^*_{(U(n),\sigma_{\mathbb{F}})}(U(n), a_{\mathbb{F}})$.*

Proof. This follows from Propositions 3.2.4 and 3.2.5 and Lemma 3.2.6. □

The above results finally culminate in the main theorem of this Chapter.

Theorem 3.2.8. *1. Let G be a Real compact Lie group, and ρ a Real (resp. Quaternionic) unitary representation of G. Then $\delta_{\mathbb{F}}^{G,inv}(\rho)^2 = 0$ in $KR_G^*(G^-)$ for $\mathbb{F} = \mathbb{R}$ (resp. $\mathbb{F} = \mathbb{H}$).*

2. In particular, if G is connected and simply-connected and $R(G, \mathbb{C}) = 0$, then $\delta_{\mathbb{R}}^{G,inv} \oplus \delta_{\mathbb{H}}^{G,inv}$ induces the following ring isomorphism

$$KR_G^*(G^-) \cong \Omega_{KR_G^*(pt)/KR^*(pt)}$$

Proof. Note that the induced map $\rho^* : KR^*_{(U(n),\sigma_{\mathbb{F}})}(U(n), a_{\mathbb{F}}) \to KR^*_G(G^-)$ sends $\delta^{G,\mathrm{inv}}_{\mathbb{F}}(\sigma_n)$ to $\delta^{G,\mathrm{inv}}_{\mathbb{F}}(\rho)$ by the interpretation of $\delta^{G,\mathrm{inv}}_{\mathbb{F}}$ in Proposition 3.1.3. Now part (1) follows from Proposition 3.2.7. Part (2) follows from part (1) and Corollary 3.1.9. $\square$

Note that Theorems 3.1.8 and 3.2.8 give a complete description of the ring structure of $KR^*_G(G^-)$. Part (2) of Theorem 3.2.8 should be viewed as a generalization of Brylinski-Zhang's result in the context of KR-theory.

Last but not least, we obtain, as a by-product, the following

Corollary 3.2.9. *If G is a compact Real Lie group and X a compact Real G-space, then for any x in $KR^1_G(X)$ or $KR^{-3}_G(X)$, $x^2 = 0$.*

Proof. Let EG^n be the join of n copies of G, with the Real structure induced by σ_G and G-action by the left-translation of G. Let $\pi_n^* : KR^*_G(X) \to KR^*_G(X \times EG^n)$ be the map induced by projection onto X. The map

$$\pi^* := \varprojlim_n \pi_n^* : KR^*_G(X) \to \varprojlim_n KR^*_G(X \times EG^n)$$

is injective because by adapting the proof of [AS, Corollary 2.3] to the Real case, $\ker(\pi) = \bigcap_{n\in\mathbb{N}} I^n \cdot KR^*_G(X)$, where I is the augmentation ideal of $RR(G)$, and $\bigcap_{n\in\mathbb{N}} I^n = \{0\}$. Now it suffices to show that $\pi^*(x)^2 = 0$. Using Lemma 3.1.1 and compactness of $X \times EG^n/G$, $\pi_n^*(x) \in KR^*_G(X \times EG^n) = KR^*(X \times EG^n/G)$ can be represented by a Real map $f_n : X \times EG^n/G \to (U(k_n), a_{\mathbb{F}})$ for some k_n. So $\pi_n^*(x) = f_n^* \delta^{\mathrm{inv}}_{\mathbb{F}}(\sigma_{k_n})$ and $\pi_n^*(x)^2 = 0$ by Proposition 3.2.7. Since $\pi^*(x)^2$ is the inverse limit of $\pi_n^*(x)^2 = 0$, $\pi^*(x)^2 = 0$ as desired. $\square$

CHAPTER 4

THE REAL FREED-HOPKINS-TELEMAN THEOREM

4.1 Introduction

In this Chapter, we assume that G is a compact, connected, simply-connected and simple Lie group, unless otherwise specified.

Having explored the equivariant KR-theory of compact Lie groups in previous Chapters, we shall turn our attention to the case where a twist, realized by an equivariant Real DD bundle, is introduced. The case where equivariant and twist structures are present while the Real structure is not was addressed by Freed-Hopkins-Teleman Theorem (cf. [Fr], [FHT1], [FHT2], [FHT3]), which is to be discussed below. It is a recent deep result which provides a connection between the equivariant twisted K-homology of G and its more subtle representation theory.

Let $\mathcal{A}$ be an equivariant DD bundle whose DD class is the generator of $H^3_G(G,\mathbb{Z}) \cong \mathbb{Z}$. The equivariant twisted K-homology $K^G_*(G,\mathcal{A}^p)$ has a multiplicative structure induced by the crossed product (see (2) in Section 1.6).

$$K^G_*(G,\mathcal{A}^p) \otimes K^G_*(G,\mathcal{A}^p) \to K^G_*(G \times G, \pi_1^*\mathcal{A}^p \otimes \pi_2^*\mathcal{A}^p)$$

followed by the pushforward map induced by the group multiplication

$$m_* : K^G_*(G \times G, \pi_1^*\mathcal{A}^p \otimes \pi_2^*\mathcal{A}^p) \to K^G_*(G,\mathcal{A}^p)$$

Note that there is a Morita isomorphism $m^*\mathcal{A}^p \cong \pi_1^*\mathcal{A}^p \otimes \pi_2^*\mathcal{A}^p$ because $m^*[\eta_G] = \pi_1^*[\eta_G] + \pi_2^*[\eta_G]$. m_* is independent of the equivariant Hilbert space bundle $\mathcal{E}$ on $G \times G$ which witnesses this Morita isomorphism, and hence canonically defined, since $H^2_G(G \times G, \mathbb{Z}) = 0$ (cf. (1) of Section 1.6). Freed-Hopkins-Teleman Theorem asserts that

Theorem 4.1.1. *Let $h^\vee$ be the dual Coxeter number of G (for definition see Section 4.2). The equivariant twisted K-homology $K^G_*(G, \mathcal{A}^{k+h^\vee})$ is isomorphic to the level k Verlinde algebra $R_k(G)$ (to be explained below), for $k \geq 0$. More precisely, the pushforward map*

$$\iota_* : R(G) \cong K^G_*(pt) \to K^G_*(G, \mathcal{A}^{k+h^\vee})$$

is onto with kernel being I_k, the level k Verlinde ideal (for definition see Section 4.2).

The above Theorem merits some remarks. On one end of the isomorphism is Verlinde algebra, which is an object of great interest in mathematical physics. It is the Grothendieck group of the positive energy representations of the level k central extension of the (free) loop group LG, equipped with an intricately defined ring structure called the fusion product. It is known that $R_k(G)$, as an abelian group, is generated freely by the isomorphism classes of irreducible positive energy representations V_λ with highest weight λ in Λ^*_k, the set of level k weights (see Section 4.2 for definition). The fusion product rule can be stipulated by defining its structural constants $c^{\gamma^*}_{\lambda\mu}$ with respect to those generators satisfying

$$[V_\lambda] \cdot [V_\mu] = \sum_{\gamma \in \Lambda^*_k} c^{\gamma^*}_{\lambda\mu} [V_\gamma]$$

to be the dimension of a certain vector space associated with the Riemann surface of genus 0, with three punctures labelled by λ, μ and $\gamma^* := -w_0\gamma$. This vector space has its root in Conformal Field Theory (see [V]) and can be interpreted as the space of conformal block (for one of its models see [Be]), which was shown to be canonically isomorphic to the space of generalized theta functions of the moduli space of G-bundles (cf. [BL] and the references therein). Thus one of the novelties of Freed-Hopkins-Teleman Theorem is that it provides an algebro-topological perspective of the fusion product, in addition to the conformal field theory and algebro-geometric approach. An alternative definition of Verlinde algebra as a quotient ring of $R(G)$ is given in Section 4.2.

Another interesting aspect of Freed-Hopkins-Teleman Theorem is that it provides an elegant formulation of geometric quantization of q-Hamiltonian spaces (cf. [M3], [M4] and the references therein). We will discuss this idea in Section 4.7.

In this Chapter, we achieve the modest goal of proving a partial generalization of Freed-Hopkins-Teleman Theorem in the Real case (cf. Theorem 4.6.4). In fact all the works in this thesis grow out of our desire to obtain this generalization. In formulating our result, we consider G^- rather than G equipped with an involutive automorphism, as the former supports a Real equivariant DD bundle whose DD class is of infinite order, while the latter does not (cf. (2) of Remark 4.3.4). Our result essentially asserts that in the case where G admits no Representations of complex type, the equivariant twisted KR-homology of G^- is isomorphic to Verlinde algebra tensored with the coefficient ring $KR_*(\mathrm{pt})$, thus exhibiting Real formality (cf. Remark 2.3.2). Though this result is somewhat expected, in our opinion it is not obvious, and we are not able to show Real formality immediately using just the statement of Freed-Hopkins-Teleman Theorem directly. Rather, we need to follow the proof of Freed-Hopkins-Teleman as worked out in details in [M2], which involves a spectral sequence argument applied to the simplicial description of G and the twist. In adapting the proof to the Real case we need to work with a special choice of maximal tori and root systems, whereas in the original proof such a choice is unnecessary.

4.2 Notations and definitions

Let G be of rank l, T a fixed choice of maximal torus, and W the corresponding Weyl group. We let $\Lambda \in \mathfrak{t}$ be the coroot lattice and $\Lambda^* \subset \mathfrak{t}^*$ the weight lattice. We fix a choice of simple roots $\{\alpha_0, \cdots, \alpha_l\}$ (we adopt the convention that $\alpha_0 = -\alpha_{\max}$, where $\alpha_{\max}$ is the highest root) and the positive Weyl chamber $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$. We let B be the basic inner product on $\mathfrak{g}^*$, which is the bi-invariant inner product such that $B(\alpha_{\max}, \alpha_{\max}) = 2$. B is used to identify $\mathfrak{t}$ and $\mathfrak{t}^*$. The dual Coxeter number, $\mathsf{h}^\vee$, of G is then defined to be $1 + B(\rho, \alpha_{\max})$. We let Δ^k be the level k closed Weyl alcove of $\mathfrak{t}^*_+$, defined by the inequalities

$$\lambda(\alpha_i^\vee) + k\delta_{i,0} \geq 0 \text{ for } i = 0, \cdots, l$$

with vertices labelled by $\{0, \cdots, l\}$, so that the origin is labelled 0. We use Δ to denote the ordinary closed Weyl alcove Δ^1. Let Λ^*_k be $\Delta^k \cap \Lambda^*$, the level k weights.

Let $I \subseteq \{0, \cdots, l\}$, and Φ_I denote the set of simple roots $\{\alpha_i | i \notin I\}$,

$$\mathfrak{t}^*_{I,+} := \{\lambda \in \mathfrak{t}^* | \lambda(\alpha_i^\vee) + \delta_{i,0} \geq 0, \alpha \in \Phi_I\},$$

$\Lambda^*_{I,+} = \mathfrak{t}^*_{I,+} \cap \Lambda^*$ and $\rho_I = \frac{1}{2}\sum_{\alpha \in \Phi_I} \alpha$.

By abuse of notation, we also use Δ to denote $B^\sharp(\Delta) \subset \mathfrak{t}$. Let Δ_I be the closed sub simplex of Δ spanned by vertices with labels from the index set I. Let W_I be the subgroup of W fixing Δ_I. We also let G_I be the stabilizer subgroup of Δ_I and Λ_I, the coroot lattice of G_I.

Define

$$W_I^k = \begin{cases} \text{the group generated by reflections across } \ker(\alpha_i^\vee) \text{ for } i \notin I, \text{ if } 0 \in I \\ \text{the group generated by reflections across } \ker(\alpha_i^\vee) \text{ for } i \neq 0 \text{ and } i \notin I, \\ \text{and } \ker(\alpha_0^\vee) - \frac{k\alpha_0}{2}, \text{ if } 0 \notin I \end{cases}$$

If we view $R(G)$ as the ring of characters of G, then the level k Verlinde ideal I_k can be defined as the vanishing ideal of

$$\left\{ \exp_T \left(B^\sharp \left(\frac{\lambda + \rho}{k + \mathsf{h}^\vee} \right) \right) \middle| \lambda \in \Lambda^*_k \right\}$$

and the level k Verlinde algebra can be alternatively defined as $R_k(G) := R(G)/I_k$ (cf. [Be]). Finally, by abuse of notation, we denote by G^- the group G viewed as a $G \rtimes \Gamma$-space, where the first factor acts by conjugation while the second acts by the anti-involution a_G.

4.3 The group of Morita isomorphism classes of equivariant Real DD bundles over G

In this section, we shall compute $H^3_{G\rtimes\Gamma}(G^-, \mathbb{Z}_\Gamma)$, the group of Morita isomorphism classes of G-equivariant Real DD-bundles over G^-. It is well-known that $H^3_G(G, \mathbb{R}) \cong \mathbb{R}$ and the integral generator is represented by the equivariant differential form

$$\eta_G(\xi) = \frac{1}{12}B(\theta^L, [\theta^L, \theta^L]) - \frac{1}{2}B(\theta^L + \theta^R, \xi)$$

where θ^L and θ^R are the left and right Maurer-Cartan forms respectively,and $\xi \in \mathfrak{g}$ (cf. [M2, Section 3]).

Lemma 4.3.1. *$H^3_G(G, \mathbb{Z})$ is invariant under Γ.*

Proof. It suffices to show that η_G is Γ-invariant. Note that

$$\begin{aligned}
-(\Gamma^*\eta_G)(\xi) &= -\frac{1}{12}B((\sigma_G \circ \mathrm{inv})^*\theta^L, [(\sigma_G \circ \mathrm{inv})^*\theta^L, (\sigma_G \circ \mathrm{inv})^*\theta^L]) + \frac{1}{2}B((\sigma_G \circ \mathrm{inv})^*(\theta^L + \theta^R), \sigma_G(\xi)) \\
&= -\frac{1}{12}B(-\sigma_G^*\theta^R, \sigma_G^*[\theta^R, \theta^R]) + \frac{1}{2}B(-\sigma_G^*(\theta^L + \theta^R), \sigma_G(\xi)) \\
&= \frac{1}{12}B(\theta^R, [\theta^R, \theta^R]) - \frac{1}{2}B(\theta^L + \theta^R, \xi) \\
&= \eta_G(\xi)
\end{aligned}$$

The result follows. □

By Lemma 1.5.11, $H^3_{G\rtimes\Gamma}(G^-, \mathbb{R}_\Gamma) \cong H^3_G(G^-, \mathbb{R})^\Gamma \cong \mathbb{R}$. It follows that $H^3_{G\rtimes\Gamma}(G^-, \mathbb{Z}_\Gamma)$ is of rank 1 and its free part is generated by $[\eta_G]$. On the other hand, there must be a $\mathbb{Z}_2$ summand in $H^3_{G\rtimes\Gamma}(G^-,\mathbb{Z}_\Gamma)$ by the discussion in Example 1.5.12. The non-zero 2-torsion is $\mathrm{DD}_{\mathbb{R}}(\pi^* o_{\mathbb{R}^{0,4}})$. Thus we have that $H^3_{G\rtimes\Gamma}(G^-,\mathbb{Z}_\Gamma)$ contains the subgroup generated by $[\eta_G]$ and $\mathrm{DD}_{\mathbb{R}}(\pi^* o_{\mathbb{R}^{0,4}})$. In fact, it is all that this equivariant Real cohomology group contains.

Proposition 4.3.2. $H^3_{G\rtimes\Gamma}(G^-,\mathbb{Z}_\Gamma)\cong\mathbb{Z}[\eta_G]\oplus\mathbb{Z}_2 DD_{\mathbb{R}}(\pi^* o_{\mathbb{R}^{0,4}})$.

Proof. By definition,

$$H^3_{G\rtimes\Gamma}(G^-,\mathbb{Z}_\Gamma)=H^3(((G^-\times EG)/G\times E\Gamma)/\Gamma,\underline{((G^-\times EG)/G\times E\Gamma\times\mathbb{Z})/\Gamma})$$

Applying Serre spectral sequence to the fiber bundle $(G^-\times EG)/G\hookrightarrow((G^-\times EG)/G\times E\Gamma)/\Gamma\to B\Gamma$, we have that the E_2-page is

$$E_2^{p,q}=H^p(B\Gamma,H^q_G(G^-,\mathbb{Z})\times_\Gamma E\Gamma)$$

Note that

$E_2^{0,3}=H^0(B\Gamma,\mathbb{Z})\cong\mathbb{Z}$ ($H^3_G(G,\mathbb{Z})\cong\mathbb{Z}$ is invariant under the Γ-action by Lemma 4.3.1)

$E_2^{1,2}=E_2^{2,1}=0$ as $H^i_G(G,\mathbb{Z})=0$ for $i=1,2$

$E_2^{3,0}=H^3_\Gamma(\mathrm{pt},\mathbb{Z}_\Gamma)\cong\mathbb{Z}_2$ (Note that Γ acts on $H^0_G(G^-,\mathbb{Z})\cong\mathbb{Z}$ by negation)

The convergence of the spectral sequence implies that $H^3_{G\rtimes\Gamma}(G^-,\mathbb{Z}_\Gamma)$ is a certain extension of subquotients of $\mathbb{Z}_2$ by $\mathbb{Z}$. But from the discussion preceding this Proposition we have that $H^3_{G\rtimes\Gamma}(G^-,\mathbb{Z}_\Gamma)$ contains $\mathbb{Z}\oplus\mathbb{Z}_2$ as a subgroup. We conclude that indeed $H^3_{G\rtimes\Gamma}(G^-,\mathbb{Z}_\Gamma)\cong\mathbb{Z}\oplus\mathbb{Z}_2$. □

Definition 4.3.3. Any equivariant Real DD bundles over G^- whose equivariant Reall DD class is $[-\eta_G]$ is called an equivariant Real fundamental DD bundle.

Remark 4.3.4. 1. Let $\mathcal{A}$ be an equivariant Real fundamental DD bundle, whose construction is given in Section 4.5. We note that the ring structure of $KR^G_*(G^-, \mathcal{A}^{k+\mathsf{h}^\vee})$ can be defined in a similar fashion to the one spelled out in Section 4.1. In this Real situation the push forward map m_* induced by the group multiplication is a canonical one because the equivariant Real cohomology $H^2_{G\rtimes\Gamma}(G^- \times G^-, \mathbb{Z}_\Gamma) = 0$ (cf. (1) in Section 1.6), which can be proved along the lines of thought in this Section.

2. Again, using the ideas in this Section, one can show that $H^3_{G\rtimes\Gamma}(G, \mathbb{Z}_\Gamma) \cong \mathbb{Z}_2\mathrm{DD}_\mathbb{R}(\pi^* o_{\mathbb{R}^{0,4}})$ if Γ acts on G by an involutive automorphism. In this case the equivariant twisted KR-homology of G, $KR^G_*(G, \pi^*(o_{\mathbb{R}^{0,4}}))$ is isomorphic to $KR^G_{*-4}(G) \cong KH^G_*(G)$ by Proposition 1.6.3.

4.4 A distinguished maximal torus with respect to σ_G

In contructing the equivariant Real fundamental DD-bundle over G and computing the equivariant twisted KR-homology of G, which are done in Sections 4.5 and 4.6, we need to work with a particular kind of maximal torus associated with σ_G. In this Section we record the results about this maximal torus, directly taken from [OS].

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{q}$, where $\mathfrak{k}$ and $\mathfrak{q}$ are the ± 1 eigenspaces of $\sigma_\mathfrak{g}$ respectively. Let $\mathfrak{a}$ be the maximal abelian subspace of $\mathfrak{q}$, $\mathfrak{t}$ a choice of maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Let $\mathfrak{k}'$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, and $\mathfrak{t}' = \mathfrak{t} \cap \mathfrak{k}'$. It is known that $\mathfrak{t} = \mathfrak{t}' \oplus \mathfrak{a}$ (c.f. [OS], Appendix B) and σ_G respects this decomposition. Let $K' = \exp_G \mathfrak{k}'$, $T' = \exp_G \mathfrak{t}'$ and $T = \exp_G \mathfrak{t}$. T is the maximal torus we will use from now on. Note that T' is a maximal torus of K'. Let W' be the Weyl group of K' with respect to T'. Let w_0' be the longest element in W'. Define $\sigma_+ : \mathfrak{t}^* \to \mathfrak{t}^*$ by

$$\sigma_+(\lambda) = -\sigma_{\mathfrak{g}}(w_0'\lambda)$$

and $\sigma_+ : \mathfrak{t} \to \mathfrak{t}$ similarly. Let $k_0 \in N_{(K')_0}(T')$ be any representative of w_0'. Let R be the root system of $(\mathfrak{g}, \mathfrak{t})$. We define a positive root system R_+ as follows. Let

$$R' = \{\alpha \in R | \sigma_{\mathfrak{t}^*}(\alpha) = \alpha\}, \quad R^{\alpha} = \{\alpha|_{\mathfrak{a}} \ |\alpha \notin R'\}$$

R' is the root system of $(\mathfrak{k}', \mathfrak{t}')$, thus a root subsystem of R. R^{α} is the system of restricted roots of the symmetric pair $(\mathfrak{g}, \mathfrak{t})$. Define

$$R_+ := R'_+ \cup \{\alpha \in R | \ \alpha|_{\mathfrak{a}} \in R^{\mathfrak{a}}_+\}$$

Now we can fix a choice of the closed Weyl alcove Δ with respect to R_+.

Proposition 4.4.1. *1. ([[OS], Lemma 4.7(i)]) σ_+ is an involution and $\sigma_+(R_+) = R_+$. Hence σ_+ preserves $\mathfrak{t}^*_+$ and Δ.*

2. *([OS], Addendum 4.11) The irreducible representation V_λ of G with highest weight λ is in $RR(G, \mathbb{R})$ (resp. $RH(G, \mathbb{R})$ iff $\sigma_+(\lambda) = \lambda$ and $(k_0^2)^\lambda = 1$ (resp. $(k_0^2)^\lambda = -1$). $V_\lambda \oplus V_{\sigma_+(\lambda)}$ is in $RR(G, \mathbb{C})$ iff $\sigma_+(\lambda) \neq \lambda$.*

3. *([OS], Lemma 4.7(ii)) $\sigma_+(\lambda) = \lambda$ iff either $\lambda \in \mathfrak{a}^*$, or $\lambda \in (\mathfrak{t}')^*$ and $\lambda = -w_0'\lambda$.*

4. *([OS], Lemma 4.7(iii)) If $\lambda \in \mathfrak{a}^*$, then $(k_0^2)^\lambda = 1$. If $\lambda \in (\mathfrak{t}')^*$ and $\lambda = -w_0'\lambda$, then $(k_0^2)^\lambda = \pm 1$.*

5. Let

$$\sigma_{G/T} : G/T \to G/T$$
$$gT \mapsto \sigma_G(g)k_0^{-1}T$$

Then the Weyl covering map

$$(G/T, \sigma_{G/T}) \times (T, Id) \to G^-$$

is a Real map.

Proof. We only show the last part. Let $\xi = \xi_1 + \xi_2$ with $\xi_1 \in \mathfrak{t}'$ and $\xi_2 \in \mathfrak{a}$. Then

$$\begin{aligned}
&\sigma_G(g)k_0^{-1}\exp(\sigma_+(\xi_1+\xi_2))k_0\sigma_G(g)^{-1}\\
=&\sigma_G(g)\exp(w_0'^{-1}(-\sigma_{\mathfrak{g}}(w_0'(\xi_1+\xi_2))))\sigma_G(g)^{-1}\\
=&\sigma_g(g)\exp(-\xi_1+\xi_2)\sigma_G(g)^{-1}\\
=&\sigma_G(g)\sigma_G(\exp(\xi_1+\xi_2))^{-1}\sigma_G(g)^{-1}
\end{aligned}$$

□

4.5 Construction of the equivariant Real fundamental DD bundle

In this Section, we shall first review the construction, as in [M2], of the equivariant DD-bundle $\mathcal{A}$ over G whose equivariant DD-class is $[-\eta_G] \in H_G^3(G, \mathbb{Z})$.

Then we point out how to equip $\mathcal{A}$ with a suitable Real structure so that it becomes a Real equivariant DD-bundle over G^- with Real equivariant DD-class $[-\eta_G] \in H^3_{G\rtimes\Gamma}(G^-, \mathbb{Z}_\Gamma)$.

Consider the restriction of $\mathcal{A}$ to a maximal torus T, viewed as a T-equivariant DD-bundle. The central extension

$$1 \longrightarrow U(1) \longrightarrow U(\mathcal{H}) \longrightarrow \mathrm{Aut}(\mathcal{K}(\mathcal{H})) \longrightarrow 1$$

is pulled back to a central extension $\widehat{T}$ of T via the T-action on $\mathrm{Aut}(\mathcal{K}(\mathcal{H}))$. In this way $\mathcal{A}|_T$ gives rise to a family of central extensions $\coprod_{t\in T} \widehat{T}_t$. Now for any $t_1, t_2 \in T$, $\widehat{T}_{t_1} \cong \widehat{T}_{t_2} \cong T \times U(1)$ as central extensions of T up to $H^2_T(\mathrm{pt}, \mathbb{Z}) \cong \Lambda^*$. Since the latter group is discrete, any path from t_1 to t_2 defines an isomorphism $\widehat{T}_{t_1} \to \widehat{T}_{t_2}$ up to an element in $H^2_T(\mathrm{pt}, \mathbb{Z})$ which only depends on the homotopy class of the path. We therefore can define a holonomy homomorphism

$$\pi_1(T) \cong H_1(T, \mathbb{Z}) \cong \Lambda \to H^2_T(\mathrm{pt}, \mathbb{Z}) \cong \Lambda^*$$

which is an element in $H^1(T, \mathbb{Z}) \otimes H^2_T(\mathrm{pt}, \mathbb{Z}) \cong \Lambda^* \otimes \Lambda^*$.

On the other hand, the restriction map

$$i_T^* : H^3_G(G, \mathbb{Z}) \to H^3_T(T, \mathbb{Z}) \cong H^2_T(\mathrm{pt}, \mathbb{Z}) \otimes H^1(T, \mathbb{Z}) \oplus H^3(T, \mathbb{Z})$$

is injective, and sends the generator η_G to minus the basic inner product $-B|_{\mathfrak{t}} \in \Lambda^* \otimes \Lambda^* \cong H^2_T(\mathrm{pt}, \mathbb{Z}) \otimes H^1(T, \mathbb{Z})$ (cf. [M2, Proposition 3.1]). Hence $\coprod_{t\in T} \widehat{T}_t$ is the $\widehat{T}$-bundle $\mathfrak{t} \times_\Lambda \widehat{T}$ where Λ acts on $\widehat{T} \cong T \times U(1)$ by

$$\lambda \cdot (t, z) = (t, t^{-B^\flat(\lambda)} z)$$

and the Weyl group W acts on $\mathfrak{t} \times_\Lambda \widehat{T}$ by

$$w \cdot [(\xi, t, z)] = [(w\xi, wt, z)]$$

The next step is to construct a suitable family of central extensions $\coprod_{g\in G} \widehat{G}_g$ of the stabilizer subgroup G_g which extends $\mathfrak{t} \times_\Lambda \widehat{T}$ in such a way that the induced action by conjugation on this family extends that of W on $\mathfrak{t} \times_\Lambda \widehat{T}$. This is done in [M2] by way of simplicial techniques, which we shall recall here.

Note that Δ parametrizes the orbit spaces of W-action on T, and

$$T = \coprod_I W/W_I \times \Delta_I / \sim$$

where, for $J \subset I$,

$$(\varphi_I^J(w), x) \sim (w, \iota_J^I(x)), \tag{4.1}$$

where $\varphi_I^J : W/W_I \to W/W_J$ is the natural projection and $\iota_J^I : \Delta_J \to \Delta_I$ the inclusion of simplices.

Definition 4.5.1. Let $\lambda_I : W_I \to \Lambda$ be defined by the equation $\Delta_I = w \cdot \Delta_I + \lambda_I(w)$ for $w \in W_I$.

Note that λ_I is a group cocycle and $\lambda_I|_{W_J} = \lambda_J$ for $J \subset I$.

Proposition 4.5.2. *The family of central extensions* $\mathfrak{t} \times_\Lambda \widehat{T}$ *is isomorphic to* $\coprod_I (W \times_{W_I} \widehat{T}) \times \Delta_I / \sim$ *W-equivariantly, where W_I acts on $\widehat{T}$ by*

$$w \cdot (t, z) = (wt, h^{B^\flat(\lambda_I(w^{-1}))})$$

Proof. We first check that the map $\widehat{T} \times \Delta_I \to \mathfrak{t} \times_\Lambda \widehat{T}$ defined by $(t, z, \xi) \mapsto [(\xi, t, z)]$ is W_I-equivariant. Indeed, for $\xi \in \Delta_I$,

$$\begin{aligned} w \cdot [(\xi, t, z)] &= [(w\xi; wh, z)] \\ &= [(\xi - \lambda_I(w); wh, z)] \\ &= [(\xi; wh, (wh)^{B^\flat(\lambda_I(w))} z)] \\ &= [(\xi; wh, h^{-B^\flat(\lambda_I(w^{-1})} z)] \end{aligned}$$

So it extends to the map $\coprod_I (W \times_{W_I} \widehat{T}) \times \Delta_I / \sim \to \mathfrak{t} \times_\Lambda \widehat{T}$, which is W-equivariant. $\square$

Since G admits the following similar simplicial description

$$G = \coprod_I G/G_I \times \Delta_I / \sim$$

the desired family of central extension $\coprod_{g \in G} \widehat{G}_g$ should be of the form

$$\coprod_I (G \times_{G_I} \widehat{G}_I) \times \Delta_I / \sim$$

where G_I acts by conjugation on $\widehat{G}_I$, which is a central extension of G_I such that

1. $\widehat{G}_I$ contains $\widehat{T}$ as the common maximal torus,
2. there are lifted inclusions $\widehat{i}^I_J : \widehat{G}_I \hookrightarrow \widehat{G}_J$ satisfying $\widehat{i}^I_K = \widehat{i}^I_J \circ \widehat{i}^J_K$ for $K \subset J \subset I$, and
3. the Weyl group element $w \in W_I \cong N_{\widehat{G}_I}(\widehat{T})/\widehat{T} \cong N_{G_I}(T)/T$ acts on $\widehat{T}$ by the action given in Proposition 4.5.2.

Proposition 4.5.3 ([M2])**.** *The central extension*

$$\widehat{G}_{I,t} := \widetilde{G}_I \times_{\pi_1(G_I)} U(1)$$

where $\widetilde{G}_I$ is the universal cover of G_I, $t = exp(\xi)$ for $\xi \in int(\Delta_I)$ and $\pi_1(G_I) = \Lambda/\Lambda_I$ acts on $U(1)$ by

$$\lambda \cdot t = t^{-B^\flat(\lambda)} = exp(-2\pi\sqrt{-1}B(\lambda, \xi))$$

satisfies the properties stated above.

Proof. The first two properties are easy. For the last one, we only need to note that if we choose $\widehat{T}$ to be the image of the map

$$\begin{aligned} i_I : T \times U(1) &\to \widehat{G}_{I,t} \\ (\exp_T(\zeta), z) &\mapsto [(\exp_{\widetilde{G}_I}(\zeta), \exp(-2\pi\sqrt{-1}B(\zeta, \xi))z)] \end{aligned}$$

then the conjugation action by a representative of $w \in W_I$ restricted to $\widehat{T}$ is indeed the one given in Proposition 4.5.2. □

The isomorphism classes of the central extensions $\widehat{G}_{I,t}$ are independent of t because $\exp(\mathrm{int}(\Delta_I))$ is contractible. We shall simply fix a choice of t and drop t from the subscript in the notation for the central extension of G_I from now on.

Lemma 4.5.4. *([M2, Lemma 3.5]) There exists a Hilbert space $\mathcal{H}$ equipped with unitary representations of the central extensions $\widehat{G}_I$ such that the central circle acts with weight -1 and, for $J \subset I$, the action of $\widehat{G}_J$ restricts to that of $\widehat{G}_I$.*

Putting $\mathcal{A}_I = G \times_{G_I} \mathcal{K}(\mathcal{H})$ and

$$\mathcal{A} = \coprod_I (\mathcal{A}_I \times \Delta_I)/\sim$$

where the relation $\sim$ is similar to the one used in Equation 4.1, we have that the family of central extensions induced by $\mathcal{A}$ is the opposite of $\coprod_{g\in G}\widehat{G}_g = \coprod_I (G\times_{G_I}\widehat{G}_I)\times\Delta_I/\sim$. The equivariant DD-class of $\mathcal{A}$ is therefore $[-\eta_G]$.

We end this section by pointing out how to endow $\mathcal{A}$ with a suitable Real structure which descends to the anti-involution on G, as promised at the beginning of this section. We choose T to be the distinguished maximal torus with respect to σ_G as in Section 4.4. Let $\sigma_{G/G_I} : G/G_I \to G/G_I$ be defined by $gG_I \mapsto \sigma_G(g)k_0^{-1}G_I$. By (5) of Proposition 4.4.1, the simplicial piece $G/G_I\times\Delta_I$ used in the simplicial description of G induces the anti-involution on G. Setting $\mathcal{H}' := \mathcal{H}\oplus\sigma_G^*\overline{\mathcal{H}}$ with the Real structure being swapping the two summands, we obtain a Real representation of $\widehat{G}_I$ with the central circle acting by -1, and $\mathcal{A}_I = G\times_{G_I}\mathcal{K}(\mathcal{H}')$. We find that the simplicial piece $\mathcal{A}_I\times\Delta_I$ with the involution

$$([g,F],\xi)\mapsto([\sigma_G(g)k_0^{-1},\overline{F}],\sigma_+(\xi))$$

induces the desired Real structure on $\mathcal{A}$.

Remark 4.5.5. The other generator of $H^3_{G\rtimes\Gamma}(G^-,\mathbb{Z}_\Gamma)$ of infinite order, $[-\eta_G] + \mathrm{DD}_{\mathbb{R}}(o_{\mathbb{R}^{0,4}})$, is the DD class of the same DD bundle $\mathcal{A}$ except that the involution $\sigma_{\mathcal{A}}$ is induced by the involution

$$([g,F],\xi)\mapsto([\sigma_G(g)k_0^{-1},J\overline{F}J^{-1}],\sigma_+(\xi))$$

on the simplicial pieces $\mathcal{A}_I\times\Delta_I$, where J is a 'quaternionic quarter turn' on $\mathcal{H}$.

4.6 A partial generalization of Freed-Hopkins-Teleman in *KR*-homology

This Section is devoted to computing the twisted equivariant *KR*-homology $KR_*^G(G^-, \mathcal{A}^{k+\mathsf{h}^\vee})$ in the special case where $R(G, \mathbb{C}) = 0$. We mainly follow the idea in [M2] of applying Segal's spectral sequence to the simplicial descriptions of G and $\mathcal{A}$ as in Section 4.5. The result, which is recorded in Theorem 4.6.4, is analogous to Freed-Hopkins-Teleman Theorem. It basically asserts that $KR_*^G(G^-, \mathcal{A}^{k+\mathsf{h}^\vee})$ is isomorphic to Verlinde algebra tensored with the coefficient ring $KR_*(\mathrm{pt})$, thus exhibiting Real formality (cf. Remark 2.3.2).

4.6.1 Conditions equivalent to $R(G, \mathbb{C}) = 0$

Proposition 4.6.1. *Suppose G is a compact semi-simple Real Lie group. The following are equivalent.*

1. $R(G, \mathbb{C}) = 0$.
2. $\sigma_+ : \mathfrak{t}^* \to \mathfrak{t}^*$ *is identity.*
3. $-w_0'$ *acts as identity on* $\mathfrak{t}'$.
4. $\mathfrak{k}'$ *is a direct sum of simple compact Lie algebras of types* not *from the following list:*
    (a) A_l *for* $l \geq 2$,
    (b) D_{2n+1}, *and*

(c) E_6.

Proof. If V_λ is the irreducible representation with λ as the highest weight, then $\overline{\sigma_G^*}V_\lambda \cong V_{\sigma_+(\lambda)}$. Hence (1) and (2) are equivalent. That (2) and (3) are equivalent can be seen by definition of σ_+. If $-w_0'$ is identity on $\mathfrak{t}'$, then w_0' acts nontrivially. Thus $\mathfrak{k}'$ is centerless, for otherwise, w_0' would act trivially on the center. So $\mathfrak{k}'$ is semi-simple, and a direct sum of simple compact Lie algebras. According to Table 1 of [Bour], the longest element of Weyl group acts as $-\mathrm{Id}$ on the maximal toral subalgebra except those listed in (4). This shows that (3) implies (4). The converse is easy. □

4.6.2 Representation-theoretic properties of twisted *KR*-homology

From now on we assume that $R(G, \mathbb{C}) = 0$. Below is a list of representation theoretic properties of equivariant twisted *KR*-homology which are adapted from their counterparts from [M2] to the Real context. They can be mostly deduced by Poincaré duality as in Section 1.6.

1. Suppose $\mathcal{B}$ is an equivariant Real G-DD bundle over a point, and $\widehat{G}$ is the central extension of G obtained by pulling back the central extension

$$1 \longrightarrow U(1) \longrightarrow U(\mathcal{H}) \longrightarrow \mathcal{A}ut(\mathcal{K}(\mathcal{H})) \longrightarrow 1$$

through the map $G \to \mathcal{A}ut(\mathcal{K}(\mathcal{H}))$. Define $RR(\widehat{G}, \mathbb{R})_k$ (reps. $RH(\widehat{G}, \mathbb{R})_k$) to be the Grothendieck group of Real $\widehat{G}$-representations of real type (reps. quaternionic type) where the central circle acts by weight k. A twisted KR-theoretic analogue of Proposition 2.4.3 yields

$$KR_G^*(\mathrm{pt}, \mathcal{B}) \cong (RR(\widehat{G}, \mathbb{R})_1 \oplus RH(\widehat{G}, \mathbb{R})_1 \otimes KR_*(\mathrm{pt}))$$

Applying Poincaré duality gives

$$KR_*^G(\mathrm{pt}, \mathcal{B}) \cong (RR(\widehat{G}, \mathbb{R})_{-1} \oplus RH(\widehat{G}, \mathbb{R})_{-1}) \otimes KR_*(\mathrm{pt})$$

2. The irreducible complex representations of G_I are indexed by those weights in $\Lambda^*_{I,+}$. By [P, Theorem 3], $K^*(G/G_I) \cong R(G_I) \otimes_{R(G)} \mathbb{Z}$ ($\mathbb{Z}$ is an $R(G)$-module through the augmentation map) and hence G/G_I is weakly equivariantly formal G-space. In particular, $K^*(G/G_I)$ is freely generated, as an abelian group, by the isomorphism classes of homogeneous vector bundles $G \times_{G_I} V_\lambda$, $\lambda \in \Lambda^*_{I,+}$. These vector bundles are equipped with anti-linear automorphism $[(g, v)] \mapsto [(\sigma_G(g)k_0^{-1}, \overline{v})]$, which is either a Real or Quaternionic structure according as whether $(k_0^2)^\lambda = 1$ or -1. By Theorem 2.3.1, we have an isomorphism

$$I^G_{G_I} : (R(G_I)^{\mathbb{R}} \oplus R(G_I)^{\mathbb{H}}) \otimes_{RR(G)} \mathbb{Z} \otimes KR^*(\mathrm{pt}) \to KR^*(G/G_I)$$

where $R(G_I)^{\mathbb{R}}$ (reps. $R(G_I)^{\mathbb{H}}$) is the abelian group freely generated by the isomorphism classes of irreducible complex representations V_λ with $(k_0^2)^\lambda = 1$ (reps. $(k_0^2)^\lambda = -1$). Here $R(G_I)^{\mathbb{R}}$ and $R(G_I)^{\mathbb{H}}$ are assigned with degree 0 and -4 respectively. The forgetful map $KR_G^*(G/G_I) \to KR^*(G/G_I)$ is obviously onto as the homogeneous vector bundles have canonical equivariant structure compatible with the Real or Quaternionic structure. Together with the fact that G/G_I is a weakly equivariantly formal G-space, we have that G/G_I is Real equivariantly formal by Definition 2.5.2, and the isomorphism (denoted again by $I_{G_I}^G$ by abuse of notation)

$$I_{G_I}^G : (R(G_I)^{\mathbb{R}} \oplus R(G_I)^{\mathbb{H}}) \otimes KR^*(\mathrm{pt}) \to KR_G^*(G/G_I)$$

by Theorem 2.5.5. It is worthwhile to note that $KR_G^*(G/G_I)$ is not isomorphic to $RR(G_I) \otimes KR^*(\mathrm{pt})$ in general, as one might expect from the fact that $K_G^*(G/G_I) \cong R(G_I)$. A counter-example is provided by $G = SU(2m)$, T the maximal torus of diagonal matrices, σ_G and $\sigma_{G/T}$ the symplectic type involution. One can easily generalize the above result to the twisted case. We had an isomorphism

$$I_{G_I}^G : (R(\widehat{G}_I)_k^{\mathbb{R}} \oplus R(\widehat{G}_I)_k^{\mathbb{H}}) \otimes KR^*(\mathrm{pt}) \to KR_G^*(G/G_I, (\mathcal{A}_I^{\mathrm{opp}})^k)$$

which is translated by Poincaré duality to the isomorphism

$$I_{G_I}^G : (R(\widehat{G}_I)_k^{\mathbb{R}} \oplus R(\widehat{G}_I)_k^{\mathbb{H}}) \otimes KR_*(\mathrm{pt}) \to KR_*^G(G/G_I, \mathcal{A}_I^k \otimes (G \times_{G_I} \mathrm{Cl}(\mathfrak{g}/\mathfrak{g}_I)))$$

Here $\widehat{G}_I$ is the central extension defined in Section 4.5.

3. According to the proof of [M2, Theorem 4.7], if S_I is the spinor module over $\mathrm{Cl}(\mathfrak{g}/\mathfrak{g}_I)$ induced by a choice of complex structure of $\mathfrak{g}/\mathfrak{g}_I$, and $\mathcal{H}$ is the Hilbert space used to construct the equivariant fundamental DD bundle $\mathcal{A}$ as in Lemma 4.5.4, the $\mathrm{Cl}(\mathfrak{g}/\mathfrak{g}_I) - \mathcal{K}(\mathcal{H}^{\mathsf{h}^\vee})$-bimodule $\mathrm{Hom}(\mathcal{H}^{\mathsf{h}^\vee}, \mathsf{S}_I)$ is G_I-equivariant and witnesses the Morita isomorphism between $\mathrm{Cl}(\mathfrak{g}/\mathfrak{g}_I)$ and $\mathcal{K}(\mathcal{H}^{\mathsf{h}^\vee})$. We can carry this result over to the Real situation by replacing $\mathcal{H}$ with $\mathcal{H}'$, the Real G-Hilbert space used in constructing the equivariant Real fundamental DD

bundle in Section 4.5, and equipping $\mathfrak{g}/\mathfrak{g}_I$ with the Real structure induced by σ_{G/G_I}, which is compatible with the complex structure determined by our choice of positive root system as in Section 4.4. Fixing $\xi \in \mathrm{int}(\Delta_I)$ and letting $\psi_I : G/G_I \to G$ be defined by $gG_I \mapsto g\exp(\xi)g^{-1}$, we have that $G \times_{G_I} \mathrm{Cl}(\mathfrak{g}/\mathfrak{g}_I)$ and $\psi_I^* \mathcal{A}^{\mathsf{h}^\vee} \cong G \times_{G_I} \mathcal{K}(\mathcal{H}'^{\mathsf{h}^\vee})$ are Morita isomorphic as Real G-DD bundles, the isomorphism being witnessed by $G \times_{G_I} \mathrm{Hom}(\mathcal{H}'^{\mathsf{h}^\vee}, \mathsf{S}_I)$.

4. Using (1), (2) and the Morita isomorphism $\mathrm{Cl}(\mathfrak{g}/\mathfrak{g}_I) \cong \mathcal{K}(\mathcal{H}'^{\mathsf{h}^\vee})$ witnessed by $\mathrm{Hom}(\mathcal{H}'^{\mathsf{h}^\vee}, \mathsf{S}_I)$ as in (3), we get that

$$
\begin{aligned}
KR^G_*(G/G_I, \mathcal{A}_I^{k+\mathsf{h}^\vee}) &\xrightarrow{(\mathrm{Id}, G\times_{G_I}\mathrm{Hom}(\mathcal{H}'^{\mathsf{h}^\vee}, \mathsf{S}_I))_*} KR^G_*(G/G_I, \mathcal{A}_I^k \otimes (G \times_{G_I} \mathrm{Cl}(\mathfrak{g}/\mathfrak{g}_I))) \\
&\xrightarrow{(I^G_{G_I})^{-1}} (R(\widehat{G}_I)^{\mathbb{R}}_k \oplus R(\widehat{G}_I)^{\mathbb{H}}_k) \otimes KR_*(\mathrm{pt})
\end{aligned}
$$

For $J \subset I$, the natural projection $\varphi_I^J : G/G_I \to G/G_J$ induces

$$
\varphi^J_{I\,*} : KR^G_*(G/G_I, \mathcal{A}^{k+\mathsf{h}^\vee}) \to KR^G_*(G/G_J, \mathcal{A}_J^{k+\mathsf{h}^\vee})
$$

which, with the above identification, amounts to the (Real) holomorphic induction map (tensored with $\mathrm{Id}_{KR_*(\mathrm{pt})}$)

$$
\mathbb{R}\mathrm{ind}_I^J \otimes \mathrm{Id}_{KR_*(\mathrm{pt})} : (R(\widehat{G}_I)^{\mathbb{R}}_k \oplus R(\widehat{G}_I)^{\mathbb{H}}_k) \otimes KR_*(\mathrm{pt}) \to (R(\widehat{G}_J)^{\mathbb{R}}_k \oplus R(\widehat{G}_J)^{\mathbb{H}}_k) \otimes KR_*(\mathrm{pt})
$$

which is taken with respect to the complex structure of $G_J/G_I \cong \widehat{G}_J/\widehat{G}_I$ determined by the choice of positive root system of G as in Section 4.4 (cf. [M2, Proposition 4.14]).

4.6.3 Computation of $KR^G_*(G^-, \mathcal{A}^{k+\mathsf{h}^\vee})$ when $R(G, \mathbb{C}) = 0$

The Segal's spectral sequence, introduced in [S1], can be viewed as a generalization of Mayer-Vietoris sequence and is best suited to computing the (co)homology of simplicial spaces. The simplicial description of G and $\mathcal{A}$ as in Section 4.5 makes this special kind of spectral sequence an ideal tool to compute $KR^G_*(G^-, \mathcal{A}^{k+\mathsf{h}^\vee})$. Its E^1-page is

$$\begin{aligned} E^1_{p,q} &= \bigoplus_{|I|=p+1} KR^G_{p+q}(G/G_I \times \Delta_I, G/G_I \times \partial\Delta_I, \mathcal{A}^{k+\mathsf{h}^\vee} \times \Delta_I) \\ &= \bigoplus_{|I|=p+1} KR^G_{p+q}(G/G_I \times B^{0,p}, G/G_I \times S^{0,p}, \mathcal{A}_I^{k+\mathsf{h}^\vee}) \\ &= \bigoplus_{|I|=p+1} KR^G_q(G/G_I, \mathcal{A}^{k+\mathsf{h}^\vee}) \end{aligned}$$

differential $d^1 : E^1_{p,q} \to E^1_{p-1,q}$ defined on the summand $KR^G_q(G/G_I, \mathcal{A}^{k+\mathsf{h}^\vee})$ as the alternating sum $\sum_{r=0}^p (-1)^r \varphi_I^{\delta_r I}{}_*$, where $\delta_r I = I \setminus \{i_r\}$ if $i_0 < i_1 < \cdots < i_p \in I$. The identification made in (4) of Section 4.6.2 enables us to rewrite the q-th row of the E^1-page as the complex

$$0 \to \bigoplus_{|I|=l+1} (R(\widehat{G}_I)^{\mathbb{R}}_k \oplus R(\widehat{G}_I)^{\mathbb{H}}_k) \otimes KR_*(\mathrm{pt}) \to \cdots \to \bigoplus_{|I|=1} (R(\widehat{G}_I)^{\mathbb{R}}_k \oplus R(\widehat{G}_I)^{\mathbb{H}}_k) \otimes KR_*(\mathrm{pt}) \to 0 \tag{4.2}$$

where d^1 becomes $\sum_{r=0}^{|I|} (-1)^r \mathbb{R}\mathrm{ind}_I^{\delta_r I} \otimes \mathrm{Id}_{KR_*(\mathrm{pt})}$ on the summand $(R(\widehat{G}_I)^{\mathbb{R}}_k \oplus R(\widehat{G}_I)^{\mathbb{H}}_k) \otimes KR_*(\mathrm{pt})$.

Let us for the time being ignore the coefficient ring $KR_*(\mathrm{pt})$ and the grading of the representation group $R(\widehat{G}_I)_k^{\mathbb{R}} \oplus R(\widehat{G}_I)_k^{\mathbb{H}}$ (or, the type of representations). What remains of the complex (4.2) after this simplification is the complex of $R(G)$-modules

$$0 \to \bigoplus_{|I|=l+1} R(\widehat{G}_I)_k \xrightarrow{\partial_l} \cdots \xrightarrow{\partial_1} \bigoplus_{|I|=1} R(\widehat{G}_I)_k \to 0 \tag{4.3}$$

with $\partial_p : \bigoplus_{|I|=p+1} R(\widehat{G}_I)_k \to \bigoplus_{|I|=p} R(\widehat{G}_I)_k$ being the alternating sum of ordinary holomorphic induction. This is exactly the E^1-page of Segal's spectral sequence for $K_*^G(G, \mathcal{A}^{k+\mathsf{h}^\vee})$, which is known to be acyclic, except in the zeroth position where the homology is the level k Verlinde algebra $R_k(G)$ (cf. [M2, Theorem 5.3], [Dou1, Proposition 2.4] and [Dou2, Proposition 2.1]). Moreover, by [M2, Theorem 5.3], we have

Proposition 4.6.2. *The complex of $R(G)$-modules (4.3) admits homotopy operators h_p : $\bigoplus_{|I|=p} R(\widehat{G}_I)_k \to \bigoplus_{|I|=p+1} R(\widehat{G}_I)_k$, $1 \leq p \leq l$.*

Proof. We shall give a proof inspired by the descriptions of $R(\widehat{G}_I)_k$ in terms of affine Weyl invariants of weights as in [Dou1] and [Dou2]. The complex of $R(G)$-modules can be rewritten as

$$0 \longrightarrow \bigoplus_{|I|=l+1} \mathbb{Z}[\Lambda^*]^{W_I^k} \xrightarrow{\partial_l} \cdots \xrightarrow{\partial_{p+1}} \bigoplus_{|I|=p+1} \mathbb{Z}[\Lambda^*]^{W_I^k} \xrightarrow{\partial_p} \cdots \xrightarrow{\partial_1} \bigoplus_{|I|=1} \mathbb{Z}[\Lambda^*]^{W_I^k} \longrightarrow 0$$

Note that $\mathbb{Z}[\Lambda^*]^{W_I^k}$ is spanned by $\frac{A_{\mu+\rho_I}^{W_I^k}}{A_{\rho_I}^{W_I^0}}$ for $\mu \in \Lambda^*$, where A_λ^W means the skew-symmetrization of λ with respect to the reflection group W. The map ∂_p : $\bigoplus_{|I|=p+1} \mathbb{Z}[\Lambda_I^*]^{W_I^k} \longrightarrow \bigoplus_{|J|=p} \mathbb{Z}[\Lambda_J^*]^{W_J^k}$ then takes the following form

$$\frac{A_{\mu+\rho_I}^{W_I^k}}{A_{\rho_I}^{W_I^0}} \mapsto \bigoplus_{r=0}^{p} (-1)^r \frac{A_{\mu+\rho_{\delta_r I}}^{W_{\delta_r I}^k}}{A_{\rho_{\delta_r I}}^{W_{\delta_r I}^0}}$$

Define, for $0 \leq i \leq l$, $h_p^i : \bigoplus_{|I|=p} \mathbb{Z}[\Lambda^*]^{W_I^k} \to \bigoplus_{|I|=p+1} \mathbb{Z}[\Lambda^*]^{W_I^k}$ by

$$\begin{cases} \dfrac{A_{\mu+\rho_I}^{W_I^k}}{A_{\rho_I}^{W_I^0}} \mapsto (-1)^{r_0} \dfrac{A_{\mu+\rho_{I\cup\{i\}}}^{W_{I\cup\{i\}}^k}}{A_{\rho_{I\cup\{i\}}}^{W_{I\cup\{i\}}^0}} & \text{if } i_{r_0-1} < i < i_{r_0} \\ \dfrac{A_{\mu+\rho_I}^{W_I^k}}{A_{\rho_I}^{W_I^0}} \mapsto 0 & \text{otherwise} \end{cases}$$

and $h_p := \bigoplus_{i=0}^{l} h_p^i$. Then h_p is the desired homotopy operator by straightforward computations. □

Remark 4.6.3. The Segal's spectral sequence collapses on the E^1-page, and hence $K_*^G(G, \mathcal{A}^{k+\mathsf{h}^\vee})$ is isomorphic to $R_k(G)$ as $R(G)$-modules. As this module isomorphism takes the ring homomorphism $\iota_* : K_*^G(\mathrm{pt}) \cong R(G) \to K_*^G(G, \mathcal{A}^{k+\mathsf{h}^\vee})$ to the quotient map $R(G) \to R_k(G)$ (cf. [M2, Remark 5.5]), it is actually a ring isomorphism.

Returning to the E^1-page of the spectral sequence of $KR_*^G(G, \mathcal{A}^{k+\mathsf{h}^\vee})$, we observe that the complex (4.2) is simply got by tensoring the complex (4.3) with $KR_q(\mathrm{pt})$ and equipping the representation group $R(\widehat{G_I})_k$ with a grading according to the type of representations. By Proposition 4.6.2 and the right exactness of taking tensoring product, (4.2) is also acyclic except at the zeroth position, whose homology is isomorphic to the level k Verlinde algebra tensored with $KR_*(\mathrm{pt})$. Applying similar arguments in Remark 4.6.3 yields the following main result in this Chapter.

Theorem 4.6.4 (Real Freed-Hopkins-Teleman for $R(G, \mathbb{C}) = 0$)**.** *Let $R(G, \mathbb{C}) = 0$, and I_k be generated by $r_1, \cdots, r_m \in R(G)$. Let RI_k be the ideal in $KR_*^G(pt) \cong (RR(G, \mathbb{R}) \oplus RH(G, \mathbb{R})) \otimes KR_*(pt)$ generated by the same set of isomorphism classes of representations,* *with the irreducible components of each r_i assigned with appropriate gradings (either* 0 *or* -4*) according to their types. Then the pushforward map*

$$KR^G_*(pt) \to KR^G_*(G^-, \mathcal{A}^{k+\mathsf{h}^\vee})$$

is onto with kernel RI_k.

Remark 4.6.5. 1. In [Dou2], Douglas gave an explicit description of Verlinde algebra of G. In particular, he gave a list of generators of I_k for each type of simple, connected, simply-connected and compact Lie groups in such a way that the number of generators is independent of the level k. One may use this list to get RI_k and hence a description of $KR^G_*(G^-, \mathcal{A}^{k+\mathsf{h}^\vee})$.

2. By the above Theorem, the degree 0 part $KR^G_0(G^-, \mathcal{A}^{k+\mathsf{h}^\vee}) \cong RR(G)/(RI_k \cap RR(G))$ gives the *Real Verlinde algebra*, the Grothendieck group of the isomorphism classes of Real positive energy representations of the Real loop group LG, where the Real structure of LG is given by

$$\begin{aligned} \sigma_{LG} : LG &\to LG \\ \ell &\mapsto \sigma_G \circ \ell \circ c \end{aligned}$$

with c meaning reflection on the unit circle.

We are unable to apply the Segal's spectral sequence argument to get a description of $KR^G_*(G^-, \mathcal{A}^{k+\mathsf{h}^\vee})$ for $R(G, \mathbb{C}) \neq 0$. The main obstacle is the far more complicated form of the E^1-page as a result of the nontrivial Real structure on Δ_I. Nevertheless, we propose the following

Conjecture 4.6.6. *Let the level k Verlinde ideal I_k be generated by $r_1, \cdots, r_m \in R(G)$, and RI_k be the ideal in $KR^G_*(pt)$ with generators obtained from $r_1, \cdots, r_m$ by the followings.*

1. *If the irreducible component of r_i can be made a Real (resp. Quaternionic) representation, then assign it with degree 0 (resp. -4).*
2. *If the irreducible component s of r_i is a complex type representation, then replace it with the double $s + \overline{\sigma^*_G} s$, which is assigned with degree 0.*

Then the pushforward map

$$KR^G_*(pt) \to KR^G_*(G^-, \mathcal{A}^{k+h^\vee})$$

is onto with kernel RI_k.

We hope to find an analogy of the structure theorem, Theorem 2.5.5 for equivariant KR-homology, and hence an easier way to compute $KR^G_*(G^-, \mathcal{A}^{k+\mathsf{h}^\vee})$ in general cases using just the statement of Freed-Hopkins-Teleman Theorem and the knowledge of how $\overline{\sigma^*_G}$ acts on Verlinde algebra. It may be useful to find a description of the K-homology classes of $K^G_*(G, \mathcal{A}^{k+\mathsf{h}^\vee})$ using twisted Fredholm modules or twisted geometric K-homology cycles (cf. [W]), so that 'Real lifts' of those classes in $KR^G_*(G^-, \mathcal{A}^{k+\mathsf{h}^\vee})$ can be defined explicitly.

4.7 Future work

Apart from working on Conjecture 4.6.6, we would like to pursue in the direction of geometric quantization in the future.

Traditionally, geometric quantization is a mathematical procedure of defining a quantum theory corresponding to a classical phase space. In its greatest generality, this procedure associates an equivariant prequantum line bundle L on a Hamiltonian G-manifold M to the equivariant index of the Spin^c Dirac operator of the spinor bundle twisted by L, which is a G-representation (see [M3, Definition 4.3]). It was conjectured by Guillemin-Sternberg in [GS] and proved by Meinrenken in [M1] that taking the geometric quantization of the symplectic reduction of M is the same as taking the trivial subrepresentation of its geometric quantization. This is the so-called 'quantization commutes with reduction' principle. For a detailed account see [Sj].

q-Hamiltonian manifolds, introduced in [AMM], are a variant of Hamiltonian manifolds which possess G-valued moment map

$$\Phi : M \to G$$

The equivalence of categories of q-Hamiltonian G-manifolds and Hamiltonian LG-manifolds enables one to define the notion of geometric quantization of q-Hamiltonian manifolds with index of Spin^c Dirac operators. In this case the quantum theory is positive energy representations of LG. One drawback of such a formulation is that it involves analysis in infinite dimensional terms. Recently, Meinrenken proposed an elegant formulation of geometric quantization of q-Hamiltonian manifolds using Freed-Hopkins-Teleman Theorem (cf. [M3], [M4]). The prequantum line bundle is replaced by a level k prequantum datum which is a Morita morphism

$$(\Phi, \mathcal{E}) : \mathrm{Cl}(TM) \to \mathcal{A}^{k+\mathsf{h}^\vee}$$

which induces the pushforward map in K-homology

$$(\Phi, \mathcal{E})_* : K_*^G(M, \mathrm{Cl}(TM)) \to K_*^G(G, \mathcal{A}^{k+\mathsf{h}^\vee})$$

The geometric quantization of M is defined to be $(\Phi, \mathcal{E})_*([M])$, where $[M]$ denotes the fundamental class. This quantization scheme has several advantages in that it does not involve quantizing the corresponding Hamiltonian LG-space, which is an infinite dimensional Banach manifold, and does not mention any twisted Dirac operator at all. Meinrenken also proved 'quantization commutes with reduction' under this formulation.

It would be interesting to investigate the Real analogue of this quantization scheme. As a first step in this direction, we have defined Real q-Hamiltonian G-manifolds, based on the definition of Real Hamiltonian G-manifolds given in [OS]. We found that the G-moment map Φ is then a Real map from M to G, the latter of which is endowed with an anti-involution. This gives yet another (meta-) reason why one should consider $KR_*^G(G^-, \mathcal{A}^{k+\mathsf{h}^\vee})$. Owing to the richer structure of $KR_*^G(G^-, \mathcal{A}^{k+\mathsf{h}^\vee})$ than that of Verlinde algebras, it will be interesting to understand what physical interpretations the extra information gives. For example, what do the torsions and the elements in higher degrees of the twisted equivariant KR-

homology mean under the Real quantization scheme? We would like to address this kind of questions in the future.

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