COHOMOLOGY AND K-THEORY OF COMPACT LIE GROUPS

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Abstract. In this expository article, we review the computation of the (de Rham) cohomology of compact connected Lie groups and the K-theory of compact, connected and simply-connected Lie groups.

Contents

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1. Introduction

In this expository article we give an account of the computation of the (de Rham) cohomology and K-theory of compact Lie groups based on the classical work of [CE] and [A1], as well as the article [R]. These become standard results in the algebraic topology of compact Lie groups.

For the computation of the cohomology groups of compact Lie groups, we demonstrate the use of the averaging trick to show that it suffices to compute the cohomology using left-invariant differential forms, which in turn have a natural correspondence with skew-symmetric multilinear forms on the Lie algebra of the Lie group. In this way the whole situation is reduced to computing the Lie algebra cohomology. One may further restrict to the bi-invariant differential forms, the advantage of which is that these forms are automatically closed. We shall then derive some interesting results about the topology of compact Lie groups using this elegant technique.

As to the cohomology ring structure, we first review the basics of Hopf algebras, which include the cohomology of compact connected Lie groups as motivating examples. Hopf's determination of the general ring structure of cohomology of compact connected Lie groups by means of a result on the classification of Hopf algebras will then be presented. To obtain more information about the cohomology ring we shall appeal to the well-known map

$$\begin{aligned} p : G/T \times T &\to G \\ (g,t) &\mapsto gtg^{-1} \end{aligned} \tag{1}$$

where T is a maximal torus of G. Using p, we can deduce information about $H^*(G,\mathbb{R})$ by computing $H^*(G/T,\mathbb{R})$ and $H^*(T,\mathbb{R})$. More precisely, it will be shown that

$$p^* : H^*(G,\mathbb{R}) \to (H^*(G/T,\mathbb{R}) \otimes H^*(T,\mathbb{R}))^W \tag{2}$$

is a ring isomorphism, where W is the Weyl group of G. We will describe the W-module structure of $H^*(G/T,\mathbb{R})$ using Morse theory. Making use of invariant theory, in particular the famous theorem by Borel that $H^*(G/T,\mathbb{R})$ is isomorphic to the space harmonic polynomials on $\mathfrak{t}$ and Solomon's result on W-invariants of differential forms on $\mathfrak{t}$ with polynomial coefficients(c.f. [So]), Reeder interpreted the right-hand side of (2) as W-invariant subspace of differential forms with harmonic polynomial coefficients and gave a detailed description of it(c.f. [R]). It will be shown in the end that we have

Theorem 1.1. *If G is compact and connected, then $H^*(G,\mathbb{R})$ is an exterior algebra generated by elements from odd cohomology groups.*

For K-theory, we will be only concerned about simply-connected compact Lie groups. The structure of the K-theory is immediate once we know that $K^*(G)$ is torsion-free and apply the fact that rational cohomology ring and rational K-theory of a finite CW-complex are isomorphic through the Chern character(c.f. [AH]). In fact

Theorem 1.2. *If G is a compact, simply-connected Lie group, then $K^*(G)$ is an exterior algebra generated by elements in $K^{-1}(G)$ induced by fundamental representations of G.*

The proof from [A1] will be reproduced, with the technical details coming from K-theory carefully explained.

2. Cohomology of compact Lie groups

The treatment of this section is mainly based on [CE]. Let G be a compact connected Lie group. We denote the space of differential n-forms of G by $\Omega^n(G)$, and use L_g to mean the map of left multiplication by g. A differential form $\omega \in \Omega^*(G)$ is left-invariant if

$$L_g^*\omega = \omega \quad \text{for all } g \in G$$

The space of left-invariant n-forms is a subspace of $\Omega^n(G)$, which will be denoted by $\Omega_L^n(G)$.

Lemma 2.1. *If $\omega \in \Omega_L^n(G)$, then $d\omega \in \Omega_L^n(G)$.*

Proof. Note that for all $g \in G$,

$$L_g^*d\omega = dL_g^*\omega = d\omega$$

□

Lemma 2.2. *If $\omega_1 \in \Omega_L^n(G)$, $\omega_2 \in \Omega_L^m(G)$, then $\omega_1 \wedge \omega_2 \in \Omega_L^{m+n}(G)$.*

Proof. Simply note that $L_g^*(\omega_1 \wedge \omega_2) = L_g^*\omega_1 \wedge L_g^*\omega_2$. □

By Lemma 2.1, $\Omega_L^*(G)$ is a complex with differential d. Let $H_L^*(G)$ be the cohomology of $\Omega_L^*(G)$. Lemma 2.2 implies that $H_L^*(G)$ has a ring structure induced by wedge product.

Since $\Omega_L^*(G)$ is a subspace of $\Omega^*(G)$, there is an inclusion map $\iota : \Omega_L^*(G) \to \Omega^*(G)$, which induces

$$\iota_* : H_L^*(G) \to H^*(G, \mathbb{R})$$

It is plain that ι_* is a ring homomorphism. In fact we have

Theorem 2.3. *ι_* is a ring isomorphism.*

Before giving a proof of theorem we shall introduce the 'averaging trick'. Let $\omega \in \Omega^n(G)$. Define

$$I\omega = \int_G L_g^* \omega dg$$

where dg denotes the normalized Haar measure of G. By the translation invariance of dg, $I\omega \in \Omega_L^n(G)$. One can easily show that the averaging operator I is linear, and commutes with d and pullbacks.

Proof of Theorem 2.3. First we shall show that ι_* is injective. Suppose that $\omega \in \Omega_L^n(G)$ and $\omega = d\mu$ for some $\mu \in \Omega^{n-1}(G)$. Then

$$\omega = I\omega = Id\mu = dI\mu$$

So $\omega \in d(\Omega_L^{n-1}(G))$ and thus $[\omega] = 0$ in $H_L^*(G)$.

To show that ι_* is surjective, we shall, for any $\omega \in \Omega^n(G)$ with $d\omega = 0$, find $\mu \in \Omega^{n-1}(G)$ such that $\omega + d\mu \in \Omega_L^n(G)$.

Claim 2.4. *There exists $\mu \in \Omega^{n-1}(G)$ such that $\omega + d\mu = I\omega$.*

Proof. Recall the de Rham theorem, which asserts, for compact manifold M, the nondegenerate pairing between the cohomology class $[\omega] \in H^n(M, \mathbb{R})$ and the homology class $[c] \in H_n(M, \mathbb{R})$ represented by a linear combination of n-chains $c = \sum_i r_i Z_i$, where Z_i is closed n-dimensional submanifolds. The pairing is given by

$$\langle [\omega], [c] \rangle = \sum_i r_i \int_{Z_i} \omega$$

If we can show that

$$\int_Z \omega - I\omega = 0 \tag{3}$$

for any n-dimensional submanifold, then $[\omega - I\omega] = 0$ and hence $\omega - I\omega$ is exact.

Note that, since G is connected, there exists a continuous path $s : I \to G$ between e and g. Thus L_g is homotopic to Id_G. It follows that the induced map

$$L_{g_*} : H_n(G, \mathbb{R}) \to H_n(G, \mathbb{R})$$

is identity, and that $[gZ] = [Z]$ for any n-dimensional submanifold Z. With this in mind, Equation (3) can be proved as follows.

$$\begin{aligned}\int_Z (\omega - I\omega) &= \int_Z \omega - \int_Z \int_G L_g^* \omega dg \\ &= \int_Z \omega - \int_G \int_Z L_g^* \omega dg \quad \text{(Fubini's theorem)} \\ &= \int_Z \omega - \int_G \int_{gZ} \omega dg \\ &= \int_Z \omega - \int_G \int_Z \omega dg \\ &= 0\end{aligned}$$

□

□

Theorem 2.3 implies that it is sufficient to use only the left-invariant forms on G to compute its cohomology. By way of left invariance, we can further restrict our attention to skew-symmetric multilinear forms on $\mathfrak{g}$, the Lie algebra of G, as one can easily observe that

Lemma 2.5. *The map*

$$\begin{aligned}\varphi : \Omega_L^*(G) &\to (\bigwedge^* \mathfrak{g})^* \\ \omega &\mapsto \{\omega\} := \omega|_{\bigwedge^n T_e G}\end{aligned}$$

is a ring isomorphism.

Proof. Note that φ is linear. Surjectivity is easy: given $\alpha \in (\bigwedge^n \mathfrak{g})^*$, define $\omega \in \Omega^n(G)$ such that

$$\omega_g((X_1)_g, \cdots, (X_n)_g) = \alpha(L_{g^{-1}*}(X_1)_g, \cdots, L_{g^{-1}*}(X_n)_g)$$

for any vector fields X_i, $1 \leq i \leq n$, on G. $\omega \in \Omega_L^n(G)$ because

$$\begin{aligned}(L_g^*\omega)_h((X_1)_h, \cdots, (X_n)_h) &= \omega_{gh}(L_{g*}(X_1)_h, \cdots, (L_g)_*(X_n)_h) \\ &= \alpha(L_{h^{-1}*}(X_1)_h, \cdots, L_{h^{-1}*}(X_n)_h) \\ &= \omega_h((X_1)_h, \cdots, (X_n)_h)\end{aligned}$$

Let $\omega \in \Omega_L^n(G)$ such that $\{\omega\} = 0$. Then for any $(X_1)_g, \cdots, (X_n)_g \in T_gG$,

$$\begin{aligned}\omega_g((X_1)_g, \cdots, (X_n)_g) &= (L_g^*\omega)_e({L_{g^{-1}}}_*(X_1)_g, \cdots, {L_{g^{-1}}}_*(X_n)_g)\\ &= \omega_e({L_{g^{-1}}}_*(X_1)_g, \cdots, {L_{g^{-1}}}_*(X_n)_g)\\ &= 0\end{aligned}$$

Note that φ is also a ring homomorphism. Together with the above we have shown that indeed φ is a ring isomorphism. □

Recall that the differential d on $\Omega^n(G)$ can be explicitly defined by

$$\begin{aligned} d\omega(X_1, \cdots, X_{n+1}) = \sum_{i=1}^{n+1}(-1)^i X_i\omega(X_1, \cdots, \widehat{X}_i, \cdots, X_{n+1}) +\\ \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_1, \cdots, \widehat{X}_i, \cdots, \widehat{X}_j, \cdots, X_{n+1})\end{aligned} \tag{4}$$

where $X_1, \cdots, X_{n+1}$ are vector fields on G.

Lemma 2.6. *The map $\delta : (\bigwedge^n \mathfrak{g})^* \to (\bigwedge^{n+1} \mathfrak{g})^*$ which makes the following diagram commutes*

$$\begin{tikzcd} \Omega_L^n(G) \arrow[r, "d"] \arrow[d, "\varphi"'] & \Omega_L^{n+1}(G) \arrow[d, "\varphi"] \\ (\bigwedge^n \mathfrak{g})^* \arrow[r, "\delta"'] & (\bigwedge^{n+1} \mathfrak{g})^* \end{tikzcd}$$

is given by

$$\delta\alpha(\xi_1, \cdots, \xi_{n+1}) = \sum_{i<j}(-1)^{i+j}\alpha([\xi_i, \xi_j], \xi_1, \cdots, \widehat{\xi_i}, \cdots, \widehat{\xi_j}, \cdots, \xi_{n+1})$$

Proof. Let $(X_i)_g = {L_g}_*(\xi_i)$ be the left-invariant vector field which restricts to $\xi_i \in T_eG \cong \mathfrak{g}$ at the identity e. Then

$$\begin{aligned}(\varphi d\omega)(\xi_1, \cdots, \xi_{n+1}) &= d\omega(X_1, \cdots, X_{n+1})((X_i)_g = {L_g}_*(\xi_i))\\ &= \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_1, \cdots, \widehat{X}_i, \cdots, \widehat{X}_j, \cdots, X_n)\\ &\text{(The first term of (4) vanishes as both } \omega \text{ and } X_i \text{ are left invariant)}\\ &= \sum_{i<j}(-1)^{i+j}\{\omega\}([(X_i)_e, (X_j)_e], \cdots, (\widehat{X}_i)_e, \cdots, (\widehat{X}_j)_e, \cdots, (X_n)_e)\\ &= (\delta\varphi\omega)(\xi_1, \cdots, \xi_{n+1})\end{aligned}$$

□

Theorem 2.7. $H_L^*(G) \cong H^*((\bigwedge^* \mathfrak{g})^*, \delta)$

Proof. By Lemma 2.5 and Lemma 2.6, the two complexes $(\Omega_L^*(G), d)$ and $((\bigwedge^* \mathfrak{g})^*, \delta)$ are isomorphic. □

Definition 2.8. The cohomology ring $H^*((\bigwedge^* \mathfrak{g})^*, \delta)$ is called the Lie algebra cohomology of $\mathfrak{g}$, denoted by $H^*(\mathfrak{g})$.

Corollary 2.9. $H^*(G, \mathbb{R}) \cong H_L^*(G) \cong H^*(\mathfrak{g})$*, if G is a compact connected Lie group.*

Remark 2.10. Corollary 2.9 says that the cohomology of a compact connected Lie group is completely determined by local information(i.e. the Lie algebra cohomology of $\mathfrak{g}$). This is made possible by virtue of the symmetry of G, as encapsulated in the trick of averaging left multiplication and the left invariance of forms in $\Omega_L^*(G)$.

Corollary 2.11. *If $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic Lie algebras, then $H^*(G_1, \mathbb{R}) \cong H^*(G_2, \mathbb{R})$ where G_i is a compact connected Lie group such that $T_eG_i = \mathfrak{g}_i$, $i = 1, 2$.*

Example 2.12. Since $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, $H^*(SU(2), \mathbb{R}) \cong H^*(SO(3), \mathbb{R})$. Indeed, both are isomorphic to $\mathbb{R}[x]/(x^2)$.

Remark 2.13. The corollary becomes false if $H^*(G_i, \mathbb{R})$ is replaced by $H^*(G_i, \mathbb{Z})$, $i = 1, 2$. A counterexample is provided by $SU(2)$ and $SO(3)(H^2(SU(2), \mathbb{Z}) = 0$ while $H^2(SO(3), \mathbb{Z}) \cong \mathbb{Z}_2)$.

Example 2.14. Let $G = (S^1)^n$, the n-dimensional torus. Then $\mathfrak{g} \cong \mathbb{R}^n$ with trivial Lie bracket. So $\delta \equiv 0$ and

$$H^*((S^1)^n, \mathbb{R}) \cong \bigwedge^* \mathbb{R}^n$$

One can take a step further by considering the complex of bi-invariant differential forms, i.e. forms that are both left-invariant and right-invariant. Let

$$\Omega_B^*(G) = \Omega_L^*(G) \cap \Omega_R^*(G)$$

denote the space of bi-invariant forms of G. Using the averaging trick as in the proof of Theorem 2.3, we obtain

Proposition 2.15. *The inclusion $\iota : \Omega_B^*(G) \hookrightarrow \Omega^*(G)$ induces a ring isomorphism $\iota_* : H_B^*(G) \to H^*(G, \mathbb{R})$.*

Definition 2.16. Let $(\bigwedge^* \mathfrak{g})^{*G}$ denote the subspace invariant under the adjoint action. Let

$$\psi : \Omega_B^*(G) \to (\bigwedge^* \mathfrak{g})^{*G}$$

map ω to its restriction ω_e at the identity.

Proposition 2.17. *$(\bigwedge^* \mathfrak{g})^{*G} = \{\alpha \in (\bigwedge^* \mathfrak{g})^* | \sum_{i=1}^n \alpha(\xi_1, \cdots, [\xi, \xi_i], \cdots, \xi_n) = 0$ for any $\xi, \xi_i \in \mathfrak{g}, 1 \leq i \leq n\}$*

Proof. Suppose $\sum_{i=1}^n \alpha(\xi_1, \cdots, [\xi, \xi_i], \cdots, \xi_n) = 0$ for any ξ, $\xi_i \in \mathfrak{g}$ for $1 \leq i \leq n$. For any given $g \in G$, there exists $\xi \in \mathfrak{g}$ such that $\exp(t\xi)$, $0 \leq t \leq 1$, joins e and g, as G is connected. So

$$\begin{aligned} &\frac{d}{dt}\bigg|_{t=0} \alpha(\mathrm{Ad}_{\exp(t\xi)}\xi_1, \cdots, \mathrm{Ad}_{\exp(t\xi)}\xi_n) \\ =&\sum_{i=1}^n \alpha(\xi_1, \cdots, [\xi, \xi_i], \cdots, \xi_n) = 0 \end{aligned}$$

It follows that

$$\alpha(\mathrm{Ad}_g\xi_1, \cdots, \mathrm{Ad}_g\xi_n) = \alpha(\xi_1, \cdots, \xi_n)$$

and so $\alpha \in (\bigwedge^* \mathfrak{g})^{*G}$. The converse is easy. □

Proposition 2.18. *ψ is a ring isomorphism.*

Proof. Since ψ is a ring homomorphism, it remains to show that it is both injective and surjective. Injectivity can be shown in the same way as in the proof of Lemma 2.5. Given $\alpha \in (\bigwedge^* \mathfrak{g})^{*G}$, let $\omega \in \Omega^*(G)$ be such that

$$\omega_g((X_1)_g, \cdots (X_n)_g) = \alpha(L_{g^{-1}*}(X_1)_g, \cdots, L_{g^{-1}*}(X_n)_g)$$

Obviously $\omega \in \Omega_L(G)$. Moreover,

$$\begin{aligned} &\omega_{gh}(R_{h*}(X_1)_g, \cdots, R_{h*}(X_n)_g) \\ =&\alpha(L_{h^{-1}g^{-1}*}R_{h*}(X_1)_g, \cdots, L_{h^{-1}g^{-1}*}R_{h*}(X_n)_g) \\ =&\alpha(\mathrm{Ad}_{h^{-1}}L_{g^{-1}*}(X_1)_g, \cdots, \mathrm{Ad}_{h^{-1}}L_{g^{-1}*}(X_n)_g) \\ =&\alpha(L_{g^{-1}*}(X_1)_g, \cdots, L_{g^{-1}*}(X_n)_g) \\ =&\omega_g((X_1)_g, \cdots, (X_n)_g) \end{aligned}$$

Hence $\omega \in \Omega_R(G)$. □

Similar to Lemma 2.6, we have

Proposition 2.19. *The following diagram*

$$\begin{tikzcd} \Omega_B^n(G) \arrow[r, "d"] \arrow[d, "\psi"'] & \Omega_B^{n+1}(G) \arrow[d, "\psi"] \\ (\bigwedge^n \mathfrak{g})^{*G} \arrow[r, "\delta"'] & (\bigwedge^{n+1} \mathfrak{g})^{*G} \end{tikzcd}$$

commutes

Corollary 2.20. *The two complexes $(\Omega_B^*(G), d)$ and $((\bigwedge^* \mathfrak{g})^{*G}, \delta)$ are isomorphic, and hence their cohomology rings $H_B^*(G)$ and $H^*((\bigwedge^* \mathfrak{g})^{*G}, \delta)$ are isomorphic as well.*

Proposition 2.21. *For any $\alpha \in (\bigwedge^* \mathfrak{g})^{*G}$, $\delta\alpha = 0$.*

Proof. Fix k. Then

$$\begin{aligned} &\sum_{j<k}(-1)^{j+k}\alpha([X_j, X_k], X_1, \cdots, \widehat{X_j}, \cdots, \widehat{X_k}, \cdots, X_{n+1}) \\ &\quad + \sum_{j>k}(-1)^{j+k}\alpha([X_k, X_j], \cdots, \widehat{X_k}, \cdots, \widehat{X_j}, \cdots, X_{n+1}) \\ =&\sum_{j<k}(-1)^{k+1}\alpha(X_1, \cdots, X_{j-1}, [X_k, X_j], X_{j+1}, \cdots, \widehat{X_k}, \cdots, X_{n+1}) \\ &\quad + \sum_{j>k}(-1)^{k+1}\alpha(X_1, \cdots, \widehat{X_k}, \cdots, [X_k, X_j], \cdots, X_{n+1}) \\ =&0 \quad \text{by Proposition 2.17} \end{aligned}$$

It follows that

$$\begin{aligned} 0 &= \sum_{k=1}^{n+1}\left(\sum_{j<k}(-1)^{j+k}\alpha([X_j, X_k], X_1, \cdots, \widehat{X_j}, \cdots, \widehat{X_k}, \cdots, X_{n+1})\right. \\ &\quad \left. + \sum_{j>k}(-1)^{j+k}\alpha([X_k, X_j], \cdots, \widehat{X_k}, \cdots, \widehat{X_j}, \cdots, X_{n+1})\right) \\ &= 2\sum_{j<k}(-1)^{j+k}\alpha([X_j, X_k], X_1, \cdots, \widehat{X_j}, \cdots, \widehat{X_k}, \cdots, X_{n+1}) \\ &= 2(\delta\alpha)(X_1, \cdots, X_{n+1}) \end{aligned}$$

$\square$

Corollary 2.22. *$H^*((\bigwedge^* \mathfrak{g})^{*G}, \delta) \cong (\bigwedge^* \mathfrak{g})^{*G}$. Hence $H^*(G, \mathbb{R}) \cong (\bigwedge^* \mathfrak{g})^{*G}$.*

Actually, Corollary 2.22 does not come in handy for explicit computation of $H^*(G, \mathbb{R})$ in general. We will give a fuller description of $H^*(G, \mathbb{R})$ in the next Section. Nonetheless, we can still deduce lower dimensional cohomology groups of G easily from Corollary 2.22.

Theorem 2.23. *Let G be a compact connected semi-simple Lie group. Then $H^1(G, \mathbb{R}) = H^2(G, \mathbb{R}) = 0$, $H^3(G, \mathbb{R}) \neq 0$.*

Proof. If $\alpha \in \mathfrak{g}^{*G}$, then $\alpha([X_1, X_2]) = 0$ for all $X_1, X_2 \in \mathfrak{g}$. Since $\mathfrak{g}$ is semi-simple, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Hence α must be 0 and $\mathfrak{g}^{*G} = H^1(G, \mathbb{R}) = 0$.

Let $\beta \in (\bigwedge^2 \mathfrak{g})^{*G}$. Then

$$\beta([X_1, X_2], X_3) + \beta(X_2, [X_1, X_3]) = 0$$

On the other hand, Proposition 2.21 implies that

$$(\delta\beta)(X_1, X_2, X_3) = -\beta([X_1, X_2], X_3) + \beta([X_1, X_3], X_2) - \beta([X_2, X_3], X_1) = 0$$

The above two equations implies that $\beta([X_2, X_3], X_1) = 0$ for all $X_i \in \mathfrak{g}$, $1 \leq i \leq 3$. So $(\bigwedge^2 \mathfrak{g})^{*G} = H^2(G, \mathbb{R}) = 0$.

Finally, consider the 3-form

$$\gamma(X_1, X_2, X_3) = B([X_1, X_2], X_3) \in (\bigwedge^3 \mathfrak{g})^{*G}$$

where B is the Killing form of $\mathfrak{g}$. Since B is non-degenerate when $\mathfrak{g}$ is semi-simple, γ is not zero. □

Theorem 2.24. *A compact connected Lie group G is semi-simple iff $\pi_1(G)$ is finite.*

Proof. We shall first claim that $H^1(G, \mathbb{R}) = 0$ iff $\pi_1(G)$ is finite. Note that $H^1(G, \mathbb{R}) = 0$ iff $H^1(G, \mathbb{Z})$ is a torsion abelian group. Since G is compact, $H^1(G, \mathbb{Z})$ must be finitely generated. Thus $H^1(G, \mathbb{Z})$ is a finite abelian group iff $H^1(G, \mathbb{R}) = 0$. By the universal coefficient theorem,

$$H^1(G, \mathbb{Z}) \cong \mathrm{Hom}(H_1(G, \mathbb{Z}), \mathbb{Z}) \oplus \mathrm{Ext}(H_0(G, \mathbb{Z}), \mathbb{Z}) \cong \mathrm{Hom}(H_1(G, \mathbb{Z}), \mathbb{Z})$$

So $H^1(G, \mathbb{Z})$ is a finite abelian group iff $H_1(G, \mathbb{Z})$ is also a finite abelian group. But $H_1(G, \mathbb{Z}) \cong \pi_1(G)_{ab} = \pi_1(G)$(the last equality holds because $\pi_1(G)$ is abelian(c.f. [BtD], Ch. V, Thm. 7.1)). This establishes the claim. If G is compact, connected and semi-simple, then by Theorem 2.23, $H^1(G, \mathbb{R}) = 0$. So $\pi_1(G)$ is finite by the claim. If $\pi_1(G)$ is finite, then $H^1(G,\mathbb{R})=0$, which in turn implies that $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$. Note that $\mathfrak{g}$ is a reductive Lie algebra and can be written as

$$\mathfrak{g}=\mathfrak{z}(\mathfrak{g})\oplus\mathfrak{h}$$

where $\mathfrak{h}$ is semi-simple. The condition $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$ forces $\mathfrak{z}(\mathfrak{g})=0$. This completes the proof. $\square$

Remark 2.25. From the proof of Theorem 2.24, we can see that if G is compact, connected and semi-simple, then $H^1(G,\mathbb{Z})=0$. To compute $H^2(G,\mathbb{Z})$, we may again make use of the universal coefficient theorem to get

$$H^2(G,\mathbb{Z})\cong\mathrm{Hom}(H_2(G,\mathbb{Z})\oplus\mathrm{Ext}(H_1(G,\mathbb{Z}),\mathbb{Z})$$

Since $H^2(G,\mathbb{Z})$ must be a finite abelian group, and $\mathrm{Hom}(\cdot,\mathbb{Z})$ is either a free abelian group or 0, $\mathrm{Hom}(H_2(G,\mathbb{Z}),\mathbb{Z})=0$ and $H^2(G,\mathbb{Z})\cong\mathrm{Ext}(H_1(G,\mathbb{Z}),\mathbb{Z})$. But $H_1(G,\mathbb{Z})\cong\pi_1(G)$, so $H^2(G,\mathbb{Z})=\pi_1(G)$.

We are now in a position to answer the classical problem that asks which spheres afford a Lie group structure. It turns out that

Theorem 2.26. *S^0, S^1 and S^3 are the only spheres that can be given Lie group structures.*

Proof. We know that $S^0\cong\mathbb{Z}_2$ and $S^1\cong U(1)$. If $n\geq 2$ and S^n affords a Lie group structure, then it must be semi-simple because $\pi_1(S^n)$ is trivial. By Theorem 2.23, $H^3(S^3,\mathbb{R})\neq 0$, so n must be 3. As it turns out, $S^3\cong SU(2)$. $\square$

3. The cohomology ring structure

Determining the ring structure of $H^*(G,\mathbb{R})$ is a more subtle issue. Nevertheless, a general description of the algebraic structure of $H^*(G,\mathbb{R})$ has been known for a long time. In this section, we will first review the basics of Hopf algebras, of which the cohomology ring of any H-space is a motivating example, and a result, due to Hopf, on the classification of Hopf algebras over a field of characteristic 0, from which one can easily deduce that $H^*(G,\mathbb{R})$ is an exterior algebra on l generators, where l is the rank of G. Next we implement a careful study of the map (1) and $H^*(G/T,\mathbb{R})$ to obtain more information about the l generators, e.g. the degrees of the generators.

3.1. **Hopf algebras and their classification.** Most of the material in this section is based on the part of Chapter 3 of [H1] on Hopf algebras. Let X be a connected H-space equipped with a multiplication map $\mu : X \times X \to X$. For convenience of exposition, suppose further that R is a commutative ring with identity and that $H^*(X, R)$ is a free R-module. By Künneth formula, $H^*(X \times X, R)$ is isomorphic to $H^*(X, R) \otimes_R H^*(X, R)$ as R-algebras. Composing the isomorphism with

$$\mu^* : H^*(X, R) \to H^*(X \times X, R)$$

we have a graded R-algebra homomorphism

$$\Delta : H^*(X, R) \to H^*(X, R) \otimes_R H^*(X, R)$$

It is not hard to show that, if $\alpha \in H^n(X, R)$, then

$$\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha \oplus \sum_{i=1}^{k} \alpha_i' \otimes \alpha_i''$$

where the degrees of α_i' and α_i'' are less than n.

The above properties of $H^*(X, R)$ led Hopf to define an algebra named after him.

Definition 3.1. A *Hopf algebra* $\mathcal{A}$ over R is a graded commutative R-algebra which satisfies

(1) $\mathcal{A}^0 = R$.
(2) There exists a graded algebra homomorphism Δ, called comultiplication,

$$\Delta : \mathcal{A} \to \mathcal{A} \otimes_R \mathcal{A}$$

such that if $\alpha \in \mathcal{A}^n$, then

$$\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha + \sum_{i=1}^{k} \alpha_i' \otimes \alpha_i''$$

where the degrees of both α_i' and α_i'' are less than n.

If $\Delta(\alpha) = \alpha \otimes 1 + 1 \otimes \alpha$, then α is called a *primitive element*.

Example 3.2. Let $\mathcal{A} = R[\alpha]$, the polynomial algebra over R generated by an element of even degree. We shall show that it is a Hopf algebra over R. Let us assume that there exists a comultiplication Δ. Then $\Delta(\alpha) = \alpha \otimes 1 \oplus 1 \otimes \alpha$ since there is no element in $R[\alpha]$ with positive degree less than that of α. It follows that

$$\Delta(\alpha^n) = (\alpha \otimes 1 + 1 \otimes \alpha)^n = \sum_{i=1}^{n} \binom{n}{i} \alpha^i \otimes \alpha^{n-i}$$

Δ thus defined is indeed a comultiplication.

Example 3.3. Let $\mathcal{A} = \bigwedge_R(\beta)$, the exterior algebra over R generated by one odd degree element α. Assume that there exists a comultiplication map Δ. Note that

$$\begin{aligned}
0 &= \Delta(\beta^2) \\
&= \Delta(\beta)^2 \\
&= (\beta \otimes 1 + 1 \otimes \beta)^2 \\
&= \beta^2 \otimes 1 + (\beta \otimes 1) \cdot (1 \otimes \beta) + (1 \otimes \beta) \cdot (\beta \otimes 1) + 1 \otimes \beta^2 \\
&= \beta \otimes \beta - \beta \otimes \beta
\end{aligned}$$

So Δ indeed defines a comultiplication, and hence $\mathcal{A}$ is a Hopf algebra.

Example 3.4. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be Hopf algebras over R. Then so is $\mathcal{A} = \mathcal{A}_1 \otimes_R \mathcal{A}_2$, with comultiplication being

$$\Delta_{\mathcal{A}} = \Delta_{\mathcal{A}_1} \otimes_R \Delta_{\mathcal{A}_2}$$

As a result, the polynomial algebra $R[\alpha_1, \cdots, \alpha_n]$ with degree of α_i even for all i, and $\bigwedge_R(\beta_1, \cdots, \beta_m)$ with degree of β_j odd for all j, are Hopf algebras over R. So is their tensor product $R[\alpha_1, \cdots, \alpha_n] \otimes_R \bigwedge_R(\beta_1, \cdots, \beta_m)$.

The following theorem is a partial converse of Example 3.4

Theorem 3.5. *Let $\mathcal{A}$ be a Hopf algebra over a field F of characteristic 0 such that $\mathcal{A}^n$ is finite dimensional over F for each n. Then $\mathcal{A}$ must be one of the following.*

1. *A polynomial algebra $F[\alpha_1, \cdots, \alpha_n]$ with degree of α_i even for all i.*
2. *An exterior algebra $\bigwedge_F(\beta_1, \cdots, \beta_m)$ with degree of β_j odd for all j.*
3. *The tensor product a polynomial algebra as in (1) and an exterior algebra as in (2).*

Proof. Since each graded piece $\mathcal{A}^n$ is finite dimensional over F, there exist $x_1, \cdots, x_n, \cdots$ such that they generate $\mathcal{A}$ and $|x_i| < |x_j|$ if $i < j$. Let $\mathcal{A}_n$ be the subalgebra generated by $x_1, \cdots, x_n$. We may assume that $x_n \notin \mathcal{A}_{n-1}$. $\mathcal{A}_n$ is a Hopf subalgebra because $\Delta(x_i) \in \mathcal{A}_n \otimes_F \mathcal{A}_n$ for $1 \leq i \leq n$. Consider the multiplication map

$$\begin{aligned}
&f : \mathcal{A}_{n-1} \otimes_F F[x_n] \to \mathcal{A}_n && \text{if } |x_n| \text{ is even} \\
\text{or } &f : \mathcal{A}_{n-1} \otimes_F \bigwedge_F(x_n) \to \mathcal{A}_n && \text{if } |x_n| \text{ is odd}
\end{aligned}$$

f is surjective by the definition of $\mathcal{A}_n$. If we can show that f is injective, then $\mathcal{A}_n$ is a tensor product of a polynomial algebra and an exterior algebra, as $\mathcal{A}_{n-1}$ is by inductive hypothesis.

Let us consider the case where $|x_n|$ is even. Suppose that f is not injective, that is, there is a nontrivial relation $\sum_{i=0}^k \alpha_i x_n^i = 0$, with $\alpha_i \in \mathcal{A}_{n-1}$. We may assume that k is the minimal degree of the equations of any nontrivial relation between x_n and elements of $\mathcal{A}_{n-1}$. Let I be an ideal in $\mathcal{A}_n$ generated by the positive degree elements in $\mathcal{A}_{n-1}$ and x_n^2, and $q : \mathcal{A}_n \to \mathcal{A}_n/I$. Note that $x_n \notin I$. Consider the composition of maps

$$\mathcal{A}_n \xrightarrow{\Delta} \mathcal{A}_n \otimes_F \mathcal{A}_n \xrightarrow{\mathrm{Id}\otimes q} \mathcal{A}_n \otimes_F \mathcal{A}_n/I$$

We have

$$\begin{aligned}
(\mathrm{Id} \otimes q) \circ \Delta(\alpha_i) &= \alpha_i \otimes 1 \\
(\mathrm{Id} \otimes q) \circ \Delta(x_n) &= 1 \otimes q(x_n) + x_n \otimes 1 \\
\implies 0 = (1 \otimes q) \circ \Delta(\sum_{i=0}^k \alpha_i x_n^i) &= \sum_{i=0}^k (\alpha_i \otimes 1) \cdot (1 \otimes q(x_n) + x_n \otimes 1)^i \\
&= \sum_{i=0}^k \sum_{j=0}^i \begin{pmatrix} i \\ j \end{pmatrix} \alpha_i x_n^j \otimes q(x_n)^{i-j} \\
&= \sum_{i=0}^k (\alpha_i x_n^i \otimes 1 + i\alpha_i x_n^{i-1} \otimes q(x_n)) \quad (q(x_n^i) = 0 \text{ for } i \geq 2) \\
&= \left(\sum_{i=0}^k i\alpha_i \otimes x_n^{i-1} \right) \otimes q(x_n)
\end{aligned}$$

Thus we have another relation $\sum_{i=0}^k i\alpha_i x_n^{i-1} = 0$ since $q(x_n) \neq 0$. This relation is nontrivial, because $x_n \neq 0$, and $i\alpha_i \neq 0$ if α_i and $i \neq 0$ as F is a field of characteristic 0. We get another nontrivial relation of lower degree, contradicting the minimality of k. Hence the multiplication map is in fact an algebra isomorphism. The case where $|x_n|$ is odd can be dealt with in a similar way. □

Theorem 3.6. *If G is a compact connected Lie group of rank l, then $H^*(G, \mathbb{R})$ is an exterior algebra on l generators of odd degrees.*

Proof. Note that $H^*(G, \mathbb{R})$ satisfies the conditions in Theorem 3.4, and that it is finite dimensional over $\mathbb{R}$ because G is compact. Thus $H^*(G, \mathbb{R})$ must be an exterior algebra on odd degree generators. It remains to show that there are l generators.

Consider the squaring map

$$\begin{aligned} f : G &\to G \\ g &\mapsto g^2 \end{aligned}$$

Let $g_0 \in G$ be a regular element, i.e. $\overline{\langle g_0 \rangle}$ is a maximal torus of G. Then $Z_G(g_0) = \overline{\langle g_0 \rangle}$. Any preimage of g_0 under f commutes with g_0 and so $f^{-1}(g_0) \subset \overline{\langle g_0 \rangle}$. It follows that $|f^{-1}(g_0)| = 2^l$, and $\deg(f) = 2^l$, which implies that f^* amounts to multiplication by 2^l on $H^{\mathrm{top}}(G, \mathbb{R})$. Suppose $H^*(G, \mathbb{R})$ is an exterior algebra on m generators $\beta_1, \cdots, \beta_m$ of odd degrees. Assume that $|\beta_i| < |\beta_j|$ if $i < j$. Note that $f^* = d^* \circ \Delta$, where

$$d^* : H^*(G \times G, \mathbb{R}) \cong H^*(G, \mathbb{R}) \otimes H^*(G, \mathbb{R}) \to H^*(G, \mathbb{R})$$

is induced by the diagonal embedding $d : G \to G \times G$ and amounts to the wedge product map. So

$$\begin{aligned} f^*(\beta_i) &= d^* \circ \Delta(\beta_i) \\ &= d^*(1 \otimes \beta_i + \beta_i \otimes 1 + \sum_j \gamma'_{ij} \otimes \gamma''_{ij}) \quad (|\gamma'_{ij}|, |\gamma''_{ij}| < |\beta_i|) \\ &= 2\beta_i + \sum_j \gamma'_{ij}\gamma''_{ij} \end{aligned}$$

Note that $f^*(\beta_i) = 2\beta_i$ for $i = 1, 2, 3$. Since $\beta_1\beta_2 \cdots \beta_m \in H^{\mathrm{top}}(G, \mathbb{R})$, we have

$$\begin{aligned} 2^l \beta_1\beta_2 \cdots \beta_m &= f^*(\beta_1\beta_2 \cdots \beta_m) \\ &= (2\beta_1)(2\beta_2)(2\beta_3)(2\beta_4 + \sum_j \gamma'_{4j}\gamma''_{4j}) \cdots (2\beta_m + \sum_j \gamma'_{mj}\gamma''_{mj}) \\ &= 2^m \beta_1\beta_2 \cdots \beta_m \end{aligned}$$

We conclude that $m = l$ and the proof is complete. □

We will give another proof of Theorem 3.6 in Section 3.2.

3.2. **The map p^*.** Recall that, if $\mathfrak{g}$ is a Lie algebra of a compact Lie group G, then $\mathfrak{g}_{\mathbb{C}}$ is a complex reductive Lie algebra. Let $\mathfrak{t}$ be the Lie algebra of T, and $\mathfrak{t}_{\mathbb{C}}$ its complexification. The maps $\mathrm{ad}_\xi : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$, $\xi \in \mathfrak{g}_{\mathbb{C}}$ are simultaneously diagonalizable and give the eigenspace decomposition of $\mathfrak{g}_{\mathbb{C}}$

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where $\alpha \in \Delta \subset \mathfrak{t}^*_{\mathbb{C}}$ are roots of $\mathfrak{g}$ satisfying $[H, X_\alpha] = \alpha(H)X_\alpha$ for $H \in \mathfrak{t}_{\mathbb{C}}$, $X_\alpha \in \mathfrak{g}_\alpha$. Note that $\dim \mathfrak{g}_\alpha = 1$. There exists $H_\alpha \in \mathfrak{t}_{\mathbb{C}}$, $\alpha \in \Delta$ and $X_\alpha \in \mathfrak{g}_\alpha$ such that

(1) $\alpha(H_\alpha) = 2$
(2) $[X_\alpha, X_{-\alpha}] = H_\alpha$

(c.f. [Se], Ch. VI, Thm. 2), and $\mathfrak{g}$ is the real span of $\{iH_\alpha, X_\alpha - X_{-\alpha}, i(X_\alpha + X_{-\alpha})\}_{\alpha\in\Delta}$(c.f. [H2], Ch. 3, proof of Thm. 6.3). It follows that $\mathfrak{t} = \mathrm{span}_{\mathbb{R}}\{iH_\alpha\}_{\alpha\in\Delta}$, and

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{m}_\alpha$$

where Δ^+ is the set of positive root, and $\mathfrak{m}_\alpha = \mathfrak{g} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ is the real span of the basis $\{i(X_\alpha + X_{-\alpha}), X_\alpha - X_{-\alpha}\}$. We will denote $\bigoplus_{\alpha\in\Delta^+} \mathfrak{m}_\alpha$ by $\mathfrak{m}$. The matrix representation of the action $\mathrm{ad}(H)$ on $\mathfrak{m}_\alpha$ with respect to this basis is

$$\begin{pmatrix} 0 & -i\alpha(H) \\ i\alpha(H) & 0 \end{pmatrix}$$

whereas that of the adjoint action $\mathrm{Ad}_{\exp(H)}$ is

$$\begin{pmatrix} \cos i\alpha(H) & -\sin i\alpha(H) \\ \sin i\alpha(H) & \cos i\alpha(H) \end{pmatrix}$$

Theorem 3.7. *If $T_{gT}G/T$ is identified with $\mathfrak{m}$, T_tT with $\mathfrak{t}$ and $T_{gtg^{-1}}G$ with $\mathfrak{g}$ by left translation, then*

$$\begin{aligned} dp_{(gT,t)} &: \mathfrak{m} \oplus \mathfrak{t} \to \mathfrak{g} \\ &(X, T) \mapsto Ad_{gt^{-1}g^{-1}}(X) - X + Ad_g(T) \end{aligned}$$

In matrix form,

$$dp_{(gT,t)} = \begin{pmatrix} Ad_{gt^{-1}g^{-1}} - Id_{\mathfrak{m}} & 0 \\ 0 & Ad_g|_{\mathfrak{t}} \end{pmatrix}$$

Proof. Identify $(X, T) \in \mathfrak{m} \oplus \mathfrak{t}$ with $(L_{g*}X, L_{t*}T) \in T_{gT}G/T \oplus T_tT$. Then

$$\begin{aligned} dp_{(gT,t)}(L_{g*}X, L_{t*}T) &= \left.\frac{d}{ds}\right|_{s=0} g\exp(sX)t\exp(sT)g^{-1}\exp(-sX) \\ &= R_{tg^{-1}*}L_{g*}X + R_{g^{-1}*}L_{gt*}T - L_{gtg^{-1}*}X \\ &= L_{gtg^{-1}*}(\mathrm{Ad}_{gt^{-1}g^{-1}}X - X + \mathrm{Ad}_gT) \end{aligned}$$

The last line is identified with $\mathrm{Ad}_{gt^{-1}g^{-1}}X - X + \mathrm{Ad}_gT$. ∎

Corollary 3.8. $\det dp_{(gT,t)} = \det(Ad_t - Id_{\mathfrak{m}})$.

Proof. Note that $\det(\mathrm{Ad}_g) = 1$ for all $g \in G$ because $\mu : G \to \mathbb{R}^\times$ defined by $\mu(g) = \det(\mathrm{Ad}_g)$ is a group homomorphism and G is compact connected. By Theorem 3.7,

$$\begin{aligned}
\det dp_{(gT,t)} &= \det(\mathrm{Ad}_{gt^{-1}g^{-1}} - \mathrm{Id}_{\mathfrak{m}}) \det(\mathrm{Ad}_g|_{\mathfrak{t}}) \\
&= \det(\mathrm{Ad}_g(\mathrm{Ad}_{t^{-1}} - \mathrm{Id}_{\mathfrak{m}})\mathrm{Ad}_{g^{-1}}) \\
&= \det(\mathrm{Ad}_{t^{-1}} - \mathrm{Id}_{\mathfrak{m}}) \\
&= \det(\mathrm{Ad}_t(\mathrm{Ad}_{t^{-1}} - \mathrm{Id}_{\mathfrak{m}})) \\
&= \det(\mathrm{Id}_{\mathfrak{m}} - \mathrm{Ad}_t) \\
&= \det(\mathrm{Ad}_t - \mathrm{Id}_{\mathfrak{m}}) \quad (\dim \mathfrak{m} \text{ is even})
\end{aligned}$$

□

Lemma 3.9. *If* $t = \exp H$, $H \in \mathfrak{t}$, *then* $\det(Ad_t - Id_{\mathfrak{m}}) = \prod_{\alpha \in \Delta^+} 4 \sin^2 \frac{i\alpha(H)}{2}$.

Proof.

$$\begin{aligned}
\det(\mathrm{Ad}_t - \mathrm{Id}_{\mathfrak{m}}) &= \prod_{\alpha \in \Delta^+} \det \begin{pmatrix} \cos i\alpha(H) - 1 & -\sin i\alpha(H) \\ \sin i\alpha(H) & \cos i\alpha(H) - 1 \end{pmatrix} \\
&= \prod_{\alpha \in \Delta^+} 2(1 - \cos i\alpha(H)) \\
&= \prod_{\alpha \in \Delta^+} 4 \sin^2 \frac{i\alpha(H)}{2}
\end{aligned}$$

□

Suppose $g_0 \in G$ is a regular value of p. It is well-known that $p^{-1}(g_0)$ consists of $|W|$ points. By Lemma 3.9, the determinant of dp at each point in the pre-image must be positive. Hence $\deg p = |W|$. The pull-back formula for integration gives

Theorem 3.10 (Weyl integration formula)**.** *Let* ω_G, $\omega_{G/T}$ *and* ω_T *be the normalized volume form of* G, G/T *and* T *respectively, and* $f : G \to \mathbb{C}$ *a continuous complex-valued function on* G. *Then*

$$\int_G f(g)\omega_G = \frac{1}{|W|} \int_{G/T \times T} f \circ p(gT, t) \det(Ad_t - Id_{\mathfrak{m}}) \omega_{G/T} \wedge \omega_T$$

In particular, if f *is a class function on* G, *then*

$$\int_G f(g)\omega_G = \frac{1}{|W|} \int_T f(t) \det(Ad_t - Id_{\mathfrak{m}}) \omega_T$$

Lemma 3.11. $dim_{\mathbb{R}} H^*(G, \mathbb{R}) = 2^l$

Proof. By Corollary 2.22, $\dim H^*(G,\mathbb{R}) = \dim(\bigwedge^* \mathfrak{g})^{*G} = \dim(\bigwedge^* \mathfrak{g})^G$. Note that $(\bigwedge^* \mathfrak{g})^G$ is the trivial subrepresentation of the adjoint representation of G on $\bigwedge^* \mathfrak{g}$. Let $\chi_{\bigwedge^* \mathfrak{g}}$ be the character of this representation. Then

$$\begin{aligned}
\dim(\bigwedge^{*} \mathfrak{g})^G &= \int_G \chi_{\bigwedge^* \mathfrak{g}}(g)\omega_G \\
&= \int_T \chi_{\bigwedge^* \mathfrak{g}}(t)\det(\mathrm{Ad}_t - \mathrm{Id}_{\mathfrak{m}})\omega_T \\
&= \int_T \det(\mathrm{Ad}_t + \mathrm{Id}_{\mathfrak{m}})\det(\mathrm{Ad}_t - \mathrm{Id}_{\mathfrak{m}})\omega_T \\
&= \int_T \det(\mathrm{Ad}_{t^2} - \mathrm{Id}_{\mathfrak{m}})\omega_T \\
&= 2^{\dim T}\int_T \det(\mathrm{Ad}_s - \mathrm{Id}_{\mathfrak{m}})\omega_s \\
&= 2^l
\end{aligned}$$

□

Now that G and $G/T \times T$ are manifolds of the same dimension, and $\deg p = |W| \neq 0$, the induced map

$$p^* : H^*(G,\mathbb{R}) \to H^*(G/T \times T,\mathbb{R})$$

is injective. There is a W-action on $G/T \times T$ defined by

$$w \cdot (gT, t) = (gw^{-1}T, wtw^{-1})$$

and it is easy to see that $p(gT,t) = p(w\cdot(gT,t))$ for $w \in W$. Thus $\mathrm{Im}(p^*) \subseteq H^*(G/T\times T)^W$. By abuse of notation we also use p^* to mean the map

$$p^* : H^*(G,\mathbb{R}) \to H^*(G/T \times T,\mathbb{R})^W$$

We claim that

Theorem 3.12. *p^* is a ring isomorphism.*

Before giving a proof of Theorem 3.12, we shall examine $H^*(G/T,\mathbb{R})$ more closely. As a first shot, we shall employ Morse theory to compute the cohomology groups.

Let $\langle \cdot, \cdot \rangle$ be an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$. This can be obtained by averaging any inner product on $\mathfrak{g}$ over G. Let $X \in \mathfrak{t}^+$ be a regular element in the positive Weyl chamber, i.e. $\frac{\alpha(X)}{i} > 0$ for all $\alpha \in \Delta^+$. We let

$$\begin{aligned} f : G/T &\to \mathbb{R} \\ gT &\mapsto \langle \mathrm{Ad}_g(X), X\rangle \end{aligned}$$

and claim that it is a Morse function. Suppose g_0T is a critical point of f. Then for all $Y \in \mathfrak{m}$,

$$\begin{aligned} 0 &= df(L_{g_0*}Y) \\ &= \frac{d}{ds}\bigg|_{s=0} f(g_0\exp(sY)T) \\ &= \frac{d}{ds}\bigg|_{s=0} \langle \mathrm{Ad}_{g_0\exp(sY)}(X), X\rangle \\ &= \langle \mathrm{Ad}_{g_0}([Y, X]), X\rangle \\ &= \langle [Y, X], \mathrm{Ad}_{g_0^{-1}}X\rangle \\ &= \langle Y, [X, \mathrm{Ad}_{g_0^{-1}}X]\rangle \quad \text{by Ad-invariance of the inner product} \end{aligned}$$

Note that $\mathfrak{m}^\perp = \mathfrak{t}$, as

$$\langle \mathfrak{t}, \mathfrak{m}\rangle = \langle \mathfrak{t}, [\mathfrak{t}, \mathfrak{m}]\rangle = \langle [\mathfrak{t}, \mathfrak{t}], \mathfrak{m}\rangle = 0$$

Thus $[X, \mathrm{Ad}_{g_0^{-1}}(X)] \in \mathfrak{t}$, and $\mathrm{Ad}_{g_0^{-1}}X \in \mathfrak{t}$. $g_0^{-1}T$ and hence g_0T must be a Weyl group element. The critical points of f are therefore all the Weyl group elements.

The Hessian $H_w(Y, Z)$ for $Y, Z \in \mathfrak{m}$, $w = g_0T \in W$ is

$$\begin{aligned} &\frac{d}{dt}\bigg|_{t=0} \langle Y, [X, \mathrm{Ad}_{(g_0\exp(tZ))^{-1}}(X)]\rangle \\ =&\langle Y, [X, [-Z, \mathrm{Ad}_{g_0^{-1}}(X)]]\rangle \\ =&-\langle [Y, X], [Z, \mathrm{Ad}_{g_0^{-1}}(X)]\rangle \\ =&-\langle [X, Y], [\mathrm{Ad}_{g_0^{-1}}(X), Z]\rangle \end{aligned}$$

For $\alpha \in \Delta^+$,

$$H_w(X_\alpha - X_{-\alpha}, X_\alpha - X_{-\alpha}) = \alpha(X)\alpha(\mathrm{Ad}_{g_0^{-1}}(X))\langle i(X_\alpha + X_{-\alpha}), i(X_\alpha + X_{-\alpha})\rangle \neq 0$$
$$H_w(i(X_\alpha + X_{-\alpha}), i(X_\alpha + X_{-\alpha})) = \alpha(X)\alpha(\mathrm{Ad}_{g_0^{-1}}(X))\langle X_\alpha - X_{-\alpha}, X_\alpha - X_{-\alpha}\rangle \neq 0$$

So H_w is nondegenerate for all $w \in W$ and f is indeed a Morse function. The index of f at w is twice the number of positive roots α such that

$$\alpha(X)\alpha(\mathrm{Ad}_{g_0^{-1}}(X)) < 0$$

Since $\frac{\alpha(X)}{i} > 0$, and $\alpha(\mathrm{Ad}_{g_0^{-1}}(X)) = (w \cdot \alpha)(X)$, the index is also twice the number of positive roots α such that $w \cdot \alpha$ is also positive. Let $\mathrm{Index}(w) = 2m(w)$. Then the Poincaré polynomial of $H^*(G/T, \mathbb{R})$ is $P(t) = \sum_{w \in W} t^{2m(w)}$, and the Euler characteristic is $\chi(G/T) = P(-1) = |W|$.

The W-action on G/T given by $w \cdot (gT) = gw^{-1}T$ induces a W-representation on $H^*(G/T, \mathbb{R})$. The trace of w on $H^*(G/T, \mathbb{R})$ is just the Lefschetz number of the action of w because $H^*(G/T, \mathbb{R})$ is concentrated on even degrees. If $w \neq 1$, then it has no fixed points on G/T, and therefore its trace is 0 by Lefschetz Fixed Points Theorem. If $w = 1$, then the trace is just the Euler characteristic $|W|$. As a result,

Proposition 3.13. *$H^*(G/T, \mathbb{R})$ is a regular representation of W.*

Proof of Theorem 3.12. Since p^* is injective, it suffices to show that $\dim H^*(G/T \times T, \mathbb{R})^W = \dim H^*(G, \mathbb{R}) = 2^l$. By Künneth formula, $H^*(G/T \times T, \mathbb{R})^W = (H^*(G/T, \mathbb{R}) \otimes H^*(T, \mathbb{R}))^W$. So

$$\begin{aligned} & \dim(H^*(G/T, \mathbb{R} \otimes H^*(T, \mathbb{R}))^W \\ = & \frac{1}{|W|} \sum_{w \in W} \chi_{H^*(G/T,\mathbb{R}) \otimes H^*(T,\mathbb{R})}(w) \\ = & \frac{1}{|W|} \sum_{w \in W} \chi_{H^*(G/T,\mathbb{R}}(w) \chi_{H^*(T,\mathbb{R})}(w) \\ = & \frac{1}{|W|} \chi_{H^*(G/T,\mathbb{R})}(1) \chi_{H^*(T,\mathbb{R})}(1) \\ = & \dim H^*(T, \mathbb{R}) \\ = & 2^l \end{aligned}$$

□

Let $\mathcal{S}$ be the real polynomial ring $S^*(\mathfrak{t}^*)$ on $\mathfrak{t}$, and $\mathcal{I}$ be the ideal in $\mathcal{S}$ generated by W-invariant polynomials. A famous theorem of Borel describes the ring structure of $H^*(G/T, \mathbb{R})$ using

Theorem 3.14 (Borel)**.** *There is a degree-doubling W-equivariant ring isomorphism*

$$c : \mathcal{S}/\mathcal{I} \to H^*(G/T, \mathbb{R})$$

where $c(\lambda)(X, Y) = \lambda([X, Y])$ for $X, Y \in \mathfrak{m}$ and deg$(\lambda) = 1$.

We refer the reader to [R] for a proof of Theorem 3.14 using invariant theory. Here we would like to show that $H^*(G/T,\mathbb{R})$ is isomorphic to $\mathcal{S}/\mathcal{I}$ using equivariant cohomology. For a review of equivariant cohomology the reader is refer to the Appendix.

Consider G/T with T acting on it by left translation. Then

$$H^*_T(G/T,\mathbb{R}) \cong H^*_{T\times T}(G,\mathbb{R})$$

where $T\times T$ acts on G by

$$(t_1,t_2)\cdot g = t_1 g t_2$$

Next note that G is diffeomorphic to the orbit space of the G-action on $G\times G$ given by $g\cdot(g_1,g_2)=(g_1g,g^{-1}g_2)$. We get

$$H^*_{T\times T}(G,\mathbb{R}) \cong H_{T\times T\times G}(G\times G,\mathbb{R})$$

where $T\times T\times G$ acts on $G\times G$ by

$$(t_1,t_2,g)\cdot(g_1,g_2) = (t_1g_1g, g^{-1}g_2t_2)$$

So

$$\begin{aligned} H^*_{T\times T\times G}(G\times G,\mathbb{R}) &\cong H^*_G(G/T\times G/T,\mathbb{R}) \\ &\cong H^*_G(G/T,\mathbb{R})\otimes_{H^*_G(\mathrm{pt},\mathbb{R})} H^*_G(G/T,\mathbb{R}) \\ &\cong H^*_T(\mathrm{pt},\mathbb{R})\otimes_{H^*_G(\mathrm{pt},\mathbb{R})} H^*_T(\mathrm{pt},\mathbb{R}) \end{aligned}$$

It is well-known that

$$H^*_T(G/T,\mathbb{R}) \cong H^*_T(\mathrm{pt},\mathbb{R})\otimes_{\mathbb{R}} H^*(G/T,\mathbb{R})$$

as $H^*_T(\mathrm{pt},\mathbb{R})$-modules, because G/T is a T-Hamiltonian manifold. Therefore

$$\begin{aligned} H^*(G/T,\mathbb{R}) &\cong \mathbb{R}\otimes_{H^*_G(\mathrm{pt},\mathbb{R})} H^*_T(\mathrm{pt},\mathbb{R}) \\ &\cong H^*_T(\mathrm{pt},\mathbb{R})/\langle r^* H_T(\mathrm{pt},\mathbb{R})\rangle \end{aligned}$$

where $r^*: H^*_G(\mathrm{pt},\mathbb{R})\to H^*_T(\mathrm{pt},\mathbb{R})$ is the map induced by restricting G-action to T-action. By the abelianization principle(c.f. [AB]), r^* is injective and its image is $H^*_T(\mathrm{pt},\mathbb{R})^W$. Identifying $H^*_T(\mathrm{pt},\mathbb{R})$ with $\mathcal{S}$, we get Theorem 3.14. Combining Theorem 3.12 and 3.14, and regarding $H^*(T,\mathbb{R})=\bigwedge^*\mathfrak{t}^*$ as differential forms, we have

Theorem 3.15. *$H^*(G,\mathbb{R}) \cong ((\mathcal{S}/\mathcal{I})_{(2)}\otimes\bigwedge^*\mathfrak{t}^*)^W$ where the RHS is the space of W-invariant differential forms with coefficients in $\mathcal{S}/\mathcal{I}$. Here $(\mathcal{S}/\mathcal{I})_{(2)}$ means $\mathcal{S}/\mathcal{I}$ with degree of each polynomial doubled.*

It is a classical result in invariant theory, due to Chevalley, that the W-invariant polynomial $\mathcal{S}^W$ is generated by l algebraically independent polynomials $F_1, \cdots, F_l$. In other words

$$\mathcal{S}^W = \mathbb{R}[F_1, \cdots, F_l]$$

Definition 3.16. The *exponents* m_i, $1 \leq i \leq l$ of G are defined to be

$$m_i = \deg F_i - 1$$

Remark 3.17. It is known that $\sum_{i=1}^l m_i = \frac{1}{2}\dim G/T$ and $\prod_{i=1}^l (1+m_i) = |W|$.

Theorem 3.18 (Solomon [So])**.** *The space $(\mathcal{S} \otimes \bigwedge^* \mathfrak{t}^*)^W$ of W-invariant differential forms with polynomial coefficients is an exterior algebra over $\mathcal{S}^W$ generated by $dF_1, \cdots, dF_l$.*

Before proving Theorem 3.18, we need a

Lemma 3.19. *Let $J = Jac(F_1, \cdots, F_l)$. Then $w \cdot J = \det(w)J$. Here $\det(w)$ means the determinant of w as a linear transformation on $\mathfrak{t}^*$. If $R \in \mathcal{S}$ and satisfies $w \cdot R = \det(w)R$, then $R = SJ$ for some $S \in \mathcal{S}^W$.*

Proof. A classical result in invariant theory asserts that, if $\alpha_1, \cdots, \alpha_l \in \mathfrak{t}^*$ are simple roots,

$$J = c\alpha_1\alpha_2 \cdots \alpha_l$$

for some $c \in \mathbb{R}$. Let $u \in \mathcal{S}$ such that $w \cdot u = \det(w)u$. As $\{\alpha_1, \cdots, \alpha_l\}$ forms a basis for $\mathfrak{t}^*$, those simple roots can be regarded as coordinate functions on $\mathfrak{t}$, and we write $u = u(\alpha_1, \cdots, \alpha_l)$, a polynomial of $\alpha_1, \cdots, \alpha$. Note that

$$\begin{aligned} u(\alpha_1, \cdots, \alpha_{l-1}, 0) &= u(\alpha_1(H_l), \cdots, \alpha_{l-1}(H_l), \alpha_l(H_l)) \\ &= u(\alpha_1(s_{H_l}(H_l)), \cdots, \alpha_{l-1}(s_{H_l}(H_l)), \alpha_l(s_{H_l}(H_l))) \\ &= (s_{\alpha_l} \cdot u)(\alpha_1(H_l), \cdots, \alpha_l(H_l)) \\ &= -u(\alpha_1, \cdots, \alpha_{l-1}, 0) \end{aligned}$$

So $u(\alpha_1, \cdots, \alpha_{l-1}, 0) = 0$, and α_l divides u. Similar reasoning implies that α_i divides u for $1 \leq i \leq l$. Since $\alpha_1, \cdots, \alpha_l$ are coprime, J divides u. $\square$

Proof of Theorem 3.18. Let $\mathcal{L} = \mathcal{S}_{(0)}$ be the field of fraction of $\mathcal{S}$. We shall first show that $\{dF_{i_1} \wedge \cdots \wedge dF_{i_r}\}_{1 \leq r \leq l}$ is linearly independent over $\mathcal{S}$ and $\mathcal{L}$. Suppose $\sum k_{i_1 \cdots i_r} dF_{i_1} \wedge \cdots \wedge dF_{i_r} = 0$. Multiplying $dF_{j_1} \wedge \cdots \wedge dF_{j_{l-r}}$, where $\{j_1, \cdots, j_{l-r}\} = \{1, \cdots, l\} \setminus \{i_1, \cdots, i_r\}$, we have

$$\pm k_{i_1 \cdots i_r} J dx_1 \wedge \cdots \wedge dx_l = 0$$

Since $F_1, \cdots, F_l$ are algebraically independent, $J \neq 0$. Thus $k_{i_1\cdots i_r} = 0$ as both $\mathcal{S}$ and $\mathcal{L}$ have no zero divisors.

Let $\omega \in (\mathcal{S} \otimes \bigwedge^p \mathfrak{t}^*)^W$. As $\mathcal{S} \subset \mathcal{L}$, we may regard ω as a W-invariant differential p-form with coefficients being rational functions on $\mathfrak{t}$. Note that $\mathcal{L} \otimes \bigwedge^p \mathfrak{t}^*$ is an L-vector space of dimension $\binom{l}{p}$, and thus $\{dF_{i_1} \wedge \cdots \wedge dF_{i_p}\}$ spans $\mathcal{L} \otimes \bigwedge^p \mathfrak{t}^*$ over $\mathcal{L}$. Write

$$\omega = \sum \frac{s_{i_1\cdots i_r}}{t_{i_1\cdots i_r}} dF_{i_1} \wedge \cdots \wedge dF_{i_r}$$

where $s_{i_1\cdots i_r}$ and $t_{i_1\cdots i_r} \in \mathcal{S}$. By the W-invariance of ω, $\frac{s_{i_1\cdots i_r}}{t_{i_1\cdots i_r}}$ is also W-invariant. Multiplying $dF_{j_1} \wedge \cdots \wedge dF_{j_{l-r}}$ with $\{j_1, \cdots, j_{l-r}\} = \{1, \cdots, l\} \setminus \{i_1, \cdots, i_r\}$, we get

$$u_{i_1\cdots i_r} dx_1 \wedge \cdots \wedge dx_l = \frac{s_{i_1\cdots i_r}}{t_{i_1\cdots i_r}} J dx_1 \wedge \cdots \wedge dx_l$$

for some $u_{i_1\cdots i_r} \in \mathcal{S}$. Comparing coefficients, we have

$$u_{i_1\cdots i_r} t_{i_1\cdots i_r} = s_{i_1\cdots i_r} J$$

Note that $w \cdot u_{i_1\cdots i_r} = \det(w) u_{i_1\cdots i_r}$. By Lemma 3.19, $u_{i_1\cdots i_r} = P_{i_1\cdots i_r} J$ for some $P_{i_1\cdots i_r} \in \mathcal{S}^W$. It follows that $P_{i_1\cdots i_r} = \frac{s_{i_1\cdots i_r}}{t_{i_1\cdots i_r}}$ and the proof is complete. $\square$

Corollary 3.20. *$(\mathcal{S}/\mathcal{I} \otimes \bigwedge^* \mathfrak{t}^*)^W$ is an exterior algebra over $\mathbb{R}$ with generators*

$$\overline{dF_i} \in ((\mathcal{S}/\mathcal{I})^{m_i} \otimes \bigwedge^1 \mathfrak{t}^*)^W$$

where $\overline{dF_i}$ means dF_i with polynomial coefficients modulo $\mathcal{I}$.

Theorem 3.21 ([R])**.** *If G is compact and connected, then $H^*(G, \mathbb{R})$ is an exterior algebra generated by elements of degree $2m_i + 1$, which correspond to $\overline{dF_i} \in ((\mathcal{S}/\mathcal{I})^{2m_i} \otimes_{\mathbb{R}} \bigwedge^1 \mathfrak{t})^W$ with the degree of polynomial coefficients doubled.*

Example 3.22. Let $G = SU(n)$, and T be the subgroup of diagonal matrices, which is of dimension $n - 1$.

$$\mathfrak{t} = \left\{ \begin{pmatrix} ix_1 & & \\ & \ddots & \\ & & ix_n \end{pmatrix} \middle| \, x_i \in \mathbb{R}, \sum_{i=1}^n x_i = 0 \right\}$$

$$\mathcal{S} \cong \mathbb{R}[x_1, \cdots, x_n]/\langle x_1 + \cdots + x_n \rangle$$

$$\mathcal{S}^W \cong \mathbb{R}[\sigma_2, \cdots, \sigma_n]$$

where $\sigma_i = \sum_{1\leq j_1<\cdots<j_i\leq n}\prod_{k=1}^{i} x_{j_k}$ is the i-th elementary symmetric polynomial. Hence the exponents m_i are $1, 2, \cdots, n-1$. By Theorem 3.14,

$$H^*(G/T,\mathbb{R}) \cong \mathbb{R}[x_1,\cdots,x_n]/\langle\sigma_1,\cdots,\sigma_n\rangle \cong \mathbb{R}[x_1,\cdots,x_n]/\langle(1+x_1)\cdots(1+x_n)=1\rangle$$

If we think of $SU(n)/T$ as the full flag manifold $\mathcal{F}l(\mathbb{C}^n) = \{0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n|\mathrm{dim}V_i = i\}$, then $x_i = c_1(\mathcal{L}_i/\mathcal{L}_{i-1})$, where

$$\mathcal{L}_i = \{((V_0, V_1,\cdots,V_n), v) \in \mathcal{F}l(\mathbb{C}^n)\times\mathbb{C}^n|v \in V_i\}$$

and by Whitney Product Formula, the single relation $(1+x_1)\cdots(1+x_n) = 1$ translates to the fact that the direct sum $\bigoplus_{i=1}^{n}\mathcal{L}_i/\mathcal{L}_{i-1}$ is isomorphic to the trivial rank n complex vector bundle on $\mathcal{F}l(\mathbb{C}^n)$. By Theorem 3.21, $H^*(SU(n),\mathbb{R})$ is an exterior algebra on $n-1$ generators of degrees 3, 5, $\cdots, 2n-1$.

4. Elements of K-theory

Let X be a compact topological space and $\mathrm{Vect}(X)$ be the category of isomorphism classes of (finite rank) complex vector bundles over X. Note that $\mathrm{Vect}(X)$ is a monoid with direct sum being binary operation. Let

$$S(X) = \{[E]-[F]|[E],[F] \in \mathrm{Vect}(X)\}$$

We say $[E_1]-[F_1] \sim [E_2]-[F_2]$ if there exists $[G] \in \mathrm{Vect}(X)$ such that $[E_1 \oplus F_2 \oplus G] = [E_2 \oplus F_1 \oplus G]$.

Definition 4.1. $K(X) := S(X)/\sim$. In other words, $K(X)$ is the Grothendieck group of $\mathrm{Vect}(X)$.

Definition 4.2. Let $\dim : K(X) \to \mathbb{Z}$ be the group homomorphism which sends the class of a vector bundle to its rank. Define the reduced K-theory $\widetilde{K}(X)$ to be the kernel of dim.

Definition 4.3. $K^0(X) := K(X)$, $K^{-q}(X) := \widetilde{K}(S^q \wedge X)$, where $\wedge$ means the smash product.

One can make $\bigoplus_{q=0}^{\infty} K^{-q}(X)$ into a ring using tensor product of vector bundles(c.f. [H1]). The renowned *Bott periodicity* states that $K^{-q}(X) \cong K^{-q-2}(X)$ for all $q \leq 0$.

Definition 4.4. $K^*(X) := K^0(X) \oplus K^{-1}(X)$, with ring structure induced by tensor product of vector bundles.

K^* is a $\mathbb{Z}_2$-graded generalized cohomology theory. Let $U(\infty)$ be the direct limit of $U(n)$ as n tends to infinity, where the morphism $U(n) \to U(m)$ for $n \leq m$ is inclusion. It is well-known that $\mathbb{Z} \times BU(\infty)$ and $U(\infty)$ are classifying spaces of K^0 and K^{-1} respectively.

Example 4.5. $K^*(S^2) \cong \mathbb{Z}[H]/(H-1)^2$ as rings, where $1 \in \mathbb{Z}$ represents the trivial line bundle and $H = \mathcal{O}(1)$.

5. K-theory of compact Lie groups

From now on we assume that G is a simply-connected compact Lie group. This section is mainly taken from [A1] and [H3]. Let $\rho_1, \cdots, \rho_l$ be the l fundamental representations of G. A representation $\rho : G \to U(n)$, composed with the inclusion $U(n) \hookrightarrow U(\infty)$, defines an element in $K^{-1}(G)$, which we denote by $\beta(\rho)$. The main result of this section is

Theorem 5.1. *If G is a simply-connected compact Lie group, then*

$$K^*(G) \cong \bigwedge(\beta(\rho_1), \cdots, \beta(\rho_l))$$

We postpone the proof of Theorem 5.1 to the end of this section. Consider $p^* : K^*(G) \to K^*(G/T \times T)$. Note that $K^*(G/T)$ is torsion-free because G/T can be given a CW-complex structure consisting of only even-dimensional cells. The same is true of $K^*(T)$ by Lemma 5.8. By Künneth formula for K-theory, $K^*(G/T \times T) \cong K^*(G/T) \otimes K^*(T)$.

Lemma 5.2. *Consider $p^* : K^*(G) \to K^*(G/T \times T) \cong K^*(G/T) \otimes K^*(T)$. Then $p^*(\beta(\rho)) = \sum_{i=1}^n \alpha(\mu_j) \otimes \beta(\mu_j)$ where μ_j's are all the weights of ρ, and*

$$\alpha : R(T) \to K^*(G/T)$$

is defined by $\alpha(\mu) = [G \times_T \mathbb{C}_\mu]$.

Proof. Note that $K^{-1}(G) \cong \widetilde{K}(S(G))$, where $S(G)$ is the unreduced suspension of G. We would like to construct $\beta(\rho)$ explicitly as a (virtual) vector bundle over $S(G)$. Consider the principal G-bundle $G * G \to S(G)$, where $G * G = \{sg_1 + (1-s)g_2 | g_1, g_2 \in G, s \in [0,1]\}$ and G acts on $G * G$ by $g \cdot (sg_1 + (1-s)g_2) = sg_1 g + (1-s)g^{-1}g_2$. Note that it is the pullback of $EG \to BG$ through the canonical embedding

$$S(G) = G * G/G \hookrightarrow BG = \lim_{n\to\infty} \underbrace{G * \cdots * G}_{n \text{ times}}/G$$

. We have that

$$\beta(\rho) = [(G * G) \times_G V_\rho] - [\underline{\mathbb{C}}^{\dim(\rho)}]$$

Consider the following diagram

$$\begin{tikzcd}
G \times (T * T) \arrow[r, "m"] \arrow[d, "\theta"'] & G * G \arrow[d, "\psi"] \\
G/T \times S(T) \arrow[r, "f"] \arrow[d, "h"'] & G * G/T \arrow[d, "q"] \\
S(G/T \times T) \arrow[r, "Sp"] & S(G)
\end{tikzcd}$$

where m is defined by $(g, st_1 + (1-s)t_2) \mapsto sgt_1 + (1-s)t_2 g^{-1}$, the various vertical maps projection maps of fiber bundles, and f and Sp are defined in such a way that the above diagram commutes. Note that

$$f^*[(G * G) \times_T \mathbb{C}_\mu] = [(G \times (T * T) \times_{T \times T} \mathbb{C}_\mu] = \alpha(\mu) \otimes (1 + \beta(\mu))$$

$$\begin{aligned}
h^*(Sp)^*[(G * G) \times_G V_\rho] &= f^* q^*[(G * G) \times_G V_\rho] \\
&= f^* \left(\sum_{i=1}^{n} [(G * G) \times_T \mathbb{C}_{\mu_i}] \right) \\
&= \sum_{i=1}^{n} \alpha(\mu_i) \otimes (1 + \beta(\mu_i)) \\
&= [\underline{\mathbb{C}}^{\dim(\rho)}] + \sum_{i=1}^{n} \alpha(\mu_i) \otimes \beta(\mu_i)
\end{aligned}$$

□

Definition 5.3. Let $a := \prod_{i=1}^{l} \beta(\rho_i) \in K^*(G)$.

Proposition 5.4. $\displaystyle\int_{G/T \times T} ch(p^*(a)) = |W|$.

Proof. By Lemma 5.2, $p^*(\beta(\rho_i)) = \sum_{j=1}^{m} \alpha(\lambda_{ij}) \otimes \beta(\lambda_{ij})$, where λ_{ij} are the weights of the fundamental representation ρ_i and m_i is its dimension. We first prove a

Claim 5.5.

$$\prod_{i=1}^{l} \sum_{j=1}^{m_i} (\lambda_{ij} \otimes \beta(\lambda_{ij})) = \left(\sum_{w \in W} \det(w) w \cdot \rho \right) \otimes \prod_{i=1}^{l} \beta(\varpi_i)$$

where $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ *and* ϖ_i *is the i-th fundamental weight.*

We have

$$\begin{aligned}
\int_{G/T\times T}\mathrm{ch}(p^*(a)) &= \int_{G/T\times T}\mathrm{ch}\left(\prod_{i=1}^{l}\sum_{j=1}^{m_i}\alpha(\lambda_{ij})\otimes\beta(\lambda_{ij})\right)\\
&= \int_{G/T\times T}\mathrm{ch}\left((\alpha\otimes\mathrm{Id})\left(\prod_{i=1}^{l}\sum_{j=1}^{m_i}(\lambda_{ij}\otimes\beta(\lambda_{ij}))\right)\right)\\
&= \int_{G/T\times T}\mathrm{ch}\left((\alpha\otimes\mathrm{Id})\left(\sum_{w\in W}\det(w)w\cdot\rho\otimes\prod_{i=1}^{l}\beta(\varpi_i)\right)\right)\\
&= \int_{G/T\times T}\mathrm{ch}\left(\alpha\left(\sum_{w\in W}\det(w)w\cdot\rho\right)\right)\otimes\mathrm{ch}\left(\prod_{i=1}^{l}\beta(\varpi_i)\right)\\
&= \int_{G/T}\mathrm{ch}\left(\alpha\left(\sum_{w\in W}\det(w)w\cdot\rho\right)\right)\times\int_{T}\mathrm{ch}\left(\prod_{i=1}^{l}\beta(\varpi_i)\right)\\
&= \chi(G/T)\cdot 1\\
&= |W|
\end{aligned}$$

□

Proposition 5.6. $\displaystyle\int_G ch(a) = 1.$

Proof.

$$\deg(p)\int_G \mathrm{ch}(a) = \int_{G/T\times T}\mathrm{ch}(p^*(a)) = |W|$$

and $\deg(p) = |W|$. □

Lemma 5.7 (a special case of [H3], Theorem A(i))**.** *$K^*(G)$ is torsion-free.*

Before proving Lemma 5.7 we show

Lemma 5.8 ([H3])**.** *If X is a finite CW-complex, and $K^*(X)$ has p-torsion, then so does $H^*(X,\mathbb{Z})$.*

Proof. Let $\mathbb{Q}_p$ be $\mathbb{Z}$ localized at the prime p. If $H^*(X,\mathbb{Z})$ has no p-torsion, then the homomorphism of spectral sequences for $K^*(X)\otimes\mathbb{Q}_p$ and $K^*(X)\otimes\mathbb{Q}$

$$E_r(X,\mathbb{Q})\to E_r(X,\mathbb{Q})$$

is injective when $r=2$, as $E_2(X,\mathbb{Q}_p) = H^*(X,\mathbb{Q}_p)$ and $E_2(X,\mathbb{Q}) = H^*(X,\mathbb{Q})$. By a result of Atiyah-Hirzebruch's(c.f. [AH], p. 19), $E_r(X,\mathbb{Q})$ collapses on the E_2-page. Induction on

r gives that $E_r(X, \mathbb{Q}_p)$ also collapses on the E_2-page. Now the associated graded group of $K^*(X) \otimes \mathbb{Q}_p$ is $E_2(X, \mathbb{Q}_p) = H^*(X, \mathbb{Q}_p)$ which has no p-torsion. Therefore $K^*(X, \mathbb{Q}_p)$ has no p-torsion and so does $K^*(X)$. $\square$

Sketch of proof of Lemma 5.7. By virtue of Lemma 5.8, it suffices to show that, even if $H^*(G, \mathbb{Z})$ has p-torsion, then $K^*(G)$ has no p-torsion. This can be done using a result of Borel's which give an exhaustive list of prime p and simple, simply-connected compact Lie group G such that $H^*(G, \mathbb{Z})$ has p-torsion, and showing that $K^*(G)$ has no p-torsion case by case. We refer the reader to [H3], III.1 for detailed proof. $\square$

Proof of Theorem 5.1. From the proof of Theorem 3.12, we know $\dim H^*(G, \mathbb{Q}) = 2^l$. Chern character gives a ring isomorphism between $K^*(G) \otimes \mathbb{Q}$ and $H^*(G, \mathbb{Q})$(c.f. [AH]). As $K^*(G)$ is torsion-free by Lemma 5.7, it is a free abelian group of rank 2^l. Let $\Lambda = \bigwedge(e_1, \cdots, e_l)$ be the exterior algebra generated by $e_1, \cdots, e_l$ over $\mathbb{Z}$. Define

$$\begin{aligned} j : \Lambda &\to K^*(G) \\ e_i &\mapsto \beta(\rho_i) \end{aligned}$$

Proposition 5.6 implies that j is an injective ring homomorphism. Since both Λ and $K^*(G)$ have the same rank, $j(\Lambda)$ has a finite index in $K^*(G)$. Note that one can use the K-theory pushforward(or the index map)

$$f_! : K^*(G) \to K^*(\mathrm{pt}) \cong \mathbb{Z}$$

defined by $f_!([E]) = \int_G \mathrm{ch}(E)\mathrm{td}(G) = \int_G \mathrm{ch}(E)(\mathrm{td}(G) = 1$ as G is parallelizable), to define

$$k : K^*(G) \to \mathrm{Hom}(\Lambda, \mathbb{Z})$$

such that

$$k(x)(y) = f_!(xj(y)), x \in K^*(G), y \in \Lambda$$

We have $k \circ j(\bigwedge_{i \in I} e_i)(\bigwedge_{j \in J} e_j) = \pm 1$ if $I \cup J = \{1, \cdots, l\}$. So $k \circ j$ is an isomorphism. It follows that $j(\Lambda)$ is a direct summand of $K^*(G)$. Being of finite index in $K^*(G)$, $j(\Lambda)$ is actually isomorphic to $K^*(G)$. This completes the proof. $\square$

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