Horn's Problem

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Notations

- Let A, B and C be Hermitian matrices

- Let α, β and γ be respectively the sets of eigenvalues of A, B and C (arranged in decreasing order)
- Let I, J and K be multi-indices $I = \{i_1 < i_2 < \cdots < i_r\}$, $J = \{j_1 < \cdots < j_r\}$, $K = \{k_1 < \cdots < k_r\}$

- We say λ, μ, ν <u>associate</u> to I, J, K resp. if

$$\lambda = \{ i_r - r, \cdots, i_1 - 1 \}$$

$$\mu = \{ j_r - r, \cdots, j_1 - 1 \}$$

$$\nu = \{ k_r - r, \cdots, k_1 - 1 \}$$

$\lambda = s(I)$ if λ assoc. to I.

- Let $\mathcal{O}_\alpha$ be the space of Hermitian matrices with spectrum α.

e.g. $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \in \mathcal{O}_{\{5,0\}}$

Horn's Problem

Given α and β, what are the necessary and sufficient conditions for γ s.t.
$\exists$ $A \in \mathcal{O}_\alpha$ and $B \in \mathcal{O}_\beta$ and

$$C = A + B \in \mathcal{O}_\gamma$$

Some easy necessary conditions

- $$\sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{n} \beta_j = \sum_{k=1}^{n} \gamma_k$$

- $$\gamma_1 \leq \alpha_1 + \beta_1$$

$$\left(\because \gamma_1 = \sup_{\|x\|=1} \langle Cx, x \rangle \right)$$

(H. Weyl, 1912)

$$\gamma_{p+q+1} \leq \alpha_{p+1} + \beta_{q+1}$$

This is a very good linear algebra exercise !

Horn's Conjecture

(α, β, γ) occurs as eigenvalues of A, B and C with $C = A + B$

iff ① $\sum_{k=1}^{n} \gamma_k = \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{n} \beta_j$

② $\sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j$

$\forall I, J, K$ of length $r < n$, s.t.

$$c_{\lambda\mu}^{\nu} \neq 0$$

$c_{\lambda\mu}^{\nu}$ is the Littlewood-Richardson coefficient. (To be explained later)

The conjecture was proved to be true by Klyachko, Totaro, ...

A major ingredient in the proof of "$\Longrightarrow$" is Schubert Calculus

Schubert Calculus

Def The **Grassmannian** $\mathbb{G}(r, \mathbb{C}^n)$ is the algebraic variety (in fact, projective variety)

$$\{L \subset \mathbb{C}^n \text{ vect. subsp.} \mid \dim(L) = r\}$$

Rmk $\mathbb{G}(r, \mathbb{C}^n)$ is a complex manifold of complex dimension $r(n-r)$.

Def A **complete flag** of $\mathbb{C}^n$ is

$$F_\bullet = \{0 = F_0 \subset F_1 \subset \cdots \subset F_n = \mathbb{C}^n\}$$

where F_i is a vect. subsp. of dim. i.

Def Let $I = \{i_1 < \cdots < i_r\} \subset \{1, \cdots, n\}$

The **Schubert variety** $\Omega_I(F_\bullet)$ is

$$\Omega_I(F_\bullet) = \{L \in \mathbb{G}(r, \mathbb{C}^n) \mid \dim(L \cap F_{i_p}) \geq p \text{ for } 1 \leq p \leq r\}$$

It is a closed analytic subvariety of $\mathbb{G}(r, \mathbb{C})$.

$$\dim_{\mathbb{C}} \Omega_I(F_\bullet) = \sum_{p=1}^{r} (i_p - p)$$

$$\Omega_I^{\circ}(F_\bullet) = \{L \in \mathbb{G}(r, \mathbb{C}) \mid \dim(L \cap F_{i_p}) = p \text{ for } 1 \leq p \leq r\}$$

- $\{\Omega^\circ_I(F_\bullet)\}_I$ gives a cell decomposition of $\mathbb{G}(r, \mathbb{C}^n)$

- $\overline{\Omega^\circ_I(F_\bullet)} = \Omega_I(F_\bullet)$

- The homology classes $[\Omega_I(F_\bullet)]$ form a basis for $H_*(\mathbb{G}(r, \mathbb{C}); \mathbb{Z})$.

- $[\Omega_I(F_\bullet)]$ is independent of $F_\bullet$. $\omega_I := [\Omega_I(F_\bullet)]$

<u>Def</u> Let λ be a sequence of weakly decreasing integers:

$$n-r \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_r \geqslant 0.$$

Let $|\lambda| = \sum_{p=1}^{r} \lambda_p$. Define

$$\sigma_\lambda \in H^{2|\lambda|}(\mathbb{G}(r, \mathbb{C}^n); \mathbb{Z})$$

to be the Poincaré dual of ω_I, where

$$i_p = n - r + p - \lambda_p, \quad 1 \leq p \leq r$$

We will identify σ_λ with ω_I.

σ_λ are called Schubert classes

By intersection theory,

$$\sigma_\lambda \cdot \sigma_\mu = [\Omega_I(F_\bullet) \cap \Omega_J(F_\bullet)]$$

with

$$i_p = n - r + p - \lambda_p$$

$$j_p = n - r + p - \mu_p$$

<u>Example</u> $\mathbb{X} = \mathbb{G}(2, \mathbb{C}^4)$.

$F_i = \mathrm{span}\{e_1, \cdots, e_i\}$

$$\sigma_{1,1} = \omega_{2,3}$$

$$= \{L \in \mathbb{X} \mid L \subset F_3\}$$

$$\sigma_{2,0} = \omega_{1,4}$$

$$= \{L \in \mathbb{X} \mid L \supset F_1\}$$

By projectivizing $\mathbb{C}^4$,

$\mathbb{G}(2, \mathbb{C}^4)$ = set of lines in $\mathbb{CP}^3$

So

$$\sigma_{1,1} = \{ \ell \in \mathbb{X} \mid \ell \subset P \}$$

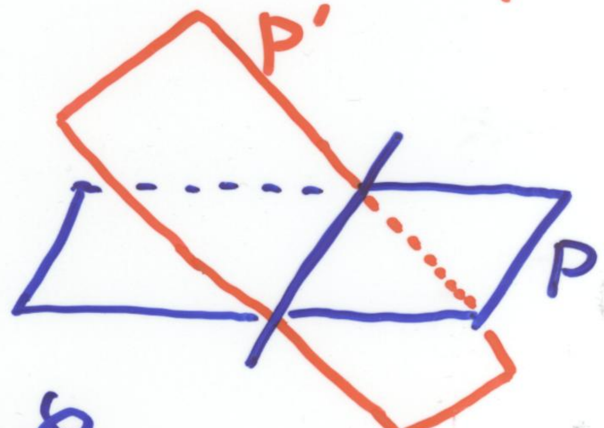
A plane $\in \mathbb{C}P^3$

$$\sigma_{2,0} = \{ \ell \in \mathbb{X} \mid \wp \in \ell \}$$

A pt. $\in \mathbb{C}P^3$

$$\sigma_{1,1} \cdot \sigma_{2,0} = 0$$

$$\sigma_{1,1}^2 = \sigma_{1,1}(P) \cdot \sigma_{1,1}(P')$$
$$= \sigma_{2,2}$$



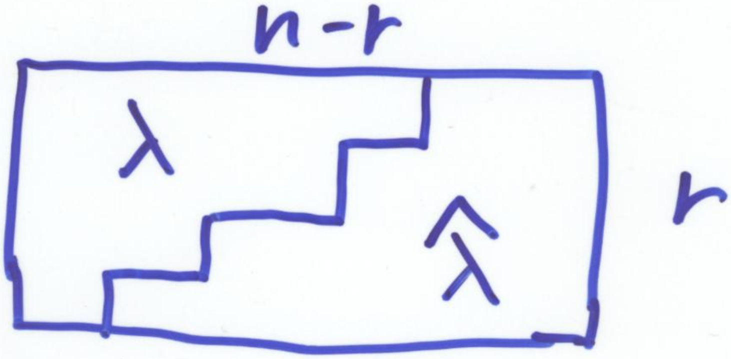
$$\sigma_{2,0}^2 = \sigma_{2,0}(\wp) \cdot \sigma_{2,0}(\wp')$$
$$= \sigma_{2,2}$$

FACT: If $|\lambda| + |\mu| = r(n-r)$,

then $\sigma_\lambda \cdot \sigma_\mu = \delta_{\lambda,\hat{\mu}}\, \sigma_{n-r,n-r,\dots,n-r}$

↑ representing a pt.

$\hat{\mu} = \{n-r-\mu_1, \cdots, n-r-\mu_r\}$



<u>Def</u> Let

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c_{\lambda\mu}^{\nu} \sigma_\nu$$

$c_{\lambda\mu}^{\nu}$ is the Littlewood-Richardson coefficient.

The Littlewood-Richardson coefficients also appear in representation theory of $GL_n(\mathbb{C})$

Proof of "$\Rightarrow$":

<u>Def</u> Let $L \in \mathbb{G}(r, \mathbb{C}^n)$. Define the **Rayleigh trace** $R_A(L)$ to be

$$\sum_{p=1}^{r} \langle A u_p, u_p \rangle$$

where $\{u_1, \cdots, u_r\}$ is an O.N.B. of L.

<u>Def</u> $F_p(A) = \text{span}\{v_1, \cdots, v_p\}$, where v_i corr. to α_i

Lemma $I = \{i_1 < \cdots < i_r\}$

$$\sum_{i \in I} \alpha_i = \min_{L \in \Omega_I(F_\bullet(A))} R_A(L)$$

$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ eigvals of A

$$\Leftrightarrow \quad \begin{array}{ccccc} \alpha_1' \geq & \alpha_2' \geq & \cdots \geq & \alpha_n' & \text{eigvals of } -A \\ \parallel & \parallel & & \parallel & \\ -\alpha_n & -\alpha_{n-1} & & -\alpha_1 & \end{array}$$

Let $I' = \{n+1-i_r < \cdots\cdots < n+1-i_1\}$

Then

$$-\sum_{i \in I} \alpha_i - \sum_{j \in J} \beta_j + \sum_{k \in K} \gamma_k$$

$$= \sum_{i \in I'} \alpha_i' + \sum_{j \in J'} \beta_j' + \sum_{k \in K} \gamma_k \quad \text{——} \quad (*)$$

Lemma $\lambda = s(I)$.

$\Rightarrow \sigma_\lambda = \omega_{I'}$

Pf: p-th elt of λ

$= i_{r+1-p} - r - 1 + p$

p-th elt of I'

$= n+1 - i_{r+1-p}$

$i_{r+1-p} - r - 1 + p = n - r + p - (n+1 - i_{r+1-p})$

∎

Lemma $\sigma_{s(I)} \cdot \sigma_{s(I')} = \sigma_{n-r, \dots, n-r}$

Pf $s(I)$ and $s(I')$ are complementary partition of $r \times (n-r)$

∎

We have TFAE

1. $c_{\lambda\mu}^{\nu} \neq 0$, $\lambda = s(I)$, $\mu = s(J)$, $\nu = s(K)$

2. $\sigma_{s(K)}$ occurs $\sigma_{s(I)} \cdot \sigma_{s(J)}$
   ($\lambda = s(I)$, $\mu = s(J)$, $\nu = s(K)$)

3. $\sigma_{s(I)} \cdot \sigma_{s(J)} \cdot \sigma_{s(K')} \neq 0$

4. $\omega_{I'} \cdot \omega_{J'} \cdot \omega_K \neq 0$.

Condition 4 says that

$$\Omega_{I'}(F_\bullet(-A)) \cap \Omega_{J'}(F_\bullet(-B)) \cap \Omega_K(F_\bullet(C))$$

$$\neq \emptyset$$

Let L be in the intersection.

$$\sum_{i \in I'} \alpha_i + \sum_{j \in J'} \beta_j + \sum_{k \in K} \gamma_k$$

$$\leq R_{-A}(L) + R_{-B}(L) + R_C(L) = 0 \quad \square$$

Why are linear inequalities sufficient?

It turns out that it is a consequence of convexity theorem in symplectic geometry.

Sketch:

(M, ω) a cpt symplectic mfd.

G a cpt Lie group

$G \circlearrowright M$ preserving ω

$\mu : M \longrightarrow \mathfrak{g}^*$ moment map

if ① G-equivariance

② $d\langle \mu(m), X \rangle = \iota_{X^\#} \omega$

Thm (Kirwan)

$$M \xrightarrow{\mu} \mathfrak{g}^* \xrightarrow{\phi} \mathfrak{t}^*_+$$

$\mathrm{Im}(\phi \circ \mu)$ is convex.

Let $M = \mathcal{O}_\alpha \times \mathcal{O}_\beta$ (symplectic)

$G = U(n)$.

$G \circlearrowright M$ diagonally.

$\mathfrak{u}(n)^* \cong \mathcal{H}$ = space of Hermitian matrices

$$\mu : \mathcal{O}_\alpha \times \mathcal{O}_\beta \longrightarrow \mathcal{H}$$

$$(A, B) \longmapsto A+B$$

is the moment map.

ϕ corr. to a map taking $A \in \mathcal{H}$ to the diag. matrix with A's eigvals decreasing down the diag