The Real $K$-theory of compact Lie groups

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Goal

Throughout the talk, $G$ is assumed to be compact, connected and simply-connected unless otherwise specified. We always view $G$ as a $G$-space equipped with conjugation action. Let $\rho_1, \cdots, \rho_\ell$ be the fundamental representations of $G$.

**Goal:** To give a description of the ring structure of $KR_G^*(G)$, the equivariant Real $K$-theory of $G$. 
**Complex $K$-theory of compact Lie groups**

**Definition**

Let $\delta_G : R(G) \to K_G^{-1}(G)$ be defined by

$$\delta_G(\rho) = [0 \to G \times \mathbb{R} \times V \xrightarrow{\alpha} G \times \mathbb{R} \times V \to 0]$$

where $V$ is the underlying complex vector space of $\rho$, and

$$\alpha(g, t, v) = \begin{cases} (g, t, t\rho(g)v) & \text{if } t \geq 0 \\ (g, t, tv) & \text{if } t < 0 \end{cases}$$

Let $\delta : R(G) \to K^{-1}(G)$ be $\delta(\rho) = f(\delta_G(\rho))$, where $f : K_G^{-1}(G) \to K^{-1}(G)$ is the forgetful map.

**Definition**

Let $\Omega^*_R(G)_{/\mathbb{Z}}$ be the exterior algebra over $R(G)$ of the module of Kähler differentials of $R(G)$ over $\mathbb{Z}$, i.e.

$$\Omega^*_R(G)_{/\mathbb{Z}} := \bigwedge_{R(G)} (d\rho_1, \cdots, d\rho_\ell)$$
Complex $K$-theory of compact Lie groups

**Theorem (Atiyah, ’65, Hodgkin, ’67)**

$$K^*(G) \cong \bigwedge_{\mathbb{Z}}(\delta(\rho), \cdots, \delta(\rho_\ell))$$

**Theorem (Brylinski-Zhang, 2000)**

Let

$$\varphi : \Omega^*_{R(G)/\mathbb{Z}} \to K^*_G(G)$$

$$\rho_V \mapsto [G \times V]$$

$$d\rho_V \mapsto \delta_G(\rho_V)$$

Then $\varphi$ is an algebra isomorphism.

**Corollary**

$$K^*_G(G) \cong R(G) \otimes K^*(G)$$
**KR-theory**

KR-theory was first introduced by Atiyah and motivated by the index theory of real elliptic operators.

**Definition**

1. $(X, \iota_X)$ is a *Real space* if $\iota_X$ is a homeomorphism on $X$ such that $\iota_X^2 = \text{Id}_X$.
2. $(G, \iota_G)$ is a *Real Lie group* if $\iota_G$ is an automorphism on $G$ such that $\iota_G^2 = \text{Id}_G$.
3. $(X, \iota_X, G, \iota_G)$ is a *Real $G$-space* if $\iota_X(g \cdot x) = \iota_G(g) \cdot \iota_X(x)$.
4. $(E, \iota_E)$ is a *Real $G$-vector bundle* on $(X, \iota_X)$ if
   1. $E$ is a complex $G$-vector bundle on $X$.
   2. Both $E$ and $X$ are Real $G$-spaces.
   3. $\iota_X \circ \pi = \pi \circ \iota_E$.
   4. $\iota_E$ maps $E_x$ to $E_{\iota_X(x)}$ anti-linearly.

5. The equivariant *KR*-theory, $\text{KR}_{(G, \iota_G)}(X, \iota_X)$ is the Grothendieck group of the commutative monoid of the isomorphism classes of Real $G$-vector bundles over $(X, \iota_X)$.

**Definition**

Let $r : K^*_G(X) \to \text{KR}^*_G(X)$ be the *realification* map defined by $[E] \mapsto [E \oplus \iota_G^* \iota_X^* E]$. Here $K^*_G(X)$ is extended to a $\mathbb{Z}_8$-graded ring by complex Bott periodicity.
Example

\[ KR^*(pt) \cong \mathbb{Z}[\eta, \mu]/(2\eta, \eta^3, \mu\eta, \mu^2 - 4), \] where \( \eta \in KR^{-1}(pt), \mu \in KR^{-4}(pt) \) represent the reduced Hopf bundles of \( \mathbb{RP}^1 \) and \( \mathbb{HP}^1 \) respectively.

What about \( KR_G^*(pt) \)?
Real representation rings

Definition

1. A *Real representation* $V$ of $(G, \iota_G)$ is a finite dimensional complex representation of $G$ equipped with an anti-linear involution $\iota_V$ such that $\iota_V(gv) = \iota_G(g)\iota_V(v)$.  

2. The *Real representation ring* $\mathbb{R}R(G, \iota_G)$ is the Grothendieck group of isomorphism classes of Real representations of $(G, \iota_G)$ with multiplication being tensor product over $\mathbb{C}$.  

3. Similarly, define *Quaternionic representation* $V$ of $(G, \iota_G)$ to be a finite dimensional complex representation of $G$ equipped with an anti-linear $J_V \in \text{End}(V)$ such that $J_V^2 = -\text{Id}_V$ and $J_V(gv) = \iota_G(g)J_V(v)$. Let $\mathbb{R}H(G, \iota_G)$ be the Quaternionic representation group.  

4. $R(G, \mathbb{C}) :=$ abelian group freely generated by the isomorphism classes of irreducible complex representations $V$ where $V \not\cong \iota_G^*V$.  

Let $F = \mathbb{R}$ or $\mathbb{H}$. If $V \in RF(G, \iota)$ is irreducible, then $K := \text{Hom}_{(G, \iota_G)}(V, V)$ is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We call $V$ a $F$-representation of $K$-type.

Definition

$RF(G, \iota_G, K)$ is defined to be the free abelian group generated by the isomorphism classes of irreducible $F$-representation of $K$-type.
The ring structure

Proposition

There is an isomorphism of graded abelian groups

\[(\mathbb{R}(G, \mathbb{R}) \oplus \mathbb{H}(G, \mathbb{R})) \otimes K^{\ast}(pt) \oplus r(\mathbb{R}(G, \mathbb{C}) \otimes K^{\ast}(pt)) \cong K^{\ast}_{G}(pt)\]

If \(\mathbb{R}(G, \mathbb{C}) = 0\), then the above is an isomorphism of graded rings.

Theorem (F)

There is an isomorphism of graded abelian groups

\[(\mathbb{R}(G, \mathbb{R}) \oplus \mathbb{H}(G, \mathbb{R})) \otimes K^{\ast}(G) \oplus r(\mathbb{R}(G, \mathbb{C}) \otimes K^{\ast}(G)) \cong K^{\ast}_{G}(G)\]

If \(\mathbb{R}(G, \mathbb{C}) = 0\), then the above is a \(K^{\ast}_{G}(pt)\)-module isomorphism.

Definition

Let \(\delta^{G}_{\mathbb{R}} : \mathbb{R}(G) \rightarrow K^{\ast}_{G}(G)\) and \(\delta^{G}_{\mathbb{H}} : \mathbb{H}(G) \rightarrow K^{\ast}_{G}(G)\) be defined similarly to \(\delta_{G}\).
The ring structure

Using the results of Seymour's on $KR^*(G)$ and Brylinski-Zhang's, we have

**Corollary**

$KR^*_G(G)$ is generated, as a $KR^*_G(pt)$-algebra, by
\[
\{ \delta^G_R(\varphi) | \varphi \in \{ \text{fundamental reps} \} \cap R_R(G, \mathbb{R}) \},
\{ \delta^G_H(\theta) | \theta \in \{ \text{fundamental reps} \} \cap R_H(G, \mathbb{R}) \}
\]
and some other generators associated to $R(G, \mathbb{C})$.

**Theorem (F)**

If $\varphi \in R_R(G)$ and $\theta \in R_H(G)$, then
\[
\delta^G_R(\varphi)^2 = \eta(\varphi \cdot \delta^G_R(\varphi) - \delta^G_R(\wedge^2 \varphi))
\]
\[
\delta^G_H(\theta)^2 = \eta(\theta \cdot \delta^G_H(\theta) - \delta^G_R(\wedge^2 \theta))
\]

So $KR^*_G(G)$ is not an exterior algebra!

**Question:** Is there a better description of the ring $KR^*_G(G)$? An exterior algebra with deformation?
Examples

Example

Let $\iota_{\mathbb{R}}$ be complex conjugation on $G = SU(n)$. Then all the fundamental representations $\wedge^k \sigma_n$ are in $R(\mathbb{R}(SU(n), \iota_{\mathbb{R}}, \mathbb{R}))$, and $R(SU(n), \iota_{\mathbb{R}}, \mathbb{C}) = 0$. So $KR_{(SU(n), \iota_{\mathbb{R}})}(SU(n), \iota_{\mathbb{R}}) \cong \Omega_{KR_{(SU(n), \iota_{\mathbb{R}})}}^{*}(pt)/KR^{*}(pt)$ as $KR_{(SU(n), \iota_{\mathbb{R}})}(pt)$-modules. We have

$$\delta^{\mathbb{G}}_{\mathbb{R}}(\wedge^k \sigma_n)^2 = \eta \sum_{i=1}^{2k} \wedge^{2k-i} \sigma_n \cdot \delta^{\mathbb{G}}_{\mathbb{R}}(\wedge^i \sigma_n)$$

In $KR^{*}(SU(n), \iota_{\mathbb{R}})$, we have

$$\delta_{\mathbb{R}}(\wedge^k \sigma_n)^2 = \eta \sum_{i=1}^{2k} \binom{n}{2k - i} \cdot \delta_{\mathbb{R}}(\wedge^i \sigma_n)$$

Example

Both $K_{SU(n)_{\text{triv}}}^{*}(SU(n))$ and $K_{SU(n)_{\text{conj}}}^{*}(SU(n))$ are isomorphic to $\Omega_{R(SU(n))/\mathbb{Z}}^{*}$ as rings. On the other hand, $KR_{(SU(n)_{\text{triv}}, \iota_{\mathbb{R}})}^{*}(SU(n), \iota_{\mathbb{R}})$ are not isomorphic to $KR_{(SU(n)_{\text{conj}}, \iota_{\mathbb{R}})}^{*}(SU(n), \iota_{\mathbb{R}})$ as rings, though both are isomorphic to $\Omega_{KR_{(SU(n), \iota_{\mathbb{R}})}}^{*}(pt)/KR^{*}(pt)$ as $KR_{(SU(n), \iota_{\mathbb{R}})}(pt)$-modules.