EQUIVARIANT FORMALITY OF HOMOGENEOUS SPACES

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Abstract. Let G be a compact connected Lie group and K a connected Lie subgroup. In this paper, we study the equivariant formality of the isotropy action of K on G/K. We introduce an analogue of equivariant formality in K-theory called rational K-theoretic equivariant formality (RKEF) and show that it is equivalent to equivariant formality in the usual sense. Using RKEF, we give a more uniform proof of the main results in [Go] and [GoNo] that the isotropy actions on (generalized) symmetric spaces are equivariantly formal, without appealing to the classification theorem of those spaces. We also give a representation theoretic condition which is equivalent to G/K being formal and the isotropy action being equivariantly formal.

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1. Introduction

Equivariant formality, first defined in [GKM], is a special property of topological spaces with group actions which allows for easy computation of their equivariant cohomology. Roughly speaking, equivariant formality amounts to the existence of equivariant extension in the equivariant cohomology theory of any element in the ordinary cohomology theory. Equivalently, a G-space X is equivariantly formal if and only if the Leray-Serre spectral sequence of the fiber bundle $X \hookrightarrow X \times_G EG \to BG$ collapses on the E_2-page. The latter is also equivalent to $H^*_G(X) \cong H^*_G(\mathrm{pt}) \otimes H^*(X)$ as $H^*_G(\mathrm{pt})$-modules. There are various

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examples of interest which are known to be equivariantly formal, e.g. Hamiltonian manifolds, smooth complex projective varieties with linear algebraic torus actions (cf. [GKM, Section 1.2 and Theorem 14.1]).

It would be desirable to have a classification of equivariantly formal spaces, but the task is still formidable even if we restrict our attention to those with compact Lie group actions. An easier question would be to determine if, for a compact Lie group G and a closed subgroup K, the K-action on the homogeneous space G/K by isotropy action (i.e. the action $g \cdot hK = ghK$) is equivariantly formal. A pair (G, K) is said to be an *isotropy formal* pair if the isotropy action on G/K is equivariantly formal. It is of interest to determine if a pair (G, K) is isotropy formal, and sufficient conditions for (G, K) to be isotropy formal pairs and certain classes of such pairs have been studied by various authors. The homogeneous spaces they considered are all *formal* in the sense of Sullivan (cf. Definition 3.3). For instance, Shiga and Takahashi obtained the following

Theorem 1.1 ([Sh] and [ShTa]). *If G/K is formal, then $i^*H^*_G(pt) = H^*_K(pt)^N$, where $N = N_G(K)/K$, if and only if (G, K) is an isotropy formal pair.*

Recently, Goertsches and Noshari showed that

Theorem 1.2 ([Go] and [GoNo]). *The classes of formal homogeneous spaces in Example 3.4 all admit equivariantly formal isotropy action.*

The first author of this paper recently showed the following

Theorem 1.3 ([Ca], Theorem 1.1). *(G, K) is an isotropy formal pair if and only if (G, S) is, where S is a maximal torus of K.*

Theorem 1.4 ([Ca], Theorem 1.2). *Let G be a compact connected Lie group and S a torus subgroup. Let $\widetilde{G}$ be a finite central covering of G and $\widetilde{S}$ the identity component of the preimage of S. Then (G, S) is isotropy formal if and only if $(\widetilde{G}, \widetilde{S})$ is.*

Thus the problem of determining if an isotropy action is equivariantly formal boils down to the one involving torus isotropy action and G which is a product of a torus and a compact simply-connected Lie group. He also gave an explicit algorithm of determining if (G, S) is isotropy formal when S is a circle subgroup (cf. [Ca, Algorithm 1.4]).

One feature in common in the main arguments used by [Ca], [Go], [GoNo] and [ShTa] to establish equivariant formality of isotropy action is the application of the condition of dim $H^*(M) =$ dim $H^*(M^T)$ which is equivalent to T acting on M equivariantly formally. For instance, to show that compact (generalized) symmetric spaces admit isotropy formal action, the authors of [Go] and [GoNo] invoked the classification theorem for such spaces and verify the equality of cohomological dimensions case by case. In our opinion, while checking the equality of cohomological dimensions to prove equivariant formality is not as straightforward as checking the surjectivity of the forgetful map from the equivariant cohomology of the homogeneous space to its ordinary cohomology, the latter approach does not come in handy either. Besides to us appealing to the classification theorem in the proof of isotropy formality of (generalized) symmetric pairs in [Go] and [GoNo] is not satisfactory.

This paper is a continuation of the paper [Ca] which made the first attempt of characterizing the subgroup K such that (G, K) is an isotropy formal pair. Instead of working in equivariant cohomology, we apply K-theory to study the equivariant formality of isotropy actions. Inspired by the notion of weakly equivariant formality, introduced in [HL], we define the similar notion of *rational K-theoretic equivariant formality* (RKEF for short, see Definition 2.2). The use of K-theory is feasible because of the following result which is crucial in our work.

Theorem 1.5 ([F2]). *Let X be a finite CW-complex with an action by a torus group T. X is a RKEF T-space if and only if it is an equivariantly formal T-space.*

The proof is reproduced in Section 2. Using the above Theorem we translate the whole problem to the context of K-theory. One advantage of working in K-theory is that it is more straightforward to check if the forgetful map is onto, since this amounts to determining if a vector bundle can be equipped with a T-action so as to become an equivariant T-vector bundle. After obtaining the K-theory ring of compact homogeneous spaces (cf. Proposition 3.7) and topological realizations of each element in the K-theory ring when the homogeneous spaces are formal manifolds, we get a simple criterion for a formal homogeneous space to be isotropy formal (cf. Proposition 4.2), which we apply to give a more uniform, simpler proof of the aforementioned result that the homogeneous spaces (3)-(4) in Example 3.4 are isotropy formal (cf. Proposition 4.4). Furthermore we also get the following sufficient condition for isotropy formality.

Theorem 1.6 (Theorem 4.9). *Let G be a compact connected Lie group and S a torus subgroup. G/S is both isotropy formal and formal in the sense of Sullivan if and only if the image of the restriction map $i^*: R(G) \to R(S)$ is regular at the augmentation ideal.*

Theorem 1.6 provides a uniform proof of the fact that homogeneous spaces (1)-(4) of Example 3.4 are both formal *and* isotropy formal in one fell swoop (that these spaces are formal were first proved in [GHV] and [T]). The regularity criterion in the Theorem can be conveniently verified using computer algebra packages such as Macaulay2 and SAGE. Two examples are worked out in Section 4.4 to demonstrate the usefulness of Theorem 1.6; one of the examples is both formal and isotropy formal, but does not belong to the classes of homogeneous spaces in Example 3.4.

2. Equivariant formality in K-theory

The notion of equivariant formality in K-theory was first defined and explored by Harada and Landweber in [HL], where they instead used the term ‘weak equivariant formality’.

Definition 2.1 (Definition 4.1 in [HL]). A G-space M is weakly equivariantly formal if the map

$$K_G^*(M) \otimes_{R(G)} \mathbb{Z} \to K^*(M)$$

induced by the forgetful map

$$f: K_G^*(M) \to K^*(M)$$

is a ring isomorphism. Here in the above tensor product, $\mathbb{Z}$ is regarded as a $R(G)$-module through augmentation homomorphism.

Recall that in cohomology, one of the equivalent conditions for a G-space M to be equivariantly formal is that the Leray-Serre spectral sequence for the (rational) cohomology of the Borel fibration $M \hookrightarrow M \times_G EG \to BG$ collapses on the E_2-page. This leads to the equivalent condition that $H^*_G(M)$ be isomorphic to $H^*(BG)\otimes H^*(M)$ as $H^*(BG)$-modules. Nevertheless, Harada and Landweber settled for Definition 2.1 as the K-theoretic analogue of equivariant formality, instead of the seemingly obvious candidate $K^*_G(M)\cong K^*_G(\mathrm{pt})\otimes K^*(M)$, citing the lack of the Leray-Serre spectral sequence for Atiyah-Segal's equivariant K-theory. The term 'weak' is in reference to the condition in Definition 2.1 being weaker than $K^*_G(M)\cong K^*_G(\mathrm{pt})\otimes K^*(M)$ because of the possible presence of torsion. We would like to define another version of K-theoretic equivariant formality in exact analogy with another cohomological equivariant formality condition that the forgetful map $H^*_G(M)\to H^*(M)$ be onto, and discuss its relation to Definition 2.1 under some conditions. The main result Theorem 1.5 in this Section comes from work in preparation ([F2]). For convenience of the reader we reproduce its proof here.

From now on, unless otherwise specified, X is a finite CW-complex equipped with an action by a torus T or more generally a compact Lie group G. We use K-theory with complex coefficient, and denote such with $K^*(X)$ and $K^*_G(X)$ by abuse of notation. Along the same vein, $R(G)$ is used to denote the representation ring with complex coefficient, thus $\mathbb{C}$ is regarded as a $R(G)$-module through the augmentation homomorphism. We use the decoration $\mathbb{Z}$ if integral coefficient is used. For instance integral K-theory of M is denoted by $K^*(M,\mathbb{Z})$. The term 'rational weakly equivariantly formal' is used to refer to, in view of Definition 2.1, the condition that

$$K^*_G(M)\otimes_{R(G)}\mathbb{C}\to K^*(M)$$

is a ring isomorphism.

Definition 2.2. We say M is a rational K-theoretic equivariantly formal (RKEF for short) G-space if the forgetful map

$$f: K^*_G(M)\to K^*(M)$$

is onto.

Under the condition of weak equivariant formality, [HL, Proposition 4.2] asserts that the kernel of f is $I(G)\cdot K^*_G(M)$, where $I(G)$ is the augmentation ideal of $K^*_G(M)$. We find that under the conditions that G be a torus group T, M be compact, and using complex coefficient, the weak equivariant formality condition can be removed.

Lemma 2.3. *Let ET^m be the join of m copies of T. Viewing $K^*(X\times ET^m/T)$ as a module over $K^*(ET^m/T)=\mathbb{C}[t_1,\cdots,t_n]/((t_1-1)^m,\cdots,(t_n-1)^m)$, we have that the kernel of the forgetful map*

$$f_m: K^*(X\times ET^m/T)\to K^*(X)$$

is $I(T)\cdot K^*(X\times ET^m/T)$, where $I(T)=(t_1-1,\cdots,t_n-1)$.

Proof. Viewing $X\times_T ET^m$ and X as fiber bundle over BT^m and a point respectively, the inclusion map $X\to X\times_T ET^m$ induces a $H^*(BT^m)$-module homomorphism between the E_2-pages of the Leray-Serre spectral sequences

$$E_2^{p,q}(X\times_T ET^m)=H^p(BT^m,H^q(X))\to E_2^{p,q}(X)=H^p(\mathrm{pt},H^q(X))$$

The kernel of this homomorphism is $J\cdot E_2^{p,q}(X\times_T ET^m)$, where $J=(u_1,\cdots,u_m)$. Hence the kernel of the homomorphism in the abutment $g_m: H^*(X\times_T ET^m)\to H^*(X)$ is $J\cdot H^*(X\times_T ET^m)$.

Consider the commutative diagram

$$
\begin{tikzcd}
K^*(X \times ET^m/T) \arrow[r, "f_m"] \arrow[d, "\mathrm{ch}"'] & K^*(X) \arrow[d, "\mathrm{ch}"] \\
H^*(X \times ET^m/T) \arrow[r, "g_m"] & H^*(X)
\end{tikzcd} \tag{1}
$$

We have that

$$
\begin{aligned}
\ker(f_m) &= \mathrm{ch}^{-1}(\ker(g_m)) \\
&= \mathrm{ch}^{-1}(J \cdot H^*(X \times ET^m/T)) \\
&= \mathrm{ch}^{-1}(J) \cdot K^*(X \times ET^m/T)
\end{aligned}
$$

It suffices to show that $\mathrm{ch} : K^*(ET^m/T) \to H^*(ET^m/T)$ maps I onto J. Note that

$$\mathrm{ch}(t_i - 1) = e^{u_i} - 1 = u_i + \frac{u_i^2}{2} + \cdots + \frac{u_i^{m-1}}{(m-1)!}$$

which is the product of u_i and a unit. That finishes the proof. $\square$

Proposition 2.4. *Let X be a finite CW-complex with T-action. Then*

$$\mathit{ker}(f) = I(T) \cdot K_T^*(X)$$

Proof. Using Lemma 2.3 and the left exactness of inverse limit, the kernel of the map

$$f_\infty : K_T^*(X \times ET) \to K^*(X)$$

is $\varprojlim_m I(T) \cdot K_T^*(X \times ET^m) = I(T) \cdot K_T^*(X \times ET)$. As the forgetful map $f : K_T^*(X) \to K^*(X)$ factors through $K_T^*(X \times ET)$, and the map $K_T^*(X) \to K_T^*(X \times ET)$ is an injection by [AS, Corollary 2.3] and the fact that $\bigcap_{m=1}^{\infty} I(T)^m = (0)$ in $K_T^*(\mathrm{pt})$, we have that $\ker(f) = I(T) \cdot K_T^*(X)$. $\square$

Proposition 2.5. *With the same condition as in Proposition 2.4, X is a RKEF T-space if and only if X is a rational weakly equivariantly formal T-space.*

Proof. Using Proposition 2.4, the forgetful map f is onto if and only if $K_T^*(X)/I(T) \to K^*(X)$ is an isomorphism. Note that the map

$$
\begin{aligned}
K_T^*(X) \otimes_{R(T)} \mathbb{C} &\to K_T^*(X)/I(T) \cdot K_T^*(X) \\
\alpha \otimes z &\mapsto \overline{\alpha} z
\end{aligned}
$$

is well-defined and an isomorphism. This completes the proof. $\square$

Corollary 2.6. *Let G be a compact Lie group with $\pi_1(G)$ torsion-free and act on a finite CW-complex X, and T a maximal torus of G. Then*

1. *X is a RKEF G-space if and only if it is a rational weakly equivariantly formal G-space.*
2. *X is a RKEF G-space if and only if it is a RKEF T-space.*

Proof. That X being a rational weakly equivariantly formal G-space implies it is a RKEF G-space is immediate. On the other hand, if X is a RKEF G-space, then it is a RKEF T-space since the forgetful map $K_G^*(X) \to K^*(X)$ factors through $K_T^*(X)$. By Proposition 2.5, X is a rational weakly equivariantly formal T-space. Invoking [HL, Lemma 4.4] (it is where the torsion-freeness of $\pi_1(G)$ is used), X is a rational weakly equivariantly formal G-space, as desired. This completes the proof of (1). (2) follows from (1), [HL, Lemma 4.4] and Proposition 2.5. $\square$

Proposition 2.5 says that under the compactness assumption on X, using torus action and complex coefficient (in fact any field coefficient of characteristic 0), rational weak equivariant formality is equivalent to our definition of K-theoretic equivariant formality, an exact analogue of another condition for X to be cohomological equivariantly formal, namely, that the forgetful map $H_T^*(M) \to H^*(M)$ is onto. It turns out that more is true.

Theorem 2.7 (Theorem 1.5)**.** *Let X be a finite CW-complex with an action by a torus group T. X is a RKEF T-space if and only if it is an equivariantly formal T-space.*

First we need a result on comparison of dimensions of K-theory of X and its fixed point set.

Lemma 2.8. *For any finite CW-complex X with a T-action, we have the following string of (in)equalities:*

$$dim\ K^*(X^T) = rank_{R(T)} K_T^*(X) \leq dim\ K_T^*(X)/I(T) \cdot K_T^*(X) \leq dim\ K^*(X)$$

Proof. The first equality follows from Segal's localization theorem (cf. [S2, Proposition 4.1]) which, when applied to torus group action, says that the restriction map $K_T^*(X) \to K_T^*(X^T)$ becomes an isomorphism after localizing at the zero prime ideal, i.e. to the field of fraction of $R(T)$. Next, we let $n = \dim\ K_T^*(X)_{I(T)}/I(T) \cdot K_T^*(X)_{I(T)}$ and $K_T^*(X)I(T)/I(T) \cdot K_T^*(X)_{I(T)}$ be spanned by $x_1, \cdots, x_n$ as a vector space over $R(T)_{I(T)}/I(T) \cdot R(T)_{I(T)} \cong \mathbb{C}$. Seeing that $K_T^*(X)$ is a finitely generated module over the Noetherian ring $R(T)$, we invoke Nakayama lemma, and have that there exist lifts $\widehat{x}_1, \cdots, \widehat{x}_n \in K_T^*(X)_{I(T)}$ that generate $K_T^*(X)_{I(T)}$ as a $R(T)_{I(T)}$-module. It follows, after further localization to the field of fraction of $R(T)$, that $\widehat{x}_1, \cdots, \widehat{x}_n$ span $K_T^*(X)_{(0)}$ as a $R(T)_{(0)}$-vector space. Noting the isomorphism $K_T^*(X)/I(T) \cdot K_T^*(X) \cong K_T^*(X)_{I(T)}/I(T) \cdot K_T^*(X)_{I(T)}$, we arrive at the first inequality. Finally, the last inequality follows from Proposition 2.4. $\square$

Proof of Theorem 1.5. If X is cohomological equivariantly formal, then $\dim H^*(X) = \dim H^*(X^T)$. The Chern character isomorphism implies that $\dim K^*(X^T) = \dim K^*(X)$ which, together with Lemma 2.8, yields $\dim K_T^*(X)/I(T) \cdot K_T^*(X) = \dim K^*(X)$ or, equivalently, that X is RKEF.

Assume on the other hand that X is RKEF. Consider the following commutative diagram

$$\begin{array}{ccc} K_T^*(X) & \xrightarrow{f} & K^*(X) \\ {\scriptstyle \mathrm{ch}_T}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}} \\ H_T^{**}(X) & \xrightarrow{g} & H^*(X) \end{array} \tag{2}$$

where $H_T^{**}(X)$ is the completion of $H_T^*(X)$ at the augmentation ideal ideal J. Since f is onto and ch is an isomorphism, g is onto. Since $H_T^*(X)$ is a finitely generated module over the Noetherian ring $H_T^*(\mathrm{pt})$, a simple result on completions (cf. [Ma, Theorem 55]) implies that $H_T^{**}(X) \cong H_T^*(X) \otimes_{H_T^*(\mathrm{pt})} H_T^{**}(\mathrm{pt})$. Applying the forgetful map g gives $H^*(X) = \mathrm{Im}(g) = \mathrm{Im}(g|_{H_T^*(X)}) \otimes_{\mathbb{C}} \mathbb{C} = \mathrm{Im}(g|_{H_T^*(X)})$. Hence X is cohomological equivariantly formal. □

Corollary 2.9 ([F2])**.** *(1) X is RKEF iff $dim K^*(X) = dim K^*(X^T)$.*

(2) Let G be a compact connected Lie group with torsion-free fundamental group acting on X. Then X is a cohomological equivariantly formal G-space iff X is a RKEF G-space.

Proof. (1) follows from Theorem 1.5 and Chern character isomorphism for compact spaces. Let T be a maximal torus of G. (2) follows from the following equivalences:

(a) X is a cohomological equivariantly formal G-space if and only if it is as a T-space.
(b) (Theorem 1.5) X is a cohomological equivariantly formal T-space if and only if it is a RKEF T-space.
(c) (Corollary 2.6 (2)) X is a RKEF T-space if and only if it is as a G-space.

□

3. K-theory of compact homogeneous spaces

In this section, we assume that G and K are compact connected Lie groups unless otherwise specified. Viewing a compact homogeneous space G/K as the base of the principal K-bundle G, the cohomology of G/K can be computed using the subcomplex of basic forms of the complex $\bigwedge^* \mathfrak{g}^*$, which can be seen as a variant of the Lie algebra cohomology. This complex is shown to be isomorphic to a Koszul complex called *Cartan algebra*:

$$(S(\mathfrak{k}^*)^{\mathrm{Ad}_K} \otimes (\bigwedge^* \mathfrak{g}^*)^{\mathrm{Ad}_G}, \nabla)$$

Here ∇ is a derivation satisfying

$$\nabla(s \otimes x_0 \wedge \cdots \wedge x_p) = \sum_{j=0}^{p} (-1)^j s \cdot i^* x_j \otimes x_0 \wedge \cdots \wedge \widehat{x}_j \wedge \cdots \wedge x_p$$

For details we refer the reader to [GHV], where general results on cohomology of compact homogeneous spaces are given. We summarize their results below.

Definition 3.1. (1) The Samelson subspace $\widehat{P}$ for the compact homogeneous space G/K is defined to be the primitive vector space of the image of the map

$$p^* : H^*(G/K) \to H^*(G)$$

(2) We denote the ring $H_K^*(\mathrm{pt})/i^* H_G^+(\mathrm{pt})$ by $H_K^*(\mathrm{pt}) /\!\!/ H_G^*(\mathrm{pt})$.

Theorem 3.2 ([GHV])**.** *For any compact homogeneous space G/K,*

$$H^*(G/K) \cong (H_K^*(pt) /\!\!/ H_G^*(pt) \oplus \mathcal{I}) \otimes \bigwedge^* \widehat{P}$$

where the elements in $H^*_K(pt)/\!\!/H^*_G(pt)$ are cohomology classes represented by elements in the Cartan algebra of zero degree in the exterior algebra part, whereas $\mathcal{I}$ is an ideal of $H^*_K(pt)/\!\!/H^*_G(pt)$ whose nonzero elements are represented by elements of positive degree in the exterior algebra part. We also have that

$$\begin{aligned}&\textit{Maximal degree in the exterior algebra part of elements in Cartan algebra}\\&\textit{representing nonzero cohomology class in } \mathcal{I}\\=&\textit{rank } G - \textit{rank } K - \textit{dim } \widehat{P}\end{aligned}$$

In particular, dim $\widehat{P}$ = rank G − rank K if and only if $\mathcal{I} = 0$.

We call $H^*_K(\mathrm{pt})/\!\!/H^*_G(\mathrm{pt})$ and $\bigwedge^* \widehat{P}$ the polynomial part and the exterior algebra part of $H^*(G/K)$, respectively.

Definition 3.3. A manifold X is defined to be *formal* in the sense of Sullivan (or simply formal) if the de Rham complex of differential forms on X is quasi-isomorphic to the complex $(H^*(X, \mathbb{R}), d = 0)$.

In some literature, e.g. [GHV], the pair (G, K) is called a *Cartan pair* if G/K is a formal manifold. The notion of formality stems from Sullivan's rational homotopy theory and minimal model.

Example 3.4. The compact homogeneous space G/K is formal if ([GHV])

(1) (Equal rank pair) rank G = rank K,
(2) (Cohomological surjective pair) the restriction map $H^*(G) \to H^*(K)$ is onto, or
(3) G/K is a symmetric space.
(4) (cf. [T], [KT] and [St]) G/K is a generalized symmetric space, i.e. $G^\sigma_0 \subset K \subset G^\sigma$ for some automorphism σ of G of finite order, then it is formal.

More generally, we have the following

Theorem 3.5. *The following are equivalent.*

(1) A compact homogeneous space G/K is formal.
(2) $\mathcal{I} = 0$, or equivalently, dim $\widehat{P}$ = rank G − rank K.
(3) The ring $H^*_K(pt)/\!\!/H^*_G(pt)$ is a complete intersection ring, i.e. the ideal $(i^*H^+_G(pt)) \subset H^*_K(pt)$ is regular.

Proof. For the equivalence of (1) and (2), see [GHV]. For the equivalence of (1) and (3), see [On, Theorem 12.2, p.211]. □

The above results on the cohomology of compact homogeneous spaces carry over to the context of K-theory by means of Chern character isomorphism. Let $\alpha : R(K) \to K^0(G/K)$ be the map which sends a K-representation ρ to the K-theory class of the associated vector bundle $G\times_K V_\rho$. Then the kernel of α is $i^*I(G)$ for the following reason. In the commutative diagram

$$
\begin{tikzcd}
R(K) \arrow[r, "\alpha"] \arrow[d] & K^*(G/K) \arrow[d, "\mathrm{ch}"] \\
H_K^*(\mathrm{pt}) \arrow[r] & H^*(G/K)
\end{tikzcd} \tag{3}
$$

The bottom map is the transgression map for the fibration $K \hookrightarrow G \to G/K$ and so its kernel is $i^* H_G^+(\mathrm{pt})$. The left vertical map, which is an inclusion of $R(K)$ into its completion at the augmentation ideal I_K (it is an inclusion because the I_K-adic topology of the completion is Hausdorff if K is connected (cf. the Note immediately preceding [AH, Section 4.5])) followed by the Chern character map to $H_K^*(\mathrm{pt}) = H^*(BK)$, is injective. So is the right vertical map. The kernel of α is then the pre-image of $i^* H_G^+(\mathrm{pt})$ under the left vertical map, which is exactly $i^* I(G)$. The quotient ring $R(K)/i^* I(G)$, which we denote similarly by $R(K)/\!\!/R(G)$, can then be identified with a subring of $K^*(G/K)$. The Chern character isomorphism $K^*(G/K) \to H^*(G/K)$ restricts to a map from $R(K)/\!\!/R(G)$ into $H_K^*(\mathrm{pt})/\!\!/H_G^*(\mathrm{pt})$.

Lemma 3.6. *For any compact homogeneous space G/K with G and K being connected Lie groups, the Chern character map*

$$ch : R(K)/\!\!/R(G) \to H_K^*(pt)/\!\!/H_G^*(pt)$$

is a ring isomorphism.

Proof. ch is injective because it is a restriction of the Chern character isomorphism $K^*(G/K) \to H^*(G/K)$. As to surjectivity of ch, we first deal with the special case where K is a torus subgroup S of G. In this case ch is given by

$$
\begin{aligned}
\mathrm{ch} : R(S)/\!\!/R(G) &\to H_S^*(\mathrm{pt})/\!\!/H_G^*(\mathrm{pt}) \\
\prod_{j=1}^{n} t_j^{a_j} &\mapsto \exp\left(\sum_{j=1}^{n} a_j u_j\right)
\end{aligned}
$$

Note that $H_S^*(\mathrm{pt})/\!\!/H_G^*(\mathrm{pt})$ is a finite dimensional vector space and hence any element of positive degree is nilpotent. Let m_j be the order of nilpotency of u_j. Consider the system of equations

$$
\begin{aligned}
\mathrm{ch}(t_j) &= 1 + u_j + \cdots + \frac{u_j^{m_j-1}}{(m_j-1)!} \\
\mathrm{ch}(t_j^2) &= 1 + 2u_j + \cdots + \frac{2^{m_j-1} u_j^{m_j-1}}{(m_j-1)!} \\
&\vdots \\
\mathrm{ch}(t_j^{m_j-1}) &= 1 + (m_j-1)u_j + \cdots + \frac{(m_j-1)^{m_j-1} u_j^{m_j-1}}{(m_j-1)!}
\end{aligned}
$$

They can be viewed as a system of linear equations in the unknowns $u_j, u_j^2, \cdots, u_j^{m_j-1}$ and the coefficient matrix is invertible. It follows that u_j can be expressed as a linear combination of $\mathrm{ch}(t_j), \cdots, \mathrm{ch}(t_j^{m_j-1})$, and ch in this special case is surjective, and hence an isomorphism. For a general compact homogeneous space G/K, one can take S to be a maximal torus of K, and observe the isomorphisms between $R(K)/\!\!/R(G)$ (resp. $H_K^*(\mathrm{pt})/\!\!/H_G^*(\mathrm{pt})$) and the Weyl group invariants $(R(S)/\!\!/R(G))^{W_K}$ (resp. $(H_S^*(\mathrm{pt})/\!\!/H_G^*(\mathrm{pt}))^{W_K}$). As the Chern character map is W_K-equivariant, it restricts to an isomorphism on the Weyl group invariants. □

By abuse of notation, we also use $\widehat{P}$ to denote the K-theoretic Samelson subspace of G/K, i.e. the primitive vector space of the image of the map $K^*(G/K) \to K^*(G)$.

Proposition 3.7. *For any compact homogeneous space G/K with K being a connected Lie subgroup, its K-theory is*

$$K^*(G/K) \cong (R(K)/\!\!/R(G) \oplus \mathcal{J}) \otimes \bigwedge\nolimits^* \widehat{P}$$

where $\mathcal{J}$ is an ideal of $R(K)/\!\!/R(G) \oplus \mathcal{J}$. G/K is formal if and only if $\mathcal{J} = 0$ if and only if $R(K)/\!\!/R(G)$ is a complete intersection ring.

Proof. This follows immediately from applying the inverse of the Chern character isomorphism to the description of $H^*(G/K)$ as in Theorem 3.2, Theorem 3.5 (3) and Lemma 3.6. □

4. Equivariant formality of compact homogeneous spaces

Throughout this section, we assume that both G and K are compact and connected Lie groups.

4.1. **A reduction.** Suppose G/K is formal and we would like to determine if it is isotropy formal. By [Ca, Theorem 1.1] it suffices to consider the isotropy action on G/S where S is a maximal torus of K. By Proposition 3.7, its K-theory is the tensor product of the polynomial part $R(S)/\!\!/R(G)$ and the exterior algebra part $\bigwedge^* \widehat{P}$. The K-theory classes in the polynomial part are represented by vector bundles of the form $G \times_S V_\rho$, where $\rho \in R(S)$. These vector bundles obviously can be made S-equivariant, covering the isotropy action on G/S. So the polynomial part of $K^*(G/S)$ admits equivariant lifts in the equivariant K-theory $K_S^*(G/S)$. By Theorem 1.5, determining if (G, S) is an isotropy formal pair then amounts to determining if the K-theoretic Samelson subspace $\widehat{P}$ admits equivariant lift in $K_S^*(G/S)$ as well

Definition 4.1. (cf. [F, Proposition 2.2]) Let $\rho_1, \rho_2 \in R(G)$ be representations of G which become the same representation on restriction to S, i.e. $\rho_1 - \rho_2 \in \ker(R(G) \to R(S))$, and V be the underlying complex vector space of the restricted representation. Define a map $\delta : \ker(R(G) \to R(S)) \to K^{-1}(G/S)$ (resp. $\delta_S : \ker(R(G) \to R(S)) \to K_S^{-1}(G/S)$) which sends $\rho_1 - \rho_2$ to the (equivariant) K-theory class represented by the complex of vector bundles

$$
\begin{aligned}
0 \longrightarrow G/S \times \mathbb{R} \times V &\longrightarrow G/S \times \mathbb{R} \times V \longrightarrow 0 \\
(gS, t, v) &\mapsto (gS, t, t\rho_1(g)\rho_2(g^{-1})v) \quad \text{if } t \leq 0 \\
(gS, t, v) &\mapsto (gS, t, tv) \quad \text{if } t \geq 0
\end{aligned}
$$

The S-action on the complex of vector bundles is given by $s \cdot (gS, t, v) = (sgS, t, \rho_1(s)v)$.

Note that $\delta(\rho_1 - \rho_2)$ admits equivariant lift $\delta_S(\rho_1 - \rho_2)$.

Proposition 4.2. *If G/S is formal and the K-theoretic Samelson subspace is spanned by elements of the form $\delta(\rho_1 - \rho_2)$ where $\rho_1 - \rho_2 \in \ker(i^* : R(G) \to R(S))$, then it is isotropy formal.*

4.2. **Isotropy formality of pairs arising from Lie group automorphisms.** In [GoNo], the authors showed the following main result.

Theorem 4.3. *([GoNo]) Let K be a Lie subgroup of G and both K and G be compact and connected. If there exists a Lie group automorphism on G such that the Lie algebra of the fixed point subgroup coincides with the Lie algebra $\mathfrak{k}$ of K, then (G, K) is an isotropy formal pair.*

Their proof consists of reductions to the special case where G is simple and $\mathfrak{k}$ is the Lie algebra of the fixed point subgroup of a Lie group automorphism σ of finite order (i.e. G/K is a (generalized) symmetric space), and showing this special case by verifying the cohomological dimension equality for a list of (generalized) symmetric spaces given by the classification theorem of such spaces. We would like to give a more uniform alternative proof of this special case.

Proposition 4.4. *If G is compact and connected, K a connected Lie subgroup, and there exists a Lie group automorphism σ on G of finite order such that the Lie algebra of the fixed point subgroup coincides with the Lie algebra $\mathfrak{k}$ of K, then (G, K) is an isotropy formal pair.*

Proof. We first consider the case where G is further assumed to have torsion-free fundamental group, so that $R(G)$ is a polynomial ring. Let G be of rank l and K of rank m, and S be a maximal torus of K. The finite-order automorphism σ of G is induced by a graph automorphism on its Dynkin diagram and the quotient graph is the Dynkin diagram of K. Moreover, the fundamental representations $\rho_1, \cdots, \rho_l$ of G are represented by the vertices of the Dynkin diagram of G, and the fundamental representations corresponding to the vertices in the same orbit of the group action of $\langle\sigma\rangle$ restrict to the same representation of K, and hence S. Let

$$
\bigcup_{k=1}^{m} \{\rho_{i_{1,k}}, \rho_{i_{2,k}}, \cdots, \rho_{i_{j_k,k}}\}
$$

be the partition of $\{\rho_1, \cdots, \rho_l\}$ corresponding to the orbit decomposition of the set of vertices of the Dynkin diagram of G. Then

$$
\bigcup_{\substack{1 < t \leq j_k \\ 1 \leq k \leq m}} \{\rho_{i_{1,k}} - \rho_{i_{t,k}}\} \subset \ker(R(G) \to R(S))
$$

and consists of $l-m$ elements. The span of the image of $\{\delta(\rho_{i_{1,k}}-\rho_{i_{t,k}})\}_{\substack{1<t\leq j_k\\1\leq k\leq m}}$ under the map $K^*(G/S)\to K^*(G)$ is an $(l-m)$-dimensional subspace of the K-theoretic Samelson subspace of $K^*(G/S)$, but by Theorem 3.2, the subspace is actually the Samelson subspace itself and $\mathcal{I}=0$. So G/S is formal by Theorem 3.5. Moreover, Proposition 4.2 implies that G/S is isotropy formal. Hence G/K is isotropy formal as well.

For the more general case where G is only assumed to be connected, we may use Theorem 1.4 to reduce to the case where G has torsion-free fundamental group. $\square$

4.3. **A sufficient condition for isotropy formality.** In this section we assume that G has torsion-free fundamental group unless otherwise specified. By the main result of [BZ] and its variant, the equivariant K-theory $K^*_{S_{\mathrm{Ad}}}(G)$ (the subscript 'Ad' means that S acts on G by the conjugation action) is isomorphic to

$$\bigwedge\nolimits^*_{R(S)}(\delta_S(\rho_1),\cdots,\delta_S(\rho_l))$$

where $\rho_1,\cdots,\rho_l$ are fundamental representations and $\delta_S:R(G)\to K^*_S(G)$ satisfies

$$\delta_S(\rho_1\otimes\rho_2)=i^*\rho_1\delta_S(\rho_2)+i^*\rho_2\delta_S(\rho_1)$$

In particular, $K^*_S(G)$ is a free $R(S)$-module. The image $\mathrm{Im}(p^*)$ of the map

$$p:K^*_S(G/S)\to K^*_{S_{\mathrm{Ad}}}(G)$$

therefore is a free $R(S)$-submodule. Since p^* maps the identity element of $K^*_S(G/S)$ to that of $K^*_{S_{\mathrm{Ad}}}(G)$, a module basis of $\mathrm{Im}(p^*)$ can be chosen to contain 1. The other basis elements are $R(S)$-linear combinations of 1 and products of $\delta_S(\rho_1),\cdots,\delta_S(\rho_l)$. By Gaussian elimination the module basis can be chosen to consist of 1 and $R(S)$-linear combinations of products of $\delta_S(\rho_1),\cdots,\delta_S(\rho_l)$. We come to the conclusion that $\mathrm{Im}(p^*)$ is an exterior algebra over $R(S)$, and now it makes sense to introduce the

Definition 4.5. The conjugation Samelson subspace $\widehat{P}_{\mathrm{conj}}$ of G/S is the free $R(S)$-module generated by the primitive elements of the image of the map

$$p^*:K^*_S(G/S)\to K^*_{S_{\mathrm{Ad}}}(G)$$

Unlike the dimension of the ordinary Samelson subspace the rank of the conjugation Samelson subspace is much more predictable. Consider the commutative diagram

$$\begin{tikzcd} K^*_S(G/S)_{(0)} \arrow[r] \arrow[d, "\cong"] & K^*_{S_{\mathrm{Ad}}}(G)_{(0)} \arrow[d, "\cong"] \\ K^*_S(Z_G(S)/S)_{(0)} \arrow[r] & K^*_S(Z_G(S))_{(0)} \end{tikzcd} \tag{4}$$

The two vertical maps are isomorphisms by virtue of Segal's localization theorem (cf. [S2]). Since S acts trivially on $Z_G(S)/S$ and $Z_G(S)$, the bottom map is equivalent to

$$\mathrm{Id}_{R(S)}\otimes p^*:R(S)_{(0)}\otimes K^*(Z_G(S)/S)\to R(S)_{(0)}\otimes K^*(Z_G(S))$$

It is known that $p^*:K^*(Z_G(S)/S)\to K^*(Z_G(S))$ is injective, and the dimension of the primitive subspace of $K^*(Z_G(S)/S)$ is rank $Z_G(S)/S=$ rank $G-$ rank S. We then have

Proposition 4.6. *The rank of the conjugation Samelson subspace of any compact homogeneous space G/S is rank G − rank S.*

In fact the conjugation Samelson subspace can be described more explicitly as follows.

Proposition 4.7. *The conjugation Samelson subspace $\widehat{P}_{conj}$ is the free $R(S)$-module generated by $\{\delta_S(\rho) \in K^*_{S_{Ad}}(G) | \rho \in ker(i^* : R(G) \to R(S))\}$.*

Proof. That $\{\delta_S(\rho) \in K^*_{S_{\mathrm{Ad}}}(G) | \rho \in \ker(i^* : R(G) \to R(S))\} \subseteq \widehat{P}_{\mathrm{conj}}$ follows from the construction in Definition 4.1. Now observe that the composition of maps

$$\widetilde{K}^*_S(G/S) \xrightarrow{p^*} \widetilde{K}^*_{S_{\mathrm{Ad}}}(G) \xrightarrow{q^*} \widetilde{K}^*_{S_{\mathrm{Ad}}}(S)$$

is 0. So $\widehat{P}_{\mathrm{conj}}$ is contained in the intersection of $\ker(q^*)$ and the $R(S)$-module generated by the primitive elements of $K^*_{S_{\mathrm{Ad}}}(G)$, the latter being exactly $\{\delta_S(\rho) \in K^*_{S_{\mathrm{Ad}}}(G) | \rho \in \ker(i^* : R(G) \to R(S))\}$. □

Remark 4.8. There is an alternative way of showing Proposition 4.6 using Proposition 4.7. Let the kernel of the map $i^* : R(G) \to R(S)$ be the ideal $(k_1, \cdots, k_p)$, where $p \geq l - m$ and m is the dimension of S. Then $i^*R(G) \cong \mathbb{C}[\rho_1, \cdots, \rho_l]/(k_1, \cdots, k_p)$ and $\bigoplus_{i=1}^p R(S) \cdot \delta_S(k_i)$ is the conjugation Samelson subspace by Proposition 4.7. Note that

$$\begin{aligned}
m &= \dim R(S) \\
&= \dim i^*R(G) \quad (R(S) \text{ is a finitely generated module over } i^*R(G)) \\
&= \mathrm{rank}_{i^*R(G)} \Omega_{i^*R(G)/\mathbb{C}} \\
&= \mathrm{rank}_{R(S)} \Omega_{i^*R(G)/\mathbb{C}} \otimes_{i^*R(G)} R(S) \\
&= \mathrm{rank}_{R(S)} \frac{\bigoplus_{i=1}^l R(S) \cdot \delta_S(\rho_i)}{\bigoplus_{i=1}^p R(S) \cdot \delta_S(k_i)} \\
&= l - \mathrm{rank}_{R(S)} \bigoplus_{i=1}^p R(S) \cdot \delta_S(k_i)
\end{aligned}$$

So $\mathrm{rank}_{R(S)} \bigoplus_{i=1}^p R(S) \cdot \delta_S(k_i) = l - m$.

Theorem 4.9. *Let G be a compact and connected Lie group and S a torus subgroup. The image of the map $i^* : R(G) \to R(S)$ is regular at the augmentation ideal I if and only if (G, S) is both an isotropy formal pair and a formal pair.*

Proof. We first deal with the case where G has torsion-free fundamental group, so that $R(G)$ is a polynomial ring and the main result of [BZ] can be applied. We shall show that regularity of the image of i^* at the augmentation ideal I is equivalent to the condition that the dimension of the image of $\widehat{P}_{\mathrm{conj}}$ under the forgetful map be $l - m$. We have

$$\Omega_{i^*R(G)/\mathbb{C}} \cong \frac{\bigoplus_{i=1}^l i^*R(G) \cdot \delta_S(\rho_i)}{\bigoplus_{i=1}^p i^*R(G) \cdot \delta_S(k_i)}$$

On the one hand, since $i^*R(G)$ is an integral finitely generated algebra over $\mathbb{C}$, we have $\dim\, i^*R(G)_I = \dim\, i^*R(G)$. The latter equals $\dim\, \Omega_{i^*R(G)/\mathbb{C}} \otimes_{i^*R(G)} i^*R(G)_{(0)}$. On the other hand, that $i^*R(G)$ is regular at I is tantamount to

$$\dim\, i^*R(G)_I = \dim_{\mathbb{C}}\Omega_{i^*R(G)/\mathbb{C}} \otimes_{i^*R(G)} \mathbb{C}$$

It follows that

$$\begin{aligned} &\dim_{i^*R(G)_{(0)}}\Omega_{i^*R(G)/\mathbb{C}} \otimes_{i^*R(G)} i^*R(G)_{(0)} = \dim_{\mathbb{C}}\Omega_{i^*R(G)/\mathbb{C}} \otimes_{i^*R(G)} \mathbb{C} \\ \iff &\dim_{i^*R(G)_{(0)}} \frac{\bigoplus_{i=1}^{l} i^*R(G)_{(0)} \cdot \delta_S(\rho_i)}{\bigoplus_{i=1}^{p} i^*R(G)_{(0)} \cdot \delta_S(k_i)} = \dim_{\mathbb{C}} \frac{\bigoplus_{i=1}^{l} \mathbb{C} \cdot \delta(\rho_i)}{\bigoplus_{i=1}^{p} \mathbb{C} \cdot \delta(k_i)} \\ \iff &\dim_{i^*R(G)_{(0)}}\mathrm{span}\{\delta_S(k_1), \cdots, \delta_S(k_p)\} = \dim_{\mathbb{C}}\mathrm{span}\{\delta(k_1), \cdots, \delta(k_p)\} \end{aligned}$$

The last equality is equivalent to saying that the dimension of the image of the conjugation Samelson subspace under the forgetful map, which is the RHS, is exactly $l - m$, which is the LHS by Propositions 4.7 and 4.6. Our first claim hence is established.

Now note that the image of the conjugation Samelson subspace under the forgetful map is contained in the ordinary Samelson subspace. If the dimension of the image is exactly $l-m$, then the image must be the whole of the ordinary Samelson subspace, whose dimension a priori does not exceed $l - m$ by Theorem 3.2. Since the dimension of the ordinary Samelson subspace is $l - m$, the pair (G, S) is formal. By Proposition 4.2, the pair is isotropy formal as well.

Conversely, suppose (G, S) is both isotropy formal and formal. Isotropy formality and Theorem 1.5 implies that the ordinary Samelson subspace admits equivariant lifts in the conjugation Samelson subspace. Formality implies that the dimension of the image of the conjugation Samelson subspace under the forgetful map is $l - m$, and hence $i^*R(G)$ is regular at I. This finishes the proof of the Theorem in the case where G has torsion-free fundamental group. For the more general case where G is only assumed to be connected, we reduce to the previous case by using Theorem 1.4, the fact that (G, S) is a formal pair if and only if $(\widetilde{G}, \widetilde{S})$ is, and that the image of $i^* : R(G) \to R(S)$ is regular at the augmentation ideal if and only the image of $\widetilde{i} : R(\widetilde{G}) \to R(\widetilde{S})$ is. $\square$

Remark 4.10. (1) There is another way of interpreting the condition that $i^*R(G)$ is regular at I. If we write $k_1, \cdots, k_p$ as polynomials in terms of the 'reduced representations' $\widetilde{\rho_i} := \rho_i - \dim\, \rho_i$, $1 \le i \le l$, then $i^*R(G)$ is regular at I if and only if each polynomial has a nonzero linear term, i.e. each k_i is not in I^2, or they are indecomposables.

(2) Theorem 4.9, together with Theorem 1.3 and the assertion that if S is a maximal torus of K, (G, S) is formal if and only if (G, K) is formal (cf. [On, Rmk., p. 212]), allows us to give a shorter and uniform proof that the classes of homogeneous spaces in Example 3.4 are both isotropy formal and formal in one fell swoop, at least if we assume that G has a torsion-free fundamental group. For (generalized) symmetric spaces, $i^*R(G)$ is simply a polynomial ring by the analysis in the proof of Proposition 4.4 and so regular at I in particular. For equal rank pairs, $i^* : R(G) \to R(K)$ is injective, and the image $i^*R(G) \subseteq R(K)$ further injects into $R(S)$. So the image is isomorphic to $R(G)$, again a polynomial ring. For cohomological surjective pairs, both of their cohomology and K-theory are exterior algebras (cf. [GHV] and use

Chern character isomorphism). Proposition 3.7 then implies that $R(K)/\!\!/R(G) \cong \mathbb{C}$ and that $i^*R(G) = R(K)$, which further injects through restriction onto $R(S)^{W_K}$, which in turn is regular at I.

4.4. **Some worked examples.** In the following examples, the torus subgroup S is 1-dimensional. So $R(S)$ is a principal ideal domain and $R(S)/\!\!/R(G)$ is a complete intersection ring. By Theorem 3.5 (3) and Lemma 3.6, G/S is formal.

4.4.1. $(G,S) = (SU(4),(t,t^{-1},t^2,t^{-2}))$. Let $G = SU(4)$, $S = \begin{pmatrix} t & & & \\ & t^{-1} & & \\ & & t^2 & \\ & & & t^{-2} \end{pmatrix}$. By [Ca, Algorithm 1.4], since S is reflected, the pair (G,S) is isotropy formal. Another way to show isotropy formality is to verify Shiga-Takahashi condition that (G,S) be a formal pair and $H^*_S(\mathrm{pt}) \cong H^*_G(\mathrm{pt})^N$, where $N = N_G(S)/S$. Note that the restriction map $i^* : H^*_G \to H^*_S \cong \mathbb{C}[s]$ sends the universal Chern classes to the following:

$$
\begin{aligned}
&c_1 \text{ to } 0,\\
&c_2 \text{ to } s(-s)+s(2s)+s(-2s)+(-s)(2s)+(-s)(-2s)+2s(-2s) = -5s^2\\
&c_3 \text{ to } (-s)(2s)(-2s)+s(2s)(-2s)+s(-s)(-2s)+s(-s)(2s) = 0\\
&c_4 \text{ to } 4s^4
\end{aligned}
$$

So indeed $i^*H^*_G \cong \mathbb{C}[-5s^2,4s^4] \cong \mathbb{C}[s^2] \cong (H^*_S)^N$. We note moreover that $H^*_S/\!\!/H^*_G \cong \mathbb{C}[s]/(s^2)$.

Finally, we use Theorem 4.9 to show isotropy formality of (G,S). Let $x := \sigma_4 - 4$, $y := \bigwedge^2 \sigma_4 - 6$ and $z := \bigwedge^3 \sigma_4 - 4$, which are 'reduced' fundamental representations generating the augmentation ideal I. Then $R(G) \cong \mathbb{C}[x,y,z]$, and the map

$$i^* : R(G) \to R(S) \cong \mathbb{C}[t,t^{-1}]$$

sends the x, y and z to the following:

$$
\begin{aligned}
&x \text{ to } t+t^{-1}+t^2+t^{-2}-4 = a+a^2-6\\
&y \text{ to } t^3+t^{-3}+t+t^{-1}-4 = a^3-2a-4\\
&z \text{ to } t+t^{-1}+t^2+t^{-2}-4 = a+a^2-6
\end{aligned}
$$

where $a = t+t^{-1}$. So

$$i^*R(G) \cong \mathbb{C}[a^3-2a-4, a^2+a-6]$$

It should be noted that $i^*R(G) \ncong R(S)^N$, and so Shiga-Takahashi condition for isotropy formality does not carry over to the K-theory context.

$$\begin{aligned} R(S)/\!\!/R(G) &\cong \frac{\mathbb{C}[t,t^{-1}]}{(a^3-2a-4, a^2+a-6)} \\ &= \frac{\mathbb{C}[t,t^{-1}]}{((a-2)(a^2+2a+2),(a-2)(a+3))} \\ &= \frac{\mathbb{C}[t,t^{-1}]}{(a-2)} \\ &= \frac{\mathbb{C}[t,t^{-1}]}{(t+t^{-1}-2)} \\ &\cong \frac{\mathbb{C}[u]}{((u-1)^2)} \end{aligned}$$

which is isomorphic to $H^*_S/\!\!/H^*_G$, verifying Lemma 3.6. The kernel of i^* is

$$(x-z, -x^3-14x^2+3xy-50x+y^2+25y)$$

Therefore

$$i^*R(G) \cong \frac{\mathbb{C}[x,y,z]}{(x-z, -x^3-14x^2+3xy-50x+y^2+25y)}$$

Though $i^*R(G)$ is not a free polynomial ring, it is regular at I because the kernel of i^* is generated by indecomposables. So (G,S) is isotropy formal.

In fact

$$K^*(G/S) \cong \frac{\mathbb{C}[u]}{((u-1)^2)} \otimes \bigwedge{}^{*}(\delta(x-z), \delta(-50x+25y))$$

and both $\delta(x-z)$ and $\delta(-50x+25y)$ admit equivariant lift in $K_S(G/S)$. For instance, the equivariant lift of $\delta(-50x+25y)$ is $\delta_S(-x^3-14x^2+3xy-50x+y^2+25y)$.

4.4.2. $(G,S) = (SU(3), (t,t,t^{-2}))$. Using [Ca, Algorithm 1.4] again, we have (G,S) is not isotropy formal because S in this case is not reflected. Alternatively, to show that (G,S) is not isotropy formal, we use Theorem 4.9 and show that $i^*R(G) \subseteq R(S)$ is not regular at I. Let $x = \sigma_3 - 3$ and $y = \bigwedge^2 \sigma_3 - 3$. Then $R(G) = \mathbb{C}[x,y]$ and $\ker(i^*) = (4x^3+4y^3-x^2y^2-6x^2y-6xy^2+27x^2+27y^2-54xy)$, which intersects I^2 nontrivially. Hence $i^*R(G)$ is not regular at I.

References

[AH] M. F. Atiyah, F. Hirzebruch, *Vector bundles and homogeneous spaces*, AMS Symposium in Pure Math. III (1960).

[AS] M. F. Atiyah, G. Segal, *Equivariant K-theory and completion*, J. Differential Geometry 3, 1-18, 1969.

[BZ] J.-L. Brylinski, B. Zhang, *Equivariant K-theory of compact connected Lie groups*, K-Theory, Volume 20, Issue 1, pp. 23-36, May 2000.

[Ca] J. D. Carlson, *Equivariant formality of isotropic torus action I*, preprint, version December 21, 2014, available at `http://arxiv.org/abs/1410.5740`

[Ca2] J. D. Carlson, *On the equivariant cohomology of homogeneous spaces*, PhD thesis, Tufts University, May 2015, p. 346, available at `https://www.dropbox.com/s/04q5z83ktcnvwt2/dissertation.pdf?dl=0`

[F] C.-K. Fok, *The Real K-theory of compact Lie groups*, Symmetry, Integrability and Geometry: Methods and Applications, Vol. 10, 022, 26 pages, 2014.

[F2] C.-K. Fok, *Equivariant cohomology theory of spaces with moment maps*, in preparation.

[Go] O. Goertsches, *The equivariant cohomology of isotropy actions on symmetric spaces*, Documenta Math. 17, 79-94, 2012.

[GoNo] O. Goertsches, S. H. Noshari, *Equivariant formality of isotropy actions on homogeneous spaces defined by Lie group automorphisms*, Preprint, version April 7, 2015, available at `http://arxiv.org/abs/1405.2655`

[GHV] W. Greub, S. Halperin, R, Vanstone, *Connections, curvature, and cohomology*, Vol. 3: Cohomology of principal bundles and homogeneous spaces, Academic Press, 1976.

[GKM] M. Goresky, R. Kottwitz, R. Macpherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. 131, no. 1, 25-83, 1998.

[HL] M. Harada, G. Landweber, *Surjectivity for Hamiltonian G-spaces in K-theory*, Trans. Amer. Math. Soc., Vol. 359, No. 12, pp. 6001-6025, Dec. 2007.

[Ma] H. Matsumura, *Commutative Algebra*, Benjamin/Cummings, 1980.

[On] A. L. Onishchik, *Topology of transitive transformation groups*, Johann Ambrosia Barth, 1994.

[KT] D. Kotschick, S. Terzić, *On formality of generalized symmetric spaces*, Math. Proc. Camb. Phil. Soc., Vol. 134, 491-505, 2003.

[S2] G. Segal, *Equivariant K-theory*, Inst. Hautes Études Sci. Publ. Math., No. 34, 129-151, 1968.

[Sh] H. Shiga, *Equivariant de Rham cohomology of homogeneous spaces*, J. Pure Appl. Algebra, 106, no. 2, 173-183, 1996

[ShTa] H. Shiga, H. Takahashi, *Remarks on equivariant cohomology of homogeneous spaces*, Technical report 17, Technological University of Nagaoka, May 1995.

[St] Z. Stepien, *On formality of a class of compact homogeneous spaces*, Geometria Dedicata, Vol. 93, 37-45, 2002.

[T] S. Terzić, *Cohomology with real coefficients of generalized symmetric spaces*, Fundam. Prikl. Mat., Vol. 7, no.1, 131-157, 2001

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