RESEARCH STATEMENT

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My research interests lie in the intersection of algebraic topology, Lie groups and symplectic geometry. Roughly speaking, in my research I apply machinery from algebraic topology to understand spaces with symmetries and physical implications. To be more precise, my dissertation consists of two major themes, namely,

(1) the Real $K$-theory (or $KR$-theory) of compact Lie groups, and
(2) topological classification of general integrable systems.

In the future I also plan to investigate

(1) geometric $KR$-homology
(2) Real version of Freed-Hopkins-Teleman Theorem and geometric quantization of Real quasi-Hamiltonian manifolds,
(3) $K$-theory of the space of $n$-tuples of commuting elements in a compact Lie group, and
(4) equivariant $K$-theory of compact Lie groups with finite fundamental groups.

1. $KR$-theory

$KR$-theory, which was first introduced by Atiyah in the beautiful paper [At1], is a version of topological $K$-theory for the Real spaces, i.e. topological spaces equipped with a continuous involution. To be more precise, $KR$-theory of a Real space $X$ is the Grothendieck group of the category of Real vector bundles on it, i.e. complex vector bundles that are themselves Real spaces, whose involution descends to the involution on $X$ and maps fiber to fiber anti-linearly. $KR$-theory was motivated by the study of the index theory of real elliptic operators and used by Atiyah to derive 8-periodicity of $KO$-theory from the 2-periodicity of complex $K$-theory (See [At1] and [At2]). What makes $KR$-theory interesting is that it can be viewed naturally as a unifying thread of $KO$-theory, complex $K$-theory and $KSC$-theory (see [At1], Sect. 3), which are $K$-groups of categories of real, complex and quaternionic vector bundles, respectively. For instance, if the involution is trivial, then $KR$-theory is equivalent to $KO$-theory. One may go one step further and consider equivariant $KR$-theory, which is simply the Grothendieck group of the category of Real equivariant $G$-vector bundles on a Real $G$-space, where we assume a compatibility condition of the $G$-action, the involutions on the vector bundles and the base space, and the involutive automorphism on $G$ (for precise definitions and basic properties, see [AS]).

In [Se], Seymour provided a structure theorem of $KR$-theory for a certain type of spaces, which enables us to compute the $KR$-theory using the knowledge of complex $K$-theory and how the action of the pullback induced by the base space involution followed by complex conjugation on complex vector bundles act on it. I observe that the conditions of Seymour’s result are an appropriate
candidate for defining an analogue of ‘weakly equivariant formality’ à la Harada and Landweber (see Definition 4.1 of [HL]), which roughly means the condition that every vector bundle has a stable equivariant lift. Inspired by Seymour’s result and the notion of weakly equivariant formality, I introduced the notion of Real equivariant formality for equivariant KR-theory (see [Fo1], Definition 4.2). I proved the following structure theorem of equivariant KR-theory of Real equivariantly formal spaces.

**Theorem 1.1** ([Fo1], Theorem 4.5). Let $X$ be a Real equivariantly formal $G$-space. For any element $a \in K^s(X)$ (resp. $a \in KR^s(X)$), let $a_G \in K_G^s(X)$ (resp. $a_G \in KR_G^s(X)$) be a fixed choice of (Real) equivariant lift of $a$. Then the map

$$f : (RR(G, \mathbb{R}) \oplus RH(G, \mathbb{R})) \otimes KR^s(X) \oplus r(R(G, \mathbb{C}) \otimes K^s(X)) \to KR_G^s(X)$$

$$\rho_1 \otimes a_1 \oplus r(\rho_2 \otimes a_2) \mapsto \rho_1 \cdot (a_1)_G \oplus r(\rho_2 \cdot (a_2)_G)$$

is a group isomorphism. Here $RR(G, \mathbb{R}) \cong KR_G^0(pt)$, $RH(G, \mathbb{R}) \cong KR_G^4(pt)$ are the ring of Real representations of real type and the group of Real representations of quaternionic type respectively, and $r : K_G^s(X) \to KR_G^s(X)$ is the realification map. In particular, if $R(G, \mathbb{C}) = 0$, then $f$ is an $RR(G, \mathbb{R}) \oplus RH(G, \mathbb{R})$-module isomorphism.

**1.1 Equivariant KR-theory of compact Lie groups with involution automorphisms.** In the 60s, Hodgkin showed that the complex $K$-theory ring of any compact connected Lie group with torsion-free fundamental group is a $\mathbb{Z}_2$-graded exterior algebra over $\mathbb{Z}$ on the module of primitive elements, which are of degree $-1$ and associated with the representations of the Lie group (see [Ho]). Since Hodgkin’s work, there have appeared two generalizations of $K$-theory of compact Lie groups. The first such is Seymour’s work on $KR$-theory of compact, connected and simply connected Lie groups equipped with involution automorphisms (see [Se]). He obtained the $KR^s(pt)$-module structure of $KR^s(G)$ using his structure theorem. However, he was unable to give a complete description of the ring structure, and could only make some conjectures about it. The second generalization is the equivariant $K$-theory of compact Lie groups. In [BZ], Brylinski and Zhang showed that, for a compact Lie group $G$ with torsion-free fundamental group and the $G$-action being the conjugation action on itself, its equivariant $K$-theory is isomorphic to the ring of Grothendieck differentials of the complex representation ring over $\mathbb{Z}$.

In [Fo1], based on the previous results of Seymour’s and Brylinski-Zhang’s, Theorem 1.1 and a description of the coefficient ring $KR^s_G(pt)$, I gave a preliminary description of $KR^s_{(G, \sigma_G)}(G, \sigma_G)$ (where $\sigma_G$ is an involution automorphism) by listing the algebra generators associated to the Real representations of $G$ of real, complex and quaternionic types (with respect to the involution automorphism). Then I gave a full description of the ring structure of $KR^s_G(G)$ by listing all the relations among the generators. To achieve this I investigated the map of equivariant $KR$-theory induced by the Weyl covering map. Of particular interest are the squares of the real and quaternionic type generators, $\delta^R_G(\varphi) \in KR^{s-1}_G(G)$ and $\delta^Q_G(\theta) \in KR^{s+5}_G(G)$, where $\varphi$ and $\theta$ are Real representations of real and quaternionic types, respectively, and $\delta^R_G : RR(G) \to KR^{1-1}_G(G)$ and $\delta^Q_G : RH(G) \to KR^{5-1}_G(G)$ are natural maps defined in Definition 4.8 of [Fo1].

**Theorem 1.2** ([Fo1], Theorem 4.30). $\delta^R_G(\varphi)^2 = \eta(\varphi \cdot \delta^R_G(\varphi) - \delta^R_G(\varphi^2))$ and $\delta^Q_G(\theta)^2 = \eta(\theta \cdot \delta^Q_G(\theta) - \delta^Q_G(\theta^2))$, where $\eta \in KR^{-1}(pt)$ represents the reduced Hopf bundle over $\mathbb{R}P^1$.

By applying the forgetful map $KR^s_G(G) \to KR^s(G)$ to the generators and relations, I solved the problem of describing the ring structure of $KR^s(G)$, which was left open in [Se]. Theorem 1.2
shows that, unlike the (equivariant) complex $K$-theory, the (equivariant) $KR$-theory of compact Lie groups are in general not an exterior algebra. The equations in Theorem 1.2 and their analogues in the ordinary $KR$-theory case actually give extra information about the $KR$-theory, and can be used to distinguish two different $G$-actions on itself. In [Fo1], I examined the two cases where $G$ acts on itself trivially and by conjugation, respectively. While the equivariant complex $K$-theory rings of both cases are isomorphic, the equivariant $KR$-theory rings are not, thanks to the extra information given by the squares of real and quaternionic type generators. Nonetheless, we shall remark that, in the special case of no Real representations of complex type, if we invert 2 in the equivariant $KR$-theory ring of $G$, then the result is an exterior algebra over the localized coefficient ring of equivariant $KR$-theory.

1.2. Equivariant $KR$-theory of compact Lie groups with anti-involutions. Another kind of topological involution on a Lie group is anti-involution, which is the composition of an involutive automorphism and group inversion. This begs the question of what is the equivariant $KR$-theory of a compact Lie group $G$ equipped with instead an anti-involution. It turns out that the result is a little bit different from the involutive automorphism case as discussed in [Fo1] and previous section.

**Theorem 1.3** ([Fo4]). Let $G$ be a compact, connected, and simply-connected Lie group and $\sigma_G$ is an involutive automorphism on it. There exist maps

$$\delta^{\text{inv}}_R : RR(G) \to KR^1_{(G,\sigma_G)}(G, \sigma_G \circ \text{inv})$$

$$\delta^{\text{inv}}_H : RH(G) \to KR^3_{(G,\sigma_G)}(G, \sigma_G \circ \text{inv})$$

which are similar to $\delta^G_R$ and $\delta^G_H$ as defined in Definition 4.8 in [Fo1]. The square of any image of $\delta^{\text{inv}}_R$ and $\delta^{\text{inv}}_H$ is 0. In particular, if $R(G, \mathbb{C}) = 0$, then

$$KR^*(G, \sigma_G \circ \text{inv}) \cong \Omega_{KR^*(G, \sigma_G)}(pt)/KR^*(pt),$$

where the right hand side is the ring of Grothendieck differentials of the coefficient ring of the equivariant $KR$-theory over the coefficient ring of ordinary $KR$-theory, whose primitive module is generated by the image of the fundamental representations of $G$ under $\delta^{\text{inv}}_R$ and $\delta^{\text{inv}}_H$.

In some sense, equipping $G$ with an anti-involution instead of an involutive automorphism (which we do in [Fo1]) is a better direction of generalizing Brylinski-Zhang’s result. In fact, there is another piece of evidence which indicates that anti-involution is the right topological involution to consider when studying a certain $KR$-theory of compact Lie groups. For more details, see Section 3.2 under ‘Future directions’.

2. Topological classification of almost symplectic integrable systems

A symplectic manifold $M$ is a smooth manifold equipped with a nondegenerate and closed 2-form $\omega$, called the symplectic form. The study of symplectic manifolds was motivated by classical mechanics, where the phase spaces of classical mechanical systems are themselves symplectic.

Let us consider a special class of symplectic manifolds, Lagrangian (resp. isotropic) fiber bundles (abbreviated LFB and IFB respectively), which correspond to integrable systems in classical mechanics. They are fiber bundles $\pi : M^{2d} \to B^k$ with compact and connected fibers of dimension $d$ (resp. less than $d$) and a symplectic form $\omega$ which vanishes on restriction to any fiber. The base $B$ necessarily has a regular Poisson structure induced by the symplectic structure of $M$ through $\pi$,
and there is a natural action of the conormal bundle $\nu^* F$ of its symplectic foliation $F$ on $M$. The stabilizer bundle $P$, called the \textit{period bundle}, is a $\mathbb{Z}^n$-subbundle of $\nu^* F$ whose sections represent closed 1-forms of $B$ which vanish on restriction to any symplectic leaf. Therefore, $T := \nu^* F / P$, a torus fiber bundle, acts on $M$ freely, making $M$ a $T$-torsor. Conversely, if there exists a period bundle $P$ on $B$, then $B$ can support LFB (resp. IFB) where the stabilizer bundle of the $\nu^* F$-action is $P$. The LFB and IFB are locally symplectomorphic to $T$, and admit local \textit{action} and \textit{angle coordinates} by Liouville-Arnold Theorem. In other words, there is no local invariant for both LFB and IFB. It is therefore interesting to look for topological invariants which measure the obstruction of the existence of global action and angle coordinates, and a topological classification of LFB and IFB over a fixed base manifold $B$ with a period bundle $P$. The first problem was settled in the early 80s by Duistermaat, who showed that global action and angle coordinates exist if and only if the \textit{monodromy} and the \textit{Lagrangian class} of LFB vanish (see [Du]). Dazord and Delzant later on in [DD] showed that the \textit{isotropic class} completely classifies all IFB over a fixed Poisson manifold $B$ with a given period bundle $P$. They also gave a sufficient and necessary condition for a $T$-torsor to possess a compatible symplectic form so as to be an IFB.

In recent years there has been a growing interest in the study of nonholonomic systems, and as a preliminary step of investigation in this context spaces with certain generalities are considered. For instance, in [FaSa] Fassò and Sansonetto studied more general integrable systems which are IFB except that they are \textit{almost symplectic}, i.e. equipped with a nondegenerate 2-form which is not necessarily closed. They obtained a generalization of Liouville-Arnold Theorem under the condition of the existence of \textit{strongly Hamiltonian vector fields}. Moreover, \textit{twisted Poisson manifolds}, first introduced in [SW] and motivated by string theory, provides the framework for the study of such nonholonomic systems as the Veselova systems and the Chaplygin sphere (see [BG-N]).

In joint work ([FS]) in preparation with Reyer Sjamaar, we generalize Duistermaat/Dazord-Delzant’s result in the almost symplectic context as in [FaSa]. In particular, we consider \textit{almost symplectically complete isotropic realizations} (ASCIR), which are almost symplectic IFB with linearly independent, locally strongly Hamiltonian vector fields tangent to fibers. We found that the base manifold of an ASCIR is necessarily a twisted regular Poisson manifold with a twisting 3-form $\eta$ satisfying $d\omega = \pi^* \eta$. Let $\text{ASCIR}(B, P, \Pi, \Omega)$ be the category of ASCIR over the twisted Poisson manifold $(B, \Pi)$ with period bundle $P$ and a \textit{characteristic 2-form} $\Theta$, i.e. a 2-form which on restriction to any almost symplectic leaf becomes its almost symplectic form. Furthermore, we let $\mathcal{P}$ be the sheaf of local sections of $P$, $\mathcal{O}_F$ be the sheaf of \textit{tropical functions} defined by $\mathcal{O}_F(U) = \{ f \in C^\infty(U) | df \in \mathcal{P}(U) \}$. Then we have the long exact sequence of sheaf cohomology (here $H^*(B, F)$ means relative cohomology of $B$ with respect to the foliation $F$).

\[ \cdots \rightarrow H^1(B, \mathcal{P}) \xrightarrow{d_{\mathcal{P}}} H^2(B, \mathcal{O}_F) \rightarrow H^2(B, \mathcal{P}) \xrightarrow{\partial^2} H^3(B, F) \xrightarrow{i_3} H^3(B, \mathcal{O}_F) \rightarrow \cdots \]

\textbf{Theorem 2.1} ([FS]). (1) The set of \textit{isomorphism classes} of $\text{ASCIR}(B, P, \Pi, \Theta)$ can be equipped with an \textit{abelian group structure}. We denote this group by $\text{Pic}(B, P, \Pi, \Theta)$. The \textit{identity element} is represented by $T := \nu^* F / P$ with almost symplectic form $\omega_{\text{can}} + \pi^* \Theta$, where $\omega_{\text{can}}$ is the canonical 2-form on $T$. The inverse of $[(M, \omega)]$ is $[(M, -\omega + 2\pi^* \Theta)]$. The product of two ASCIRs can be defined by means of a symplectic reduction (see [Sj] for a related construction).

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\[1\] The notion of Lagrangian class was implicit in [Du] and introduced later by Dazord and Delzant in [DD].
(2) Let \( \text{ASCIR}_0(B, P, \Pi, \Theta) \) be the subcategory of \( \text{ASCIR}(B, P, \Pi, \Theta) \) whose objects are \( \text{ASCIR} \) with twisting 3-form \( d\Omega \). The map
\[
\pi_0(\text{ASCIR}_0(B, P, \Pi, \Theta)) \to H^2(B, O_F)
\]
is an isomorphism. Moreover we have the short exact sequence
\[
0 \to H^2(B, F)/dP \to H^1(B, P) \to \pi_0(\text{ASCIR}_0(B, P, \Pi, \Theta)) \to \ker(\partial^2) \to 0
\]
where the second map sends \([\gamma] \) to \([T, \omega_{\text{can}} + \pi^*(\Omega + \gamma)]\) for \( \gamma \), a closed 2-form which vanishes on restriction to any leaf of \( F \), and the third map gives the Chern class of \( \text{ASCIRs} \) as a \( T \)-torsor.

(3) Any \( T \)-torsor \( M \) can be equipped with a compatible almost symplectic form so as to be in \( \text{ASCIR}(B, P, \Pi, \Theta) \). Moreover,
\[
\partial^2 c(M) = [d\Theta - \eta] \in H^3(B, F)
\]
if and only if \( M \) is an \( \text{ASCIR} \) with an almost symplectic form such that \( d\omega = \pi^*\eta \). Here \( c(M) \in H^2(B, P) \) is the Chern class of the \( T \)-torsor \( M \).

(4) We have the exact sequence
\[
0 \to H^2(B, O_F) \to \text{Pic}(B, P, \Pi, \Theta) \to \mathbb{Z}^{3}_F(B) \to 0
\]
where the last map sends an \( \text{ASCIR} \) \( M \) with twisting 3-form \( \eta \) to \( d\Theta - \eta \). The image of the last map consists of those relative closed 3-forms whose cohomology classes in \( H^3(B, F) \) are in \( \ker(i_3) \).

(5) We have the short exact sequence
\[
0 \to \Omega^2_F/d\Omega^1_F \to \text{Pic}(B, P, \Pi, \Theta) \to H^2(B, P) \to 0
\]
where \( \Omega^k_F \) is the space of \( k \)-forms which vanish on restriction to any leaf of \( F \), and the second map sends \([\gamma] \) to \([T, \omega_{\text{can}} + \pi^*(\Omega + \gamma)]\).

As \( H^2(B, P) \) is a discrete group, the image of \( \Omega^2_F/d\Omega^1_F \) under the second map in the last exact sequence is the identity component subgroup of the Picard group. This subgroup is to the one consisting of isomorphism classes of degree 0 invertible sheaves on a scheme in algebraic geometry what \( H^2(B, P) \) is to the Néron-Severi group.

We shall remark that item (2) of Theorem 2.1 was also obtained by Sansonetto and Sepe in a very recent paper [SS], where they considered the classification of twisted isotropic realizations which we showed are actually equivalent to \( \text{ASCIR} \).

We hope to find some interesting real-life examples of \( \text{ASCIR} \) where we can apply our results to understand their topological complexity. We suspect that certain chemical systems fit into the category of \( \text{ASCIR} \).

3. Future directions

In the future I intend to follow up my work in [Fo1] in two major directions. One such is to formulate a suitable version of equivariant twisted \( KR \)-homology properly, study the equivariant twisted \( KR \)-homology of compact Lie groups and obtain an analogue of Freed-Hopkins-Teleman Theorem, which can help understand the geometric quantization of Real quasi-Hamiltonian manifolds \( \text{à la} \) Meinrenken (see [M]). Another is to extend Brylinski-Zhang’s result, work out the
equivariant $K$-theory of general compact Lie groups and even the space of commuting $n$-tuples in a compact Lie group.

3.1. **Geometric $KR$-homology.** $K$-homology is a homology theory dual to $K$-theory through the $K$-theory version of Poincaré duality, where a manifold is oriented in $K$-theory if it has a spin$^c$ structure. Inspired by the Atiyah-Singer index theorem, Kasparov gave the first definition of $K$-homology using Hilbert modules (see [Kas]). One drawback of the definition is that it is very elaborate and analytic in flavor. In some geometric context, it was desired that an alternative definition of $K$-homology be used. In [BD], Baum and Douglas introduced geometric $K$-homology, constructed out of geometric $K$-cycles. It had long been a folklore result that there is a natural isomorphism from geometric $K$-homology to Kasparov’s $K$-homology until recently when Baum, Higson and Schick proved that it is indeed the case (see [BHS]). Baum et al. also proved that such an isomorphism exists in the equivariant context (see [BO-OSW]). This geometric incarnation of $K$-homology has sparked interest in search for new proofs of the ‘quantization commutes with reduction’ (see [So]).

I had worked on a generalization of Baum-Higson-Schick’s result in the context of $KR$-homology. $KR$-homology (in the sense of Kasparov and Baum-Douglas) can be easily defined by adding Real structure throughout. For instance, rather than using spin$^c$ manifolds as part of the data in a geometric $K$-cycle, we use Real spin$^c$ manifolds. The proof in [BHS] involves exploiting the relation between $K$-homology and framed bordism. I plan to first work on the proof of the $KR$-homology generalization by following the framed bordism approach and studying the relationship between $KR$-theory and Real cobordism (see, for example, [Fu]). The next step is to generalize to the twisted equivariant setting following the ideas in [BO-OSW] and the notion of twisted geometric cycles in [W]. I successfully formulated models for Real version of twists, namely Real bundle gerbes and Real Dixmier-Douady bundles, and showed that the third $\mathbb{Z}_2$-equivariant cohomology with a certain local system classifies the Real twists (cf. [Kah] for the construction of the local system). A successful formulation of twisted equivariant geometric $KR$-homology will prove to be a useful machinery in studying geometric quantization of Real quasi-Hamiltonian manifolds, to be explained below.

3.2. **Real version of Freed-Hopkins-Teleman and geometric quantization of Real quasi-Hamiltonian manifolds.** Freed-Hopkins-Teleman Theorem (FHT) asserts that the equivariant twisted $K$-homology of a compact connected Lie group $G$ with torsion-free fundamental group (with ring structure being Pontryagin product) is isomorphic to Verlinde algebra of $G$, which is the ring of positive energy representations of the loop group $LG$, with ring structure being the fusion product (see [FHT]). Motivated by this result, Meinrenken realized the quantization of quasi-Hamiltonian $G$-manifolds (for definition see [AMM]) as equivariant $K$-homology pushforward induced by the $G$-valued moment maps. He applied his results to examples such as moduli spaces of flat connections of principal $G$-bundles over orientable compact surfaces (see [M]). This quantization scheme has several advantages in that it does not involve quantizing the corresponding Hamiltonian $LG$-space, which is an infinite dimensional Banach manifold, and does not mention any twisted Dirac operator at all.

I have been working on a generalization to the Real context of the results in [M]. By now I have formulated the notion of Real quasi-Hamiltonian manifolds, obtained extension of related results in [AMM] and a (partial) generalization of FHT in the presence of a topological involution of the compact Lie group, using twisted equivariant $KR$-homology, which will serve as the framework
for a possible generalization of Meinrenken’s quantization scheme in the Real case. In particular, the Real quasi-Hamiltonian manifolds, among other things, requires the existence of the Real $G$-valued moment map $\mu : (M, \sigma_M) \to (G, \sigma_G \circ \text{inv})$. Meinrenken’s quantization scheme prompts us to generalize FHT using anti-involutions.

**Theorem 3.1** ([Fo4]). Suppose $G$ is a compact, connected and simply-connected Lie group with an involutive automorphism $\sigma_G$. If $R(G, \mathbb{C}) = 0$, then the equivariant twisted $KR$-homology $KR^*_G(G, \sigma_G \circ \text{inv}, \mathbb{A}^{k+h^\vee})$, where $h^\vee$ is the dual Coxeter number of $G$ and $\mathbb{A}$ a Real Dixmier-Douady bundle whose Dixmier-Douady class is a generator of $H^3(G, \sigma \circ \text{inv}, \mathbb{Z}(1)) \cong \mathbb{Z}$, is isomorphic to $R_k(G) \otimes KR^*(pt)$, the tensor product of the level $k$ Verlinde algebra and the coefficient ring.

My work ([Fo1]) actually was intended to be a stepping stone to a formulation of Real version of FHT. Expectedly the twisted equivariant $KR$-homology of $G$ in general has a richer structure than the Verlinde algebras, and it will be interesting to understand what physical interpretations the extra information gives. For example, what do the torsions and the elements in higher degree of the twisted equivariant $KR$-homology mean under the Real quantization scheme? I would like to address this kind of questions in the future.

3.3. Equivariant $K$-theory of compact Lie groups with finite fundamental groups. By using Hodgkin’s spectral sequence, Brylinski-Zhang showed that the equivariant $K$-theory $K^*_G(G)$ of any compact Lie group $G$ with torsion-free fundamental group is a free module over the complex representation ring (see [BZ]). Though they worked out the example of $PSU(3)$, little is known about the equivariant $K$-theory of any compact Lie groups with fundamental groups with torsion, in particular, the torsion part of the equivariant $K$-theory. In [Fo3], I plan to first attack the special case where the compact Lie group has fundamental group of prime order. I conjecture that $K^*_G(G)$ consists of two parts, namely, the free $R(G)$-submodule which is an exterior algebra and the torsion $R(G)$-submodule generated by the twisted line bundle $G \times \pi_1(G) \mathbb{C}_\mu$, where $G$ is the universal cover of $G$ and $\mu$ is a character of $\pi_1(G)$. I verified that the conjecture is true for $SO(3)$, whose equivariant $K$-theory can be easily computed by Segal’s spectral sequence:

$$K^*_G(SO(3)) \cong \bigwedge_{R(SO(3))}(\xi) \otimes \mathbb{Z} R(SO(3))[\xi] / (\xi^3) \cong \mathbb{Z} \langle 0, 2\xi + \xi^2 \rangle \langle \sigma_3 + 1, \xi \rangle$$

Here $\xi$ is a $-1$ degree class associated with the standard representation of $SO(3)$, $\xi$ is the class of reduced line bundle $SU(2) \times \mathbb{Z}$, and $\sigma_3 \in R(SO(3))$ is the standard representation. I want to find out, in more general cases, the primitive generators of the exterior algebra part of $K^*_G(G)$ and their topological interpretations. This will involve a detail study of $R(G)$ (which is not a free polynomial ring and more complicated). I expect that the index theory argument in [Fo2] (which actually gives an alternative, shorter proof to a special case of Brylinski-Zhang’s result) will be helpful in this regard. As to the torsion part, I intend to generalize to the equivariant setting the approach in [HS], where the torsion part of the ordinary $K$-theory ring of such Lie groups was worked out by applying the Atiyah-Hirzebruch spectral sequence to a certain fiber bundle associated to the Lie group and its fundamental group.

3.4. $K$-theory of the space of $n$-tuples of commuting elements in a compact Lie group. Moduli spaces of flat bundles are important objects in physics, so it will be desirable to have more understanding about their topology. The cohomology of moduli spaces of flat $G$-bundles over a torus $(S^1)^n$, which are nothing but the space of $n$-tuples of commuting elements in a compact Lie group...
$G$ (denoted by $Y_n(G)$), was studied in [AC], [B] and [BJS]. In [B] the rational cohomology ring was obtained using an abelianization argument, whereas in [AC] and [BJS] integral cohomology groups of examples of $Y_n(G)$ were computed by analyses of its suspension. In [AG] Adém and Goméz obtained the module structure of rational equivariant $K$-theory of $Y_n(G)$ by applying their result about the more general case of $G$-spaces with maximal rank isotropy groups satisfying certain technical conditions.

To the best of my knowledge, a complete description of the torsion part of both (equivariant) cohomology and $K$-theory of $Y_n(G)$ for general compact Lie group $G$ are not known. I intend to work on the equivariant $K$-theory of $Y_n(G)$ by using an alternative approach involving index theory. I suspect that the torsion part comes from the singularities of $Y_n(G)$. Understanding the structure of the singularities and working out their local $K$-theory (in analogy with local cohomology) using index theory argument may shed some light on the torsions.

References


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