RESEARCH STATEMENT

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My research interests lie in the intersection of algebraic topology, Lie groups and symplectic geometry. Roughly speaking, in my research I apply machinery from algebraic topology to understand spaces with symmetries and physical implications. To be more precise, my work consists of two major themes, namely,

(1) the K-theory of compact Lie groups and homogeneous spaces, and
(2) topological classification of general integrable systems.

1. K-theory of compact Lie groups and homogeneous spaces

1.1. KR**-theory.** KR-theory, which was first introduced by Atiyah in the beautiful paper [At1], is a version of topological K-theory for the Real spaces, i.e. topological spaces equipped with a continuous involution. To be more precise, KR-theory of a Real space X is the Grothendieck group of the category of Real vector bundles on it, i.e. complex vector bundles that are themselves Real spaces, whose involution descends to the involution on X and maps fiber to fiber anti-linearly. KR-theory was motivated by the study of the index theory of real elliptic operators and used by Atiyah to derive 8-periodicity of KO-theory from the 2-periodicity of complex K-theory (See [At1] and [At2]). What makes KR-theory interesting is that it can be viewed naturally as a unifying thread of KO-theory, complex K-theory and KSC-theory (see [At1], Sect. 3), which are K-groups of categories of real, complex and quaternionic vector bundles, respectively. For instance, if the involution is trivial, then KR-theory is equivalent to KO-theory. One may go one step further and consider equivariant KR-theory, which is simply the Grothendieck group of the category of Real equivariant G-vector bundles on a Real G-space, where we assume a compatibility condition of the G-action, the involutions on the vector bundles and the base space, and the involutive automorphism on G (for precise definitions and basic properties, see [AS]). In recent years there is a rekindled interest in KR-theory; in particular it has found applications in string theory, as it classifies the D-brane charges in orientifold string theory (cf. [DMR]).

In [Se], Seymour provided a structure theorem of KR-theory for a certain type of spaces, which enables us to compute the KR-theory using the knowledge of complex K-theory and how the action of the pullback induced by the base space involution followed by complex conjugation on complex vector bundles act on it. I observe that the conditions of Seymour's result are an appropriate candidate for defining an analogue of 'weakly equivariant formality' *à la* Harada and Landweber (see Definition 4.1 of [HL]), which roughly means the condition that every vector bundle has a stable equivariant lift. Inspired by Seymour's result and the notion of weakly equivariant formality, I introduced the notion of *Real equivariant formality* for equivariant KR-theory (see [Fo1], Definition

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4.2). I proved the following structure theorem of equivariant KR-theory of Real equivariantly formal spaces.

Theorem 1.1 ([Fo1], Theorem 4.5)**.** *Let X be a Real equivariantly formal G-space. For any element $a \in K^*(X)$ (resp. $a \in KR^*(X)$), let $a_G \in K_G^*(X)$ (resp. $a_G \in KR_G^*(X)$) be a fixed choice of (Real) equivariant lift of a. Then the map*

$$f : (RR(G, \mathbb{R}) \oplus RH(G, \mathbb{R})) \otimes KR^*(X) \oplus r(R(G, \mathbb{C}) \otimes K^*(X)) \to KR_G^*(X)$$
$$\rho_1 \otimes a_1 \oplus r(\rho_2 \otimes a_2) \mapsto \rho_1 \cdot (a_1)_G \oplus r(\rho_2 \cdot (a_2)_G)$$

is a group isomorphism. Here $RR(G, \mathbb{R}) \cong KR_G^0(pt)$, $RH(G, \mathbb{R}) \cong KR_G^{-4}(pt)$ are the ring of Real representations of real type and the group of Real representations of quaternionic type respectively, and $r : K_G^*(X) \to KR_G^*(X)$ is the realification map. In particular, if $R(G, \mathbb{C}) = 0$, then f is an $RR(G, \mathbb{R}) \oplus RH(G, \mathbb{R})$-module isomorphism.

1.2. **Equivariant KR-theory of compact Lie groups with involutive automorphisms.** In the 60s, Hodgkin showed that the complex K-theory ring of any compact connected Lie group with torsion-free fundamental group is a $\mathbb{Z}_2$-graded exterior algebra over $\mathbb{Z}$ on the module of primitive elements, which are of degree -1 and associated with the representations of the Lie group (see [Ho]). Since Hodgkin's work, there have appeared two generalizations of K-theory of compact Lie groups. The first such is Seymour's work on KR-theory of compact, connected and simply connected Lie groups equipped with involutive automorphisms (see [Se]). He obtained the $KR^*(\mathrm{pt})$-module structure of $KR^*(G)$ using his structure theorem. However, he was unable to give a complete description of the ring structure, and could only make some conjectures about it. The second generalization is the equivariant K-theory of compact Lie groups. In [BZ], Brylinski and Zhang showed that, for a compact Lie group G with torsion-free fundamental group and the G-action being the conjugation action on itself, its equivariant K-theory is isomorphic to the ring of Grothendieck differentials of the complex representation ring over $\mathbb{Z}$.

In [Fo1], based on the previous results of Seymour's and Brylinski-Zhang's, Theorem 1.1 and a description of the coefficient ring $KR_G^*(\mathrm{pt})$, I gave a preliminary description of $KR^*_{(G,\sigma_G)}(G, \sigma_G)$ (where σ_G is an involutive automorphism) by listing the algebra generators associated to the Real representations of G of real, complex and quaternionic types (with respect to the involutive automorphism). Then I gave a full description of the ring structure of $KR_G^*(G)$ by listing all the relations among the generators. To achieve this I investigated the map of equivariant KR-theory induced by the Weyl covering map. Of particular interest are the squares of the real and quaternionic type generators, $\delta_{\mathbb{R}}^G(\varphi) \in KR_G^{-1}(G)$ and $\delta_{\mathbb{H}}^G(\theta) \in KR_G^{-5}(G)$, where φ and θ are Real representations of real and quaterionic types, respectively, and $\delta_{\mathbb{R}}^G : RR(G) \to KR_G^{-1}(G)$ and $\delta_{\mathbb{H}}^G : RH(G) \to KR_G^{-5}(G)$ are natural maps defined in Definition 4.8 of [Fo1].

Theorem 1.2 ([Fo1], Theorem 4.30)**.** $\delta_{\mathbb{R}}^G(\varphi)^2 = \eta(\varphi \cdot \delta_{\mathbb{R}}^G(\varphi) - \delta_{\mathbb{R}}^G(\wedge^2\varphi))$ *and* $\delta_{\mathbb{H}}^G(\theta)^2 = \eta(\theta \cdot \delta_{\mathbb{H}}^G(\theta) - \delta_{\mathbb{R}}^G(\wedge^2\theta))$*, where $\eta \in KR^{-1}(pt)$ represents the reduced Hopf bundle over $\mathbb{RP}^1$.*

By applying the forgetful map $KR_G^*(G) \to KR^*(G)$ to the generators and relations, I solved the problem of describing the ring structure of $KR^*(G)$, which was left open in [Se]. Theorem 1.2 shows that, unlike the (equivariant) complex K-theory, the (equivariant) KR-theory of compact Lie groups are in general not an exterior algebra. The equations in Theorem 1.2 and their analogues in the ordinary KR-theory case actually give extra information about the KR-theory, and can be used to distinguish two different G-actions on itself. In [Fo1], I examined the two cases where G

acts on itself trivially and by conjugation, respectively. While the equivariant complex K-theory rings of both cases are isomorphic, the equivariant KR-theory rings are not, thanks to the extra information given by the squares of real and quaternionic type generators.

1.3. **Equivariant KR-theory of compact Lie groups with anti-involutions.** Another kind of topological involution on a Lie group is *anti-involution*, which is the composition of an involutive automorphism and group inversion. This begs the question of what is the equivariant KR-theory of a compact Lie group G equipped with instead an anti-involution. It turns out that the result is a little bit different from the involutive automorphism case as discussed in [Fo1] and previous section.

Theorem 1.3 ([Fo4])**.** *Let G be a compact, connected, and simply-connected Lie group and σ_G is an involutive automorphism on it. There exist maps*

$$\begin{aligned}\delta_{\mathbb{R}}^{inv}:&RR(G)\to KR^1_{(G,\sigma_G)}(G,\sigma_G\circ inv)\\ \delta_{\mathbb{H}}^{inv}:&RH(G)\to KR^{-3}_{(G,\sigma_G)}(G,\sigma_G\circ inv)\end{aligned}$$

which are similar to $\delta_{\mathbb{R}}^G$ and $\delta_{\mathbb{R}}^G$ as defined in Definition 4.8 in [Fo1]*. The square of any image of $\delta_{\mathbb{R}}^{inv}$ and $\delta_{\mathbb{H}}^{inv}$ is 0. In particular, if $R(G,\mathbb{C})=0$, then*

$$KR^*_{(G,\sigma_G)}(G,\sigma_G\circ inv)\cong\Omega_{KR^*_{(G,\sigma_G)}(pt)/KR^*(pt)},$$

where the right hand side is the ring of Grothendieck differentials of the coefficient ring of the equivariant KR-theory over the coefficient ring of ordinary KR-theory, whose primitive module is generated by the image of the fundamental representations of G under $\delta_{\mathbb{R}}^{inv}$ and $\delta_{\mathbb{H}}^{inv}$.

In some sense, equipping G with an anti-involution instead of an involutive automorphism (which we do in [Fo1]) is a better direction of generalizing Brylinski-Zhang's result. In fact, there is another piece of evidence which indicates that anti-involution is the right topological involution to consider when studying a certain KR-theory of compact Lie groups, as we will see in the next section. A consequence of Theorem 1.3 is

Corollary 1.4 ([Fo4])**.** *Let G be a Real compact and connected Lie group and X a compact Real G-space. Then if $x\in KR^i_G(X)$ for $i=1$ or -3, $x^2=0$.*

Note that graded commutativity only implies that x^2 is 2-torsion. On the other hand, it is not true in general that $x^2=0$ if $x\in KR^i_G(X)$ for $i=-1$ or -5, as Theorem 1.2 shows.

1.4. **The Real Freed-Hopkins-Teleman Theorem.** Freed-Hopkins-Teleman Theorem (FHT) asserts that the equivariant twisted K-homology of a compact connected Lie group G with torsion-free fundamental group (with ring structure being Pontryagin product) is isomorphic to Verlinde algebra of G, which is the ring of positive energy representations of the loop group LG, with ring structure being the fusion product (see [Fr], [FHT1], [FHT2], [FHT3]). Verlinde algebra is an object of great interest in mathematical physics and algebraic geometry. One of the remarkable aspects of Freed-Hopkins-Teleman Theorem is that it provides an algebro-topological approach to interpreting the fusion product, which is usually defined using conformal blocks or moduli spaces of G-bundles on Riemann surfaces (cf. [Be], [BL] and [V]). Moreover, Freed-Hopkins-Teleman also provides the framework for a formulation of geometric quantization of q-Hamiltonian spaces (cf. [M2] and [M3]). This will be elaborated in Section 3.4.

Seeing the possible applications in string theory as well as moduli spaces of nonorientable surfaces, we find it is of interest to obtain a generalization of FHT in the context of KR-theory. We first set the scene by developing such preliminary material as the equivariant Real 3rd cohomology group which is shown to classify the Real Dixmier-Douady bundles representing the Real twists, and Real Spin^c structures. Our main result shows that, by incorporating a group anti-involution of G, the corresponding equivariant twisted KR-homology of G is essentially a module over the equivariant KR-homology coefficient ring, generated by the irreducible positive energy representations of real, complex and quaternionic types. Moreover, the ring structure of the equivariant twisted KR-theory induced by the Pontryagin product, when restricted to those generators of positive energy representations, is precisely the fusion product. In short, the group anti-involution as the additional Real structure in the equivariant twisted KR-homology respects the algebra structure of the Verlinde algebra. The following is the precise statement of the main result.

Theorem 1.5 ([Fo5])**.** *Suppose G is a compact, connected and simply-connected Lie group with an involutive automorphism σ_G. Let $\mathcal{A}$ be the equivariant Real fundamental DD bundle over $(G, \sigma_G \circ inv)$. Let the level k Verlinde ideal I_k be generated by $r_1, \cdots, r_m \in R(G)$, and RI_k be the ideal in $KR_*^G(pt)$ with generators obtained from $r_1, \cdots, r_m$ by the followings.*

(1) Assigning each irreducible component of r_i which is not in $R(G, \mathbb{C})$ with degree 0 (resp. -4) according as whether it can be made a Real representation (resp. Quaternionic representation), and
(2) replacing each irreducible component s of r_i which is in $R(G, \mathbb{C})$ with the double $s + \overline{\sigma_G^*}s$, which is assigned with degree 0.

Then the pushforward map

$$\iota_*^{\mathbb{R}} : KR_*^{(G,\sigma_G)}(pt) \to KR_*^{(G,\sigma_G)}(G, \sigma_G \circ inv, \mathcal{A}^{k+\mathsf{h}^\vee})$$

is onto with kernel RI_k.

Using the description of Verlinde ideal in [Dou] and the KR-homology coefficient ring, we can obtain an explicit description of $KR_*^{(G,\sigma_G)}(G, \sigma_G \circ \mathrm{inv}, \mathcal{A}^{k+\mathsf{h}^\vee})$. As a consequence, the degree zero piece of the equivariant Real twisted KR-homology of G gives the Real Verlinde algebra, the Grothendieck group of the isomorphism classes of Real positive energy representations of the Real loop group LG, where the involution is induced by the Lie group involution on G and reflection on the loop.

1.5. **Adams operations on classical compact Lie groups.** Adams operations are important cohomological operations on K-theory which are utilized with great success by Adams in solving the famous problem of finding parallelizable spheres (cf. [Ad]). Another application of Adams operations is the extraction of certain information of homotopy groups of H-spaces and Lie groups in particular (cf. [Bou], [D], [DP]). It would be of interest to find out Adams operations on compact Lie groups. This task was first carried out in the paper [N], which unfortunately is strewn with many typos, and whose results on classical compact Lie groups are incomplete, not so concise and not explicit. Adams operations ψ^l on rank 2 compact Lie groups, and on $SU(n)$ for $l = -1, 2, 3$ were obtained in [Wa2], [Wa3] and [DP]. Ever since then no further formulas for Adams operations on other classical compact Lie groups have appeared. However, Adams operations on exception Lie groups and their eigenvectors were completely settled in [D]. In [Fo8], I settled the remaining cases by giving explicit formulas of Adams operations on all types of classical compact Lie groups, and eigenvectors of Adams operations on $U(n)$.

Theorem 1.6 ([Fo8]). *(1) Let l be a positive integer, σ_n the standard representation of $U(n)$, and $\delta : R(G) \to K^{-1}(G)$ the derivation of $R(G)$ defined in* [Ho]. *For $G = U(n)$, we have*

$$\psi^l(\delta(\bigwedge^k \sigma_n)) = (-1)^k l \sum_{p=1}^{n} \sum_{q=0}^{k-1} (-1)^{p+q} \binom{n}{q} \binom{n + l(k-q) - p - 1}{n-1} \delta(\bigwedge^p \sigma_n).$$

In particular, when $l = 2$,

$$\psi^2(\delta(\bigwedge^k \sigma_n)) = (-1)^k \cdot 2 \sum_{p=1}^{2k} (-1)^p \binom{n}{2k-p} \delta(\bigwedge^p \sigma_n).$$

The formula for $G = SU(n)$ is the same except that $\delta(\bigwedge^n \sigma_n)$ becomes 0 in this case.

(2) Let $\{p_j(y)\}_{j=0}^{\infty}$ be the sequence of polynomials which are coefficients of the Taylor series of $\left(\frac{t}{\sinh t}\right)^y$, i.e.

$$\left(\frac{t}{\sinh t}\right)^y = \sum_{j=0}^{\infty} p_j(y) t^{2j}.$$

Then $p_j(y)$ is of degree j and satisfies the following recurrent relation

$$p_0(y) = 1, p_j(y) = -\frac{y}{2j} \sum_{k=1}^{j} \frac{2^{2k} B_{2k}}{(2k)!} p_{j-k}(y)$$

where B_{2k} is the $2k$-th Bernoulli number. Moreover,

$$\sum_{i=1}^{n} (-1)^{i-1} \left(\sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{p_j(n)}{(k-2j)!} (n-2i)^{k-2j} \right) \delta(\bigwedge^i \sigma_n) \tag{1}$$

is an eigenvector of $\psi^l \otimes Id_{\mathbb{Q}} : K^*(U(n)) \otimes \mathbb{Q} \to K^*(U(n)) \otimes \mathbb{Q}$ (for l any integer) corresponding to the eigenvalue l^{n-k}, for $k = 0, 1, \cdots, n-1$.

Adams operations on $Sp(n)$ and $Spin(n)$ can be found in [Fo8, Theorem 1.1]. I also use Theorem 1.6 (1) to recover Adams operations on the exceptional Lie group G_2, which was obtained previously in [Wa3] by the indirect means of appealing to Chern character isomorphism.

1.6. **Equivariant formality of isotropic action of homogeneous spaces.** Equivariant formality is a special property of topological spaces with group actions which allows for easy computation of their equivariant cohomology. Roughly speaking, equivariant formality amounts to the existence of equivariant extension in the equivariant cohomology theory of any element in the ordinary cohomology theory. Equivalently, a G-space X is equivariantly formal if and only if $H_G^*(X) \cong H_G^*(\mathrm{pt}) \otimes H^*(X)$ as $H_G^*(\mathrm{pt})$-modules. It would be desirable to have a classification of equivariantly formal spaces, but the task is still too ambitious even if we restrict our attention to those with compact Lie group actions. An easier question would be to determine if, for a compact Lie group G and a closed subgroup K, the K-action on G/K by isotropy action (i.e. left multiplication) is equivariantly formal. We call (G, K) an *isotropy formal pair* if K acts on G/K equivariantly formally. There were some known partial results. For instance, in [Sh] and [ShTa] it was shown that if G/K is a formal manifold in the sense of Sullivan (or equivalently (G,K) is a Cartan pair. See [GHV] for definition) and satisfies an injectivity condition on cohomology, then (G,K) is an isotropy formal pair. In [Go] and [GoNo] it is proved that if (G,K) is a pair satisfying

(1) rank $G =$ rank K, i.e. equal rank pair,
(2) the restriction map $H^*(G) \to H^*(K)$ is onto, i.e. cohomologically surjective pair,
(3) G/K is a symmetric space, or
(4) G/K is a generalized symmetric space, i.e. $G_0^\sigma \subset K \subset G^\sigma$ for some automorphism σ of G of finite order,

then (G,K) is isotropy formal. Carlson showed in [Ca] that if T is a maximal torus of K, then (G,K) is an isotropy formal pair if and only if (G,T) is. This result enables us to reduce the whole problem to the case where K is a torus subgroup S of G. Carlson also gave a sufficient and necessary condition for (G,S) to be an isotropy formal pair when S is a circle subgroup (cf. [Ca, Algorithm 1.4]).

One feature in common in the main arguments used by [Ca], [Go], [GoNo] and [ShTa] to establish equivariant formality of isotropy action is the application of the condition of dim $H^*(M) =$ dim $H^*(M^T)$ which is equivalent to T acting on M equivariantly formally. In our opinion, while checking the equality of cohomological dimensions to prove equivariant formality is not as straightforward as checking the surjectivity of the forgetful map from the equivariant cohomology of the homogeneous space to its ordinary cohomology, the latter approach does not come in handy either, as this involves solving tedious ODEs if we work in the equivariant de Rham model. Besides to us appealing to the classification theorem in the proof of isotropy formality of (generalized) symmetric pairs in [Go] and [GoNo] is not satisfactory.

In joint work [CF] with Carlson, we apply K-theory instead to approach this problem and try to find alternative sufficient conditions for isotropy formality. Inspired by the notion of weakly equivariant formality, introduced in [HL], we study the similar notion of *rational K-theoretic equivariant formality* (RKEF for short).

Definition 1.7. X is a RKEF G-space if the forgetful map

$$f : K_G^*(X) \otimes \mathbb{Q} \to K^*(X) \otimes \mathbb{Q}$$

is onto.

The use of K-theory is feasible in this problem on cohomological equivariant formality of homogeneous spaces because of the following result which is crucial in our work.

Theorem 1.8 ([Fo9]). *Let X be a finite CW-complex with an action by a torus group T. X is a RKEF T-space if and only if it is an equivariantly formal T-space.*

Using the above Theorem we translate the whole problem to the context of K-theory. One advantage of working in K-theory is that it is more straightforward to check if the forgetful map is onto, since this amounts to determining if a vector bundle can be equipped with a T-action so as to become an equivariant T-vector bundle. After obtaining the K-theory ring of compact homogeneous spaces and topological realizations of each element in the K-theory ring when the homogeneous spaces are formal, we get a simple criterion for a formal homogeneous space to be isotropy formal, which we apply to give a more uniform proof of the aforementioned result that the homogeneous spaces (3)-(4) are isotropy formal. Furthermore we also get

Theorem 1.9 ([CF]). *Let G be a compact connected Lie group and S a torus subgroup. G/S is both isotropy formal and formal in the sense of Sullivan if and only if the image of the restriction map $i^*: R(G) \to R(S)$ is regular at the augmentation ideal.*

Theorem 1.9 provides a uniform proof of the fact that homogeneous spaces (1)-(4) above are both formal and isotropy formal in one fell swoop (that these spaces are formal were first proved in [GHV], [KT] and [St]). The regularity criterion in the Theorem can be conveniently verified using computer algebra packages such as Macaulay2 and SAGE. Two examples are worked out in [Fo9] to demonstrate the usefulness of Theorem [Fo9]; one of the examples is both formal and isotropy formal, but does not belong to the classes of homogeneous spaces (1)-(4).

2. Topological classification of almost symplectic integrable systems

A *symplectic manifold* M is a smooth manifold equipped with a nondegenerate and closed 2-form ω, called the symplectic form. The study of symplectic manifolds was motivated by classical mechanics, where the phase spaces of classical mechanical systems are themselves symplectic. Any smooth function H on M induces a vector field X_H, called the *Hamiltonian vector field*, by the Hamiltonian equation

$$\iota_{X_H}\omega = dH$$

H is correspondingly called a *Hamiltonian function* of the vector field. The terminology comes from the physical quantities conserved in classical mechanical systems in Hamiltonian mechanics, as H is constant on the flow line of X_H. We say that two Hamiltonian functions H_1 and H_2 *Poisson commute* if X_{H_1} and X_{H_2} commute. It is easy to show that, for a $2d$-dimensional symplectic manifolds, there are at most d Poisson-commuting, functionally independent Hamiltonian functions. If the maximum number of independent commuting Hamiltonian functions is achieved, then the symplectic manifold is called a *completely integrable system.* The famous Liouville-Arnold Theorem gives a description of the local symplectic structure of completely integrable systems. It asserts that, for a completely integrable system with Hamiltonian functions $H_1, \cdots, H_d$, if the level sets of $(H_1, \cdots, H_d) : M \to \mathbb{R}^d$ are compact and connected, then M is a torus fiber bundle, and symplectomorphic to the trivial torus fiber bundle $(U \times (S^1)^n, \sum_{i=1}^n dq_i \wedge dp_i)$, where U is an open subset in $\mathbb{R}^n$. In other words, M locally admits *action coordinates* p_i and *angle coordinates* q_i.

Let us consider a special class of symplectic manifolds, *symplectically complete isotropic realizations* (SCIR), which correspond to integrable systems in classical mechanics. They are fiber bundles $\pi : M^{2d} \to B^k$ with compact and connected fibers and a symplectic form ω which vanishes on restriction to any fiber, and the base B has a regular Poisson structure Π induced by the symplectic structure of M through π. There is a natural action of the conormal bundle $\nu^*\mathcal{F}$ of the symplectic foliation $\mathcal{F}$ of B on M. The stabilizer bundle P, called the *period bundle*, is a $\mathbb{Z}^n$-subbundle of $\nu^*\mathcal{F}$ whose sections represent some closed 1-forms of B which vanish on restriction to any symplectic leaf. Therefore, $T := \nu^*\mathcal{F}/P$, a torus fiber bundle, acts on M freely, making M a *T-torsor.* Conversely, if there exists a period bundle P on B, then B can support a SCIR where the stabilizer bundle of the $\nu^*\mathcal{F}$-action is P. The SCIR are locally symplectomorphic to T, and admit local *action* and *angle coordinates* when $d = k$ by Liouville-Arnold Theorem. In other words, there is no local invariant for SCIR. It is therefore interesting to look for topological invariants which measure the obstruction of the existence of global action and angle coordinates, and a topological classification of SCIRs over a fixed base Poisson manifold (B, Π) with a period bundle P. The first problem was settled in the early 80s by Duistermaat in the case $d = k$, who showed that global action and angle coordinates exist if and only if the *monodromy* and the *Lagrangian class*[1] of SCIR vanish (see [Du]). Dazord and Delzant later on in [DD] showed more generally that the *isotropic class* completely classifies all SCIRs over a fixed Poisson manifold B with a given period bundle P. They also gave a sufficient and necessary condition for a T-torsor to possess a compatible symplectic form so as to be an SCIR.

In recent years there has been a growing interest in the study of nonholonomic systems, and as a preliminary step of investigation in this context spaces with certain generalities are considered. For instance, in [FaSa] Fassò and Sansonetto studied more general integrable systems which are SCIR except that they are *almost symplectic*, i.e. equipped with a nondegenerate 2-form which is not necessarily closed. They obtained a generalization of Liouville-Arnold Theorem under the condition of the existence of *strongly Hamiltonian vector fields*. Moreover, *twisted Poisson manifolds*, first introduced in [SW] and motivated by string theory, provides the framework for the study of such nonholonomic systems as the Veselova systems and the Chaplygin sphere (see [BG-N]).

In ([Fo7]), I generalize Duistermaat/Dazord-Delzant's result in the almost symplectic context as in [FaSa]. In particular, I consider *almost symplectically complete isotropic realizations* (ASCIR), which are almost symplectic with linearly independent, locally strongly Hamiltonian vector fields tangent to fibers. We found that the base manifold of an ASCIR is necessarily a twisted regular Poisson manifold with a twisting 3-form η satisfying $d\omega = \pi^*\eta$. Let ASCIR(B, P, Π, Θ) be the category of ASCIR over the twisted Poisson manifold (B, Π) with period bundle P and a *characteristic 2-form* Θ, i.e. a 2-form which on restriction to any almost symplectic leaf becomes its almost symplectic form. Furthermore, we let $\mathcal{P}$ be the sheaf of local sections of P, and $\mathcal{K}_{\mathcal{F}} := \mathcal{Z}^1_{\mathcal{F}}/\mathcal{P}$, where $\mathcal{Z}^1_{\mathcal{F}}$ is the sheaf of the closed 1-forms which vanish on each leaf. Then we have the long exact sequence of sheaf cohomology (here $H^*(B, \mathcal{F})$ means relative cohomology of B with respect to the foliation $\mathcal{F}$).

$$(2)\quad \cdots \longrightarrow H^1(B,\mathcal{P}) \xrightarrow{d_{P,*}} H^2(B,\mathcal{F}) \longrightarrow H^1(B,\mathcal{K}_{\mathcal{F}}) \longrightarrow H^2(B,\mathcal{P}) \xrightarrow{\partial^2} H^3(B,\mathcal{F}) \longrightarrow \cdots$$

Theorem 2.1 ([Fo7]). *(1) The set of isomorphism classes of ASCIR(B, P, Π, Θ) can be equipped with an abelian group structure. We denote this group by Pic(B, P, Π, Θ). The identity element is represented by $T := \nu^*\mathcal{F}/P$ with almost symplectic form $\omega_{can} + \pi^*\Theta$, where ω_{can} is the canonical 2-form on T. The inverse of $[(M, \omega)]$ is $[(M, -\omega + 2\pi^*\Theta)]$. The product of two ASCIRs can be defined by means of a symplectic reduction (see* [Sj] *for a related construction).*

(2) Let ASCIR$_0(B, P, \Pi, \Theta)$ be the subcategory of ASCIR(B, P, Π, Θ) whose objects are ASCIR with twisting 3-form $d\Theta$. The map

$$\pi_0(ASCIR_0(B,P,\Pi,\Theta)) \xrightarrow{\text{isotropic class}} H^1(B,\mathcal{K}_{\mathcal{F}})$$

is an isomorphism. Moreover we have the short exact sequence

$$0 \longrightarrow H^2(B,\mathcal{F})/d_{P,*}H^1(B,\mathcal{P}) \longrightarrow \pi_0(ASCIR_0(B,\Pi,P,\Theta)) \longrightarrow ker(\partial^2) \longrightarrow 0$$

where the second map sends $[\gamma]$ to $[(T, \omega_{can} + \pi^*(\Omega + \gamma)]$ for γ, a closed 2-form which vanishes on restriction to any leaf of $\mathcal{F}$, and the third map gives the Chern class of ASCIRs as a T-torsor.

(3) Any T-torsor M can be equipped with a compatible almost symplectic form so as to be in ASCIR(B, P, Π, Θ). We have the exact sequence

$$0 \longrightarrow H^1(B,\mathcal{K}_{\mathcal{F}}) \longrightarrow Pic(B,P,\Pi,\Theta) \longrightarrow Z^3_{\mathcal{F}}(B)$$

[1] The notion of Lagrangian class was implicit in [Du] and introduced later by Dazord and Delzant in [DD].

where the last map sends an ASCIR M with twisting 3-form η to $d\Theta - \eta$.

(4) We have the short exact sequence

$$0 \longrightarrow \Omega^2_{\mathcal{F}}/d\Omega^1_{\mathcal{F}} \longrightarrow Pic(B, \Pi, P, \Theta) \longrightarrow H^2(B, \mathcal{P}) \longrightarrow 0$$

where $\Omega^k_{\mathcal{F}}$ is the space of k-forms which vanish on restriction to any leaf of $\mathcal{F}$, and the second map sends $[\gamma]$ to $[(T, \omega_{can} + \pi^*(\Omega + \gamma)]$

As $H^2(B, \mathcal{P})$ is a discrete group, the image of $\Omega^2_{\mathcal{F}}/d\Omega^1_{\mathcal{F}}$ under the second map in the last exact sequence is the identity component subgroup of the Picard group. This subgroup is to the Jacobian variety consisting of isomorphism classes of degree 0 invertible sheaves on a scheme in algebraic geometry what $H^2(B, \mathcal{P})$ is to the Néron-Severi group.

3. Future directions

3.1. **Liouville-Arnold Theorem for ASCIRs.** In [DD], Dazord and Delzant established a non-abelian version of Liouville-Arnold Theorem. It asserts that if L is a symplectic leaf of a Poisson manifold (B, Π) which is symplectomorphic to a regular coadjoint orbit $\mathcal{O}$ of a compact simply-connected semi-simple Lie group G, and $\pi : M^{2d} \to B^k$ is a SCIR, then there is a saturated neighborhood of L in M^{2d} such that it is symplectomorphic to $G \times Z \times C$, where Z is a certain torus subgroup of G and C the positive Weyl chamber of G. When G is a torus T, we recover Liouville-Arnold Theorem which is the special case when $k = d$ (or equivalently the symplectic leaves are zero dimensional). Their proof consists of a careful study of the *Dazord-Delzant homomorphism* $\partial^2 : H^2(B, \mathcal{P}) \to H^3(B, \mathcal{F})$ as in Equation 2 in the last Section. I would like to generalize this nonabelian Liouville-Arnold Theorem in the context of ASCIRs, based on the results in [Fo7]. The generalization possibly would involve ASCIRs over *quasi-Poisson* G-manifolds (cf. [AK-SM] for definition), and the standard local model would be $G \times Z \times \Delta$, where Δ is the Weyl alcove (an example is given in [Fo7, Section 6]). This can be seen as the 'exponentiated' version of the local model $G \times Z \times C$.

3.2. **Equivariant formality in other cohomology theories.** Seeing that the two notions of equivariant formality in cohomology and K-theory are equivalent (cf. Theorem 1.8), I would like to formulate equivariant formality in other complex oriented cohomology theories (i.e. those where Chern classes and formal group laws can be defined), and determine if it is equivalent to cohomological equivariant formality. In particular I will focus on complex cobordism which is 'universal' among all complex oriented cohomology theories. To get a more concrete taste of this project, in [Fo9] I am working on other equivariant cohomology theories of spaces with moment maps (e.g. (quasi-) Hamiltonian manifolds, Hamiltonian Poisson manifolds, etc.). This work in progress is inspired by a long chain of previous work, including the well-known results on moment maps and equivariant formality of Hamiltonian manifolds by Kirwan, the established theory of GKM manifolds, and the paper [HL] on the K-theory of Hamiltonian manifolds. I would like to obtain explicit topological description of any element of the other equivariant cohomology theories of spaces with moment maps.

3.3. **Equivariant quantum Schubert calculus and equivariant Verlinde algebra.** Schubert calculus concerns the study of the cohomology of the Grassmannian $Gr(k, n)$, or more precisely, how the Schubert varieties, whose Poincaré duals (called the Schubert classes) form a $\mathbb{Z}$-basis of the cohomology group, intersect. Schubert classes can be indexed by Young diagrams within the

$k \times (n-k)$-rectangle, and there is a well-established combinatorial theory of Young diagrams which enables one to compute the structural constants of the cohomology ring of $Gr(k,n)$, which record the number of intersection points among Schubert varieties.

First studied mathematically in [Ber], the quantum version of Schubert calculus is about the more general notion of 'fuzzy intersection theory' of $Gr(k,n)$. The quantum cohomology $QH^*(Gr(k,n))$ is an algebra over the ring $\mathbb{Z}[q]$ on the quantum variable q, generated by the Schubert classes. The ring structure is dictated by the Gromov-Witten invariants, which give the number of rational curves of specified degree passing through given Schubert varieties in general position in $Gr(k,n)$. Again there is an extensive research on the quantum version of the combinatorial theory of Young diagrams which compute the ring structure of the quantum cohomology. See, for example, [BCFF].

There is a deep link between $QH^*(Gr(k,n))$ and the Verlinde algebra, first discovered and studied in physical terms by Witten in [W], and proved mathematically in [Ag]. It asserts that there is an isomorphism

$$QH^*(Gr(k,n))/(q-1) \cong R_{n-k}(U(k))$$

In light of the equivariant generalization (*a la* Borel) of quantum Schubert calculus (first studied in [Mi]), it would be of interest to get an equivariant analogue of the above isomorphism. We note that any formulation of such an equivariant generalization necessitates a notion of equivariant Verlinde algebra, of which there are a few possible candidates. We find that using equivariant twisted K-homology, inspired by Freed-Hopkins-Teleman theorem, would be the most convenient and promising for such a generalization, namely, we propose that the RHS of the above isomorphism be replaced by $K_*^{U(k)\times T}(U(k),\mathcal{A})$, where T is an n-dimensional torus acting on $U(k)$ trivially, and $\mathcal{A}$ is a certain Dixmier-Douady bundle. The LHS is conceivably the Mihalcea's equivariant quantum cohomology. We find that when $k=1$ and $\mathrm{DD}(\mathcal{A}) = nx + \sum_{i=1}^n t_i \in H^3_{U(1)\times T}(U(1))$,

$$\begin{aligned} K_*^{U(1)\times T}(U(1),\mathcal{A}) &\cong \mathbb{Z}[x, y_1, \cdots, y_n]/(x^n y_1\cdots y_n - 1) \\ QH_T^*(\mathbb{P}^{n-1},\mathbb{Z}) &\cong \mathbb{Z}[\sigma, t_1, \cdots, t_n, q]/((\sigma - t_1)\cdots(\sigma - t_n) - q) \end{aligned}$$

and we have the isomorphism

$$QH_T^*(\mathbb{P}^{n-1},\mathbb{Z})/(q-1) \cong K_*^{U(1)\times T}(U(1),\mathcal{A})$$

through the map $\sigma \mapsto x$, $t_i \mapsto x(1-y_i)$. In [Fo6] we are working on a proof of this equivariant analogue of the isomorphism.

We would also like to define equivariant Verlinde algebra by geometric means instead of representation theory. This is another candidate for formulating equivariant Verlinde algebra. For ordinary Verlinde algebra, it is well-known that the structural constants (with respect to the set of generators of irreducible positive energy representations) is the quantization of the moduli space of flat connections on a thrice-punctured Riemann sphere with prescribed holonomies along the three holes. We would like to define a suitable T-action on this moduli space and use the equivariant quantization to define the equivariant Verlinde algebra. Our goal is to show that such a definition is equivalent to the aforementioned equivariant twisted K-homology version and hence establish an equivariant version of Freed-Hopkins-Teleman.

3.4. **Geometric quantization of Real quasi-Hamiltonian manifolds.** Quasi-Hamiltonian manifolds, introduced in [AMM], are a variant of Hamiltonian manifolds which possess G-valued moment maps $\mu : M \to G$. Motivated by Freed-Hopkins-Teleman Theorem, Meinrenken realized the quantization of quasi-Hamiltonian G-manifolds as equivariant K-homology pushforward induced by the G-valued moment maps. He applied his results to examples such as moduli spaces of flat connections of principal G-bundles over orientable compact surfaces (see [M2], [M3]). This quantization scheme has several advantages in that it does not involve quantizing the corresponding Hamiltonian LG-space, which is an infinite dimensional Banach manifold, and does not mention any twisted Dirac operator at all.

I have been working on a generalization to the Real context of the results in [M3]. By now I have formulated the notion of Real quasi-Hamiltonian manifolds, obtained extension of related results in [AMM]. These, together with my work [Fo5] on the Real FHT, will serve as the framework for a possible generalization of Meinrenken's quantization scheme in the Real case. In particular, the Real quasi-Hamiltonian manifolds, among other things, requires the existence of the Real G-valued moment map $\mu : (M, \sigma_M) \to (G, \sigma_G \circ \mathrm{inv})$. Meinrenken's quantization scheme prompts us to generalize FHT using group anti-involutions. Given that the twisted equivariant KR-homology of G has a richer structure than the Verlinde algebras, it will be interesting to understand what physical interpretations the extra information gives. For example, what extra information can be gleaned from the torsions and the elements in higher degree of the twisted equivariant KR-homology under the Real quantization scheme? I would like to address this kind of questions in the future.

3.5. **Equivariant K-theory of compact Lie groups with finite fundamental groups.** By using Hodgkin's spectral sequence, Brylinski-Zhang showed that the equivariant K-theory $K^*_G(G)$ of any compact Lie group G with torsion-free fundamental group is a free module over the complex representation ring (see [BZ]). Though they worked out the example of $PSU(3)$, little is known about the equivariant K-theory of any compact Lie groups with fundamental groups with torsion, in particular, the torsion part of the equivariant K-theory. In [Fo3], I plan to first attack the special case where the compact Lie group has fundamental group of prime order. I conjecture that $K^*_G(G)$ consists of two parts, namely, the free $R(G)$-submodule which is an exterior algebra and the torsion $R(G)$-submodule generated by the twisted line bundle $\widetilde{G} \times_{\pi_1(G)} \mathbb{C}_\mu$, where $\widetilde{G}$ is the universal cover of G and μ is a character of $\pi_1(G)$. I verified that the conjecture is true for $SO(3)$, whose equivariant K-theory can be easily computed by Segal's spectral sequence:

$$K^*_{SO(3)}(SO(3)) \cong \bigwedge\nolimits_{R(SO(3))}(\varepsilon) \otimes_{\mathbb{Z}} R(SO(3))[\xi] \big/ (\varepsilon\xi, 2\xi + \xi^2, (\sigma_3 + 1)\xi)$$

Here ε is a -1 degree class associated with the standard representation of $SO(3)$, ξ is the class of reduced line bundle $SU(2) \times_{\mathbb{Z}_2} \mathbb{C}_\mu - SO(3) \times \mathbb{C}$, and $\sigma_3 \in R(SO(3))$ is the standard representation. I want to find out, in more general cases, the primitive generators of the exterior algebra part of $K^*_G(G)$ and their topological interpretations. This will involve a detailed study of $R(G)$ (which is not a free polynomial ring and more complicated). I expect that the index theory argument in [Fo2] (which actually gives an alternative, shorter proof to a special case of Brylinski-Zhang's result) will be helpful in this regard. As to the torsion part, I intend to generalize to the equivariant setting the approach in [HS], where the torsion part of the ordinary K-theory ring of such Lie groups was worked out by applying the Atiyah-Hirzebruch spectral sequence to a certain fiber bundle associated to the Lie group and its fundamental group.

3.6. **K-theory of the space of n-tuples of commuting elements in a compact Lie group.** Moduli spaces of flat bundles are important objects in physics, so it will be desirable to have more understanding about their topology. The cohomology of moduli spaces of flat G-bundles over a torus

$(S^1)^n$, which are nothing but the space of n-tuples of commuting elements in a compact Lie group G (denoted by $Y_n(G)$), was studied in [AC], [B] and [BJS]. In [B] the rational cohomology ring was obtained using an abelianization argument, whereas in [AC] and [BJS] integral cohomology groups of examples of $Y_n(G)$ were computed by analyses of its suspension. In [AG] Adém and Goméz obtained the module structure of rational equivariant K-theory of $Y_n(G)$ by applying their result about the more general case of G-spaces with maximal rank isotropy groups satisfying certain technical conditions.

To the best of my knowledge, a complete description of the torsion part of both (equivariant) cohomology and K-theory of $Y_n(G)$ for general compact Lie group G are not known. I intend to work on the equivariant K-theory of $Y_n(G)$ by using an alternative approach involving index theory. I suspect that the torsion part comes from the singularities of $Y_n(G)$. Understanding the structure of the singularities and working out their local K-theory (in analogy with local cohomology) using index theory argument may shed some light on the torsions.

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