CANONIZING RELATIONS ON NONSMOOTH SETS

CLINTON T. CONLEY

Abstract. We show that any symmetric, Baire measurable function from the complement of E_0 to a finite set is constant on an E_0-nonsmooth square. A simultaneous generalization of Galvin's theorem that Baire measurable colorings admit perfect homogeneous sets and the Kanovei-Zapletal theorem canonizing Borel equivalence relations on E_0-nonsmooth sets, this result is proved by relating E_0-nonsmooth sets to embeddings of the complete binary tree into itself and appealing to a version of Hindman's theorem on the complete binary tree. We also establish several canonization theorems which follow from the main result.

§0. Introduction. While it is well known that many Ramsey-style partition properties fail at uncountable cardinals, it is perhaps surprising that their descriptive analogs often hold. For example, using a wellordering of the reals it is easy to build a two-coloring of pairs of real numbers which admits no uncountable homogeneous set. If we restrict our attention to Baire measurable colorings of pairs, however, Galvin's theorem (see [2, Theorem 19.7]) ensures that we may always find a homogeneous perfect subset.

At first glance, this may seem as far as one could hope to push things, as the classical framework for descriptive set theory lies within the confines of Polish spaces, whose cardinalities are bounded above by that of the continuum. However, in the descriptive context we must also change our outlook on cardinality. For example, given two countable Borel equivalence relations E and F on Polish spaces X and Y, there may be no Borel function $\varphi : X \to Y$ such that $x_0 \mathrel{E} x_1 \Leftrightarrow \varphi(x_0) \mathrel{F} \varphi(x_1)$ (such a function is a *reduction* of E to F) nor a Borel function $\psi : Y \to X$ with the analogous property. In such a situation, there is no Borel way of comparing the quotient spaces X/E and Y/F, even though of course each has the cardinality of the continuum. In that sense, the "Borel cardinality" of quotient spaces can be very complicated.

As expected, the list of the "Borel cardinal numbers" begins $0, 1, 2, \dots, \aleph_0, \mathfrak{c}$. Remarkably, there is a cardinal successor of the continuum, namely $2^\omega/E_0$, where E_0 is the equivalence relation of eventual agreement of binary strings. This is the celebrated Harrington-Kechris-Louveau generalization of the Glimm-Effros dichotomy (see [1]): if a Borel equivalence relation E does not Borel reduce to equality of reals, then there is a Borel reduction of E_0 to E. One naturally wonders about the Ramsey-theoretic properties this next Borel cardinal might possess.

Supported in part by Marie Curie grant no. 249167 from the European Union.

First, we introduce some terminology. We say a Borel set $A \subseteq 2^\omega$ is *smooth* if there is a Borel reduction of $E_0 \mid A$ to the identity relation $\Delta(2^\omega)$. This is equivalent to A being contained in the union of countably many Borel sets each meeting any E_0-class in at most one point. We say a set $A \subseteq 2^\omega$ is *nonsmooth* if there is a Borel reduction of E_0 to $E_0 \mid A$ (or equivalently, if there is no Borel reduction of $E_0 \mid A$ to $\Delta(2^\omega)$). In the previous heuristic of Borel cardinality, a set A is smooth if A/E_0 is strictly smaller than $2^\omega/E_0$, and A is nonsmooth if A/E_0 is the same size as $2^\omega/E_0$.

Kanovei and Zapletal have shown the following striking canonization property on nonsmooth sets, an elementary proof of which appears in [3].

THEOREM 0.1 (Kanovei-Zapletal). *Suppose that E is a Borel equivalence relation on 2^ω. Then there exists a nonsmooth compact set K such that $E \mid K$ agrees with one of $\Delta(K)$, $E_0 \mid K$, and K^2.*

Motivated by the existence of such a small basis for equivalence relations, we wonder whether there is a partition property lurking underneath. The main goal of this paper is to prove the following.

THEOREM 0.2. *Suppose that $c : (2^\omega)^2 \to 2$ is a symmetric, Baire measurable function. Then there exists a nonsmooth, compact set K such that c is constant on $K^2 \setminus E_0$.*

In some sense, it would be a precise analog of Galvin's theorem if we could obtain that c is constant on $K^2 \setminus \Delta$, but this is generally impossible. Nevertheless, this may be viewed as the ability to find a nonsmooth set which is homogeneous up to a small, prescribed amount of error within E_0-classes. One consequence of this is the corresponding partition theorem for the quotient space $2^\omega/E_0$, i.e., any finite coloring of $[2^\omega/E_0]^2$ induced by a Baire measurable, E_0-invariant function on 2^ω admits a nonsmooth homogeneous set.

At the combinatorial core of this theorem is the ability to find certain monochromatic aligned subtrees of the complete binary tree. We say a function $f : 2^{<\omega} \to 2^{<\omega}$ is an *aligned embedding* if there exist sequences $(u_n^0)_{n\in\omega}$ and $(u_n^1)_{n\in\omega}$ of elements of $2^{<\omega}$ such that for all $n \in \omega$, $i \in 2$, and $s \in 2^{<\omega}$,

1. $|u_n^0| = |u_n^1|$;
2. $u_n^i(0) = i$; and
3. $f(s^\frown i) = f(s)^\frown u^i_{|s|}$.

THEOREM 0.3. *For any function $c : 2^{<\omega} \to 2$, there is an aligned embedding $f : 2^{<\omega} \to 2^{<\omega}$ such that for all $s, t \in 2^{<\omega}$, $c(f(s)) = c(f(t))$.*

The observant reader will notice that this is a special case of Milliken's tree theorem (see [4, Chapter 6]), which can be applied to find a monochromatic strongly embedded subtree isomorphic to $2^{<\omega}$. Nevertheless, we present a simple proof from Hindman's theorem, due both to the relative ubiquity of the latter and also to the relative difficulty of the former.

The paper is organized as follows. In §1 we present a proof of the required partition theorem from Hindman's theorem, which is otherwise self-contained. In §2 we prove the main theorem and discuss some of its further applications.

ACKNOWLEDGMENTS. The author would like to express thanks to Alexander Kechris, Benjamin Miller, Stevo Todorcevic, and Jindrich Zapletal for many enjoyable conversations and useful suggestions.

§1. A Partition Theorem. Since Ramsey's famous theorem on graph colorings, there have been many partition theorems proved on a broad class of structures. Typically, these theorems state that if a structure is partitioned into pieces, then one of these pieces is large. In this paper, the structure being partitioned is the complete binary tree $2^{<\omega}$, and the partitions are into finitely many parts. Our notion of largeness will include those subtrees of $2^{<\omega}$ where the splitting at each height occurs homogeneously across each level. This sort of homogeneity, which is useful for descriptive set-theoretic applications, is most easily formalized via a special type of embedding.

We say a function $f : 2^{<\omega} \to 2^{<\omega}$ is an *aligned embedding* if there exist sequences $(u^0_n)_{n\in\omega}$ and $(u^1_n)_{n\in\omega}$ in $2^{<\omega}$ such that for all $n \in \omega$, $i \in 2$, and $s \in 2^{<\omega}$,

1. $|u^0_n| = |u^1_n|$;
2. $u^i_n(0) = i$; and
3. $f(s^\frown i) = f(s)^\frown u^i_{|s|}$.

It is easy to see that the composition of two aligned embeddings is again an aligned embedding.

We let FIN denote the set of all nonempty finite subsets of ω. For $X, Y \in \mathrm{FIN}$, we say $X < Y$ if $\forall x \in X\ \forall y \in Y\ (x < y)$. Hindman's theorem below is a powerful tool for finding large monochromatic subsets of FIN.

THEOREM 1.1 (Hindman). *For every finite coloring of the set* FIN*, there is a sequence $\mathcal{X} = (X_n)_{n\in\omega}$ of elements of* FIN *such that $X_0 < X_1 < \cdots$ and the set $[\mathcal{X}]$ of all finite unions of members of $\mathcal{X}$ is monochromatic.*

PROOF. See [4, Theorem 2.41]. ⊣

In order to apply Hindman's theorem to finite colorings of $2^{<\omega}$, we must first develop a scheme to translate between finite subsets of the naturals and finite binary strings. We define the *support* of a string $s \in 2^{<\omega}$ by

$$\mathrm{supp}(s) = \{n \in \omega \colon s(n) = 1\}.$$

The possibility of terminal zeroes allows several strings to share the same support, but we may find a large subtree on which such strings all get the same color.

LEMMA 1.2. *For any function $c : 2^{<\omega} \to 2$, there is an aligned embedding $f : 2^{<\omega} \to 2^{<\omega}$ such that for all $s, t \in 2^{<\omega}$, $\mathrm{supp}(s) = \mathrm{supp}(t) \Longrightarrow c(f(s)) = c(f(t))$.*

PROOF. We recursively construct a decreasing sequence $(A_n)_{n\in\omega}$ of infinite subsets of ω, an increasing sequence $(a_n)_{n\in\omega}$ with each $a_n = \min(A_n)$, and a sequence of functions $(f_n : 2^n \to 2^{<\omega})_{n\in\omega}$ such that for all $n \in \omega$ and $s \in 2^n$,

1. $f_{n+1}(s^\frown 0) = f_n(s)^\frown 0^\frown 0^{a_{n+1}-a_n-1}$;
2. $f_{n+1}(s^\frown 1) = f_n(s)^\frown 1^\frown 0^{a_{n+1}-a_n-1}$; and
3. $c(f_{n+1}(s^\frown 0)) = c(f_n(s))$.

After we have completed this construction, it is easy to see that $f(s) = f_{|s|}(s)$ witnesses the conclusion of the lemma.

To begin the construction, let A_0 be an infinite subset of ω such that for all $a, b \in A_0$, $c(0^a) = c(0^b)$. Set $a_0 = \min(A_0)$, and define $f_0(\emptyset) = 0^{a_0}$.

Now suppose that we have completed the construction up to stage n. Let A_{n+1} be an infinite subset of $A_n \setminus \{a_n\}$ such that for all $a, b \in A_{n+1}$ and $s \in 2^n$,

$$c(f_n(s)^\frown 0^\frown 0^{a-a_n-1}) = c(f_n(s)^\frown 0^\frown 0^{b-a_n-1}) \text{ and}$$
$$c(f_n(s)^\frown 1^\frown 0^{a-a_n-1}) = c(f_n(s)^\frown 1^\frown 0^{b-a_n-1}).$$

As before, set $a_{n+1} = \min(A_{n+1})$, and define f_{n+1} by

$$f_{n+1}(s^\frown 0) = f_n(s)^\frown 0^\frown 0^{a_{n+1}-a_n-1}, \text{ and}$$
$$f_{n+1}(s^\frown 1) = f_n(s)^\frown 1^\frown 0^{a_{n+1}-a_n-1}.$$

Conditions 1 and 2 are met by construction, so it suffices to check the third condition. Towards this end, fix $n \in \omega$. If $n = 0$, then since $f(\emptyset) = 0^{a_0}$ and $f(0) = 0^{a_1}$, we have $c(f(0)) = c(f(\emptyset))$ as a_0 and a_1 are both elements of A_0.

The situation is similar for $n > 0$; fix $s \in 2^{n-1}$ and $i \in 2$. Unfolding the construction, we see that

$$f_n(s^\frown i) = f_{n-1}(s)^\frown i^\frown 0^{a_n - a_{n-1}-1},$$

and

$$\begin{aligned} f_{n+1}(s^\frown i^\frown 0) &= f_n(s^\frown i)^\frown 0^{a_{n+1}-a_n} \\ &= f_{n-1}(s)^\frown i^\frown 0^{a_n-a_{n-1}-1\frown} 0^{a_{n+1}-a_n} \\ &= f_{n-1}(s)^\frown i^\frown 0^{a_{n+1}-a_{n-1}-1}. \end{aligned}$$

Since a_n and a_{n+1} are both elements of A_n, we have $c(f_{n+1}(s^\frown i^\frown 0)) = c(f_n(s^\frown i))$ as required. ⊣

THEOREM 1.3. *For any function $c : 2^{<\omega} \to 2$, there is an aligned embedding $f : 2^{<\omega} \to 2^{<\omega}$ such that for all $s, t \in 2^{<\omega}$, $c(f(s)) = c(f(t))$.*

PROOF. By pulling back through the embedding granted by Lemma 1.2, we may assume that $c(s) = c(t)$ whenever $\operatorname{supp}(s) = \operatorname{supp}(t)$. The function c then induces a coloring $c' : \mathrm{FIN} \to 2$ by letting $c'(X) = c(s)$ where s is any string with $\operatorname{supp}(s) = X$. Hindman's theorem yields a sequence $\mathcal{X} = (X_n)_{n\in\omega}$ of elements of FIN such that $X_0 < X_1 < \cdots$ and the set $[\mathcal{X}]$ of all finite unions of members of $\mathcal{X}$ is c'-monochromatic of color k.

For each $n \in \omega$, set $x_n = \min(X_n)$, and define $g_n : 2^n \to 2^{x_n}$ by

$$g_n(s)(i) = 1 \Leftrightarrow i \in \bigcup_{s(j)=1} X_j.$$

We then define $g : 2^{<\omega} \to 2^{<\omega}$ by setting $g(s) = g_{|s|}(s)$, and we finally define $f : 2^{<\omega} \to 2^{<\omega}$ by $f(s) = g(1^\frown s)$. The remainder of the proof is devoted to showing that f satisfies the conclusion of the theorem.

We first show that f is an aligned embedding. Since $s \mapsto 1^\frown s$ is clearly an aligned embedding, it suffices to show that g is an aligned embedding. For each $n \in \omega$, let $u^0_n, u^1_n \in 2^{x_{n+1}-x_n}$ be defined by $u^0_n = 0^{x_{n+1}-x_n}$ and $u^1_n(i) = 1 \Leftrightarrow i + x_n \in X_n$. Then certainly $u^0_n(0) = 0$ and $u^1_n(0) = 1$. Moreover, since $X_0 < X_1 < \dots$, we see $g(s^\frown i) = g(s)^\frown u^i_{|s|}$ as needed.

For each $s \in 2^{<\omega}$,

$$\begin{aligned} \operatorname{supp}(f(s)) &= \operatorname{supp}(g(1^\frown s)) \\ &= X_0 \cup \bigcup_{s(j)=1} X_{j+1} \\ &\in [\mathcal{X}], \end{aligned}$$

and consequently, $c(f(s)) = c'(\operatorname{supp}(f(s))) = k$. This value is independent of the choice of s, as required. ⊣

§2. Applications. Recall that if E is a Borel equivalence relation on a Polish space X, then a Borel set $A \subseteq X$ is *E-smooth* when there is a Borel reduction of $E \mid A$ to the identity relation $\Delta(X)$. In the special case that E has countable classes, $A \subseteq X$ is E-smooth iff it is contained in the union of countably many Borel partial transversals of E (where a set is a *partial transversal* of E if it meets each E-class in at most one point). We view the E-smooth sets as being small, and the E-nonsmooth sets (i.e., those which are not E-smooth) as being large. We denote by $\mathcal{I}_E$ the collection of Borel E-smooth sets.

One particular equivalence relation plays a special role in the study of smoothness. We denote by E_0 the equivalence relation of eventual agreement of elements of 2^ω; more formally,

$$x \; E_0 \; y \Leftrightarrow \exists n \; \forall m > n \; (x(m) = y(m)).$$

In some sense E_0 is the simplest equivalence relation for which the smooth σ-ideal is nontrivial. For this reason, we refer to E_0-(non)smooth sets simply as *(non)smooth.* As we shall see in Corollary 2.5, the Harrington-Kechris-Louveau generalization of the Glimm-Effros dichotomy essentially allows us to find for any Borel equivalence relation E a copy of $\mathcal{I}_{E_0}$ within $\mathcal{I}_E$, provided $\mathcal{I}_E$ is nontrivial, justifying this convention.

The analysis of aligned embeddings gives insight into the structure of nonsmooth sets, since if $f : 2^{<\omega} \to 2^{<\omega}$ is an aligned embedding, the function $\varphi : 2^\omega \to 2^\omega$ defined by

$$\varphi_f(x) = \lim_{n\to\infty} f(x|n)$$

has nonsmooth image. In fact, by correctly interpreting the Glimm-Effros dichotomy, one also has that a set $A \subseteq 2^\omega$ is nonsmooth if and only if A contains the image of such an aligned embedding. Viewing E_0 as the orbit equivalence relation of $(\mathbb{Z}/2\mathbb{Z})^{<\omega}$ acting on 2^ω by coordinatewise addition mod 2, the proof of [3, Theorem 4], yields the following.

Proposition 2.1. *Suppose that $E \subseteq E_0$ is a nonsmooth equivalence relation on Cantor space. Then there is an aligned embedding $f : 2^{<\omega} \to 2^{<\omega}$ such that φ_f is an embedding of E_0 into E.*

Recall the theorems of Galvin and Kanovei-Zapletal mentioned earlier.

Theorem 2.2 (Galvin). *Suppose that $c : (2^\omega)^2 \to 2$ is a symmetric, Baire measurable function. Then there exists a perfect set K such that $K^2 \setminus \Delta$ is c-monochromatic.*

Theorem 2.3 (Kanovei-Zapletal). *Suppose that E is a Borel equivalence relation on 2^ω. Then there exists a nonsmooth compact set K such that $E \mid K$ agrees with one of $\Delta(K)$, $E_0 \mid K$, and K^2.*

The analysis of aligned embeddings in §1 is employed to establish the following simultaneous generalization, which may be viewed as our main result:

Theorem 2.4. *Suppose that $c : (2^\omega)^2 \to 2$ is a symmetric, Baire measurable function. Then there exists a nonsmooth compact set K such that c is constant on $K^2 \setminus E_0$.*

This implies Galvin's theorem since any nonsmooth Borel set contains a perfect partial transversal of E_0. On the other hand, one obtains the Kanovei-Zapletal canonization by coloring pairs in 2^ω according to whether they are E-related. This reduces the problem to canonizing subequivalence relations of E_0, applying Theorem 2.4, and then appealing to Proposition 2.1.

Recall that the *lexicographical* order on 2^ω is given by $x \leq_{\text{lex}} y$ iff $x = y$ or $x(n) < y(n)$, where n is the first coordinate on which x and y differ.

Proof of Theorem 2.4. As usual, for $s \in 2^{<\omega}$ we let $\mathcal{N}_s$ denote the basic open set of $x \in 2^\omega$ which have s as an initial segment. We first recursively construct functions $g_n : 2^n \to 2^{<\omega}$ and sequences u_n^i in $2^{<\omega}$ such that for all $n \in \omega$, $i \in 2$, and $s, t \in 2^n$,

1. $|u_n^0| = |u_n^1|$,
2. $u_n^i(0) = i$,
3. $g_{n+1}(s^\frown i) = g_n(s)^\frown u_n^i$.

As we go, we will also build functions $d_n : 2^n \to 2$ and sequences $(U_n^k)_{k\in\omega}$ of open dense subsets of $(2^\omega)^2$ such that for all $x, y \in 2^\omega$,

$$(x, y) \in (\mathcal{N}_{g_{n+1}(s^\frown 0)} \times \mathcal{N}_{g_{n+1}(s^\frown 1)}) \cap \bigcap_{k\in\omega} U_n^k \Longrightarrow c(x, y) = d_n(s).$$

To begin, set $g_0(\emptyset) = \emptyset$. As c is Baire measurable, we may find $u_0^0 \sqsupseteq 0$ and $u_0^1 \sqsupseteq 1$ of equal length so that c is constant on a (relatively) comeager subset $C_\emptyset$ of $\mathcal{N}_{u_0^0} \times \mathcal{N}_{u_0^1}$. Set $d_0(\emptyset)$ equal to this constant value, and fix open dense sets $U_0^k \subseteq (2^\omega)^2$ such that

$$(\mathcal{N}_{u_0^0} \times \mathcal{N}_{u_0^1}) \cap \bigcap_{k\in\omega} U_0^k \subseteq C_\emptyset.$$

Naturally, we may then set $g_1(i) = u_0^i$.

Now suppose that we have defined g_n. Again, as c is Baire measurable, we may find $v_n^0 \sqsupseteq 0$ and $v_n^1 \sqsupseteq 1$ of equal length so that for each $s \in 2^n$, c is constant on a comeager subset C_s of $\mathcal{N}_{g_n(s)^\frown v_n^0} \times \mathcal{N}_{g_n(s)^\frown v_n^1}$ (this is a straightforward recursive construction of length 2^n). Set $d_n(s)$ equal to this constant value, and fix open dense sets $U_n^k \subseteq (2^\omega)^2$ such that for each $s \in 2^n$,

$$\mathcal{N}_{g_n(s)^\frown v_n^0} \times \mathcal{N}_{g_n(s)^\frown v_n^1} \cap \bigcap_{k\in\omega} U_n^k \subseteq C_s.$$

Finally, choose $u_n^0 \sqsupseteq v_n^0$, $u_n^1 \sqsupseteq v_n^1$ of equal length such that for all $s, t \in 2^n$,

$$(\mathcal{N}_{g_n(s)^\frown u_n^0} \times \mathcal{N}_{g_n(t)^\frown u_n^1}) \subseteq \bigcap_{j,k<n} U_j^k, \text{ and}$$
$$(\mathcal{N}_{g_n(s)^\frown u_n^1} \times \mathcal{N}_{g_n(t)^\frown u_n^0}) \subseteq \bigcap_{j,k<n} U_j^k.$$

As before, set $g_{n+1}(s^\frown i) = g_n(s)^\frown u_n^i$, completing the recursive step of the construction.

Let $g : 2^{<\omega} \to 2^{<\omega}$ be defined by $g(s) = g_{|s|}(s)$, and similarly let $d : 2^{<\omega} \to 2$ be defined by $d(s) = d_{|s|}(s)$. By Theorem 1.3, we may find an aligned embedding $f : 2^{<\omega} \to 2^{<\omega}$ and $a \in 2$ such that for all $s \in 2^{<\omega}$, $d(f(s)) = a$. We define $\varphi : 2^\omega \to 2^\omega$ by

$$\varphi(x) = \lim_{n\to\infty} g(f(x|n)).$$

As both f and g are aligned embeddings, it follows that $K = \varphi[2^\omega]$ is a nonsmooth set. The remainder of the proof is devoted to showing that K satisfies the conclusion of the theorem.

Suppose that $(y_0, y_1) \in K^2 \setminus E_0$. By the symmetry of c, we may assume that $y_0 <_{\text{lex}} y_1$. Setting $x_0 = \varphi^{-1}(y_0)$ and $x_1 = \varphi^{-1}(y_1)$, we see $(x_0, x_1) \in (2^\omega)^2 \setminus E_0$ and $x_0 <_{\text{lex}} x_1$. Fix $s \in 2^{<\omega}$ and $x_0', x_1' \in 2^\omega$ so that

$$x_0 = s^\frown 0^\frown x_0' \text{ and}$$
$$x_1 = s^\frown 1^\frown x_1'.$$

One then sees that

$$y_0 \in \mathcal{N}_{g(f(s^\frown 0))} \subseteq \mathcal{N}_{g(f(s)^\frown 0)} = \mathcal{N}_{g(f(s))^\frown u^0_{|f(s)|}} \text{ and}$$
$$y_1 \in \mathcal{N}_{g(f(s^\frown 1))} \subseteq \mathcal{N}_{g(f(s)^\frown 1)} = \mathcal{N}_{g(f(s))^\frown u^1_{|f(s)|}},$$

since f is an aligned embedding. If we next show that $(y_0, y_1) \in \bigcap_k U^k_{|f(s)|}$, we may conclude that $(y_0, y_1) \in C_{f(s)}$ and thus $c(y_0, y_1) = d(f(s)) = a$, completing the proof.

Towards that end, fix $k \in \omega$ and fix $n \in \omega$ larger than both k and $|s|$ such that $x_0(n) \neq x_1(n)$. Let $i = x_0(n)$ and $\bar{\imath} = x_1(n) = 1 - i$, and write

$$x_0 = t_0{}^\frown i^\frown x_0'' \text{ and}$$
$$x_1 = t_1{}^\frown \bar{\imath}^\frown x_1'',$$

with $|t_0| = |t_1| = n$. Then

$$y_0 \in \mathcal{N}_{g(f(t_0{}^\frown i))} \subseteq \mathcal{N}_{g(f(t_0)^\frown i)} = \mathcal{N}_{g(f(t_0))^\frown u^i_{|f(t_0)|}} \text{ and}$$
$$y_1 \in \mathcal{N}_{g(f(t_1{}^\frown \bar{\imath}))} \subseteq \mathcal{N}_{g(f(t_1)^\frown \bar{\imath})} = \mathcal{N}_{g(f(t_1))^\frown u^{\bar{\imath}}_{|f(t_1)|}}.$$

Thus,

$$(y_0, y_1) \in \bigcap_{j,k<|f(t_0)|} U_j^k.$$

Since $|f(s)| < n \leq |f(t_0)|$ and $k < n \leq |f(t_0)|$, we conclude that $(y_0, y_1) \in U^k_{|f(s)|}$ as required. ⊣

A slightly weaker version of this theorem holds for arbitrary nonsmooth Borel equivalence relations.

COROLLARY 2.5. *Suppose that E is a Borel equivalence relation on a Polish space X, and suppose that $X \notin \mathcal{I}_E$. Suppose further that $c : X^2 \to 2$ is a symmetric Borel function. Then there exists an E-nonsmooth compact set K such that c is constant on $K^2 \setminus E$.*

PROOF. By the Harrington-Kechris-Louveau generalization of the Glimm-Effros dichotomy [1], there exists a continuous embedding $\varphi : 2^\omega \to X$ of E_0 into E. Let $c' : (2^\omega)^2 \to 2$ be defined by $c'(x, y) = c(\varphi(x), \varphi(y))$. Theorem 2.4 then grants an E_0-nonsmooth compact set $K' \subseteq 2^\omega$ which is c'-monochromatic off of E_0. It is then easy to see that $K = \varphi[K']$ satisfies the conclusion of the corollary. ⊣

REMARK 2.6. The corollary is false if we weaken the constraint on c to allow all Baire measurable functions, as the Glimm-Effros embedding might send all of 2^ω to a meager subset of X on which the coloring c is quite pathological. To avoid this, it is sufficient to assume that the coloring function is ω-universally Baire measurable (recall that a set $A \subseteq X$ is *ω-universally Baire* if for every continuous function $\varphi : \omega^\omega \to X$, the set $\varphi^{-1}(A)$ has the Baire property).

We may also use Theorem 2.4 to prove various canonization theorems for classes of relations on Cantor space. As in Corollary 2.5, analogs hold for arbitrary non-smooth Borel equivalence relations on Polish spaces, provided that "Baire property" is replaced with "ω-universally Baire."

COROLLARY 2.7. *Suppose that R is a binary relation on 2^ω with the Baire property. Then there exists a nonsmooth compact set K such that $R \mid K$ agrees with one of $\Delta(K)$, $\leq_{\mathrm{lex}} \mid K$, $\geq_{\mathrm{lex}} \mid K$, and K^2 on $K^2 \setminus E_0$.*

PROOF. By replacing R with $R \cup \Delta(2^\omega)$ if necessary, we may assume that R is reflexive. Define $c : (2^\omega)^2 \to 4$ by

$$c(x, y) = \begin{cases} 0 & \text{if } R \mid \{x, y\} = \Delta\{x, y\}, \\ 1 & \text{if } R \mid \{x, y\} = \leq_{\mathrm{lex}} \mid \{x, y\} \text{ and } x \neq y, \\ 2 & \text{if } R \mid \{x, y\} = \geq_{\mathrm{lex}} \mid \{x, y\} \text{ and } x \neq y, \\ 3 & \text{if } R \mid \{x, y\} = (2^\omega)^2 \mid \{x, y\} \text{ and } x \neq y, \end{cases}$$

so c is clearly symmetric and Baire measurable. By Theorem 2.4, we may find a nonsmooth compact set K such that $K^2 \setminus E_0$ is c-monochromatic. The four possible colors naturally yield the four possibilities for $R \mid K$. ⊣

COROLLARY 2.8. *Suppose that $\leq$ is a Baire property linear order on 2^ω. Then there exists a nonsmooth compact set K such that $\leq \mid K \in \{\leq_{\mathrm{lex}} \mid K, \geq_{\mathrm{lex}} \mid K\}$.*

PROOF. By Corollary 2.7, we may find a nonsmooth compact set K such that $\leq \mid K$ agrees with one of $\leq_{\mathrm{lex}} \mid K$ and $\geq_{\mathrm{lex}} \mid K$ off of E_0. The two cases are handled analogously, so we consider only the $\leq_{\mathrm{lex}}$ case. The proof of Theorem 2.4 also yields an aligned embedding $f : 2^{<\omega} \to 2^{<\omega}$ such that $\varphi : 2^\omega \to 2^\omega$ defined by $\varphi(x) = \lim_{n\to\infty} f(x|n)$ satisfies $\varphi[2^\omega] \subseteq K$.

Then for $(x, y) \in K^2 \setminus E_0$ we have $x \leq y \Leftrightarrow x \leq_{\text{lex}} y$. It thus suffices to check that $x \leq y \Leftrightarrow x \leq_{\text{lex}} y$ for $(x, y) \in K^2 \cap E_0$. Fix $(x, y) \in K^2 \cap E_0$ and suppose without loss of generality that $x <_{\text{lex}} y$. Fix x', y' such that $\varphi(x') = x$ and $\varphi(y') = y$, and note that $x' <_{\text{lex}} y'$. We may then find $z' \notin [x]_{E_0}$ with $x' <_{\text{lex}} z' <_{\text{lex}} y'$ since every E_0-class is $\leq_{\text{lex}}$-dense. Then $x <_{\text{lex}} \varphi(z') <_{\text{lex}} y$ and $\varphi(z') \notin [x]_{E_0}$. Consequently, $x \leq z$ and $z \leq y$, so by transitivity, $x \leq y$ as required. ⊣

This result can be used to canonize linear orders in a slightly different way.

Proposition 2.9. *Suppose that $\leq$ is a Borel linear order of a Polish space X and E is a Borel equivalence relation on X. Then exactly one of the following holds:*

1. *E is smooth.*
2. *There is a nonsmooth compact set K such that $\leq$ orders each $E \mid K$-class in order type $\mathbb{Q}$.*

In particular, a linear order can order each E-class in a scattered way if and only if E is smooth.

Proof. Suppose that E is nonsmooth, and fix a continuous embedding $\varphi : 2^\omega \to X$ of E_0 into E. Pull $\leq$ back through φ and canonize by Corollary 2.8 on a nonsmooth compact set K'. Then $K = \varphi[K']$ is as in the second alternative. ⊣

Theorem 2.4 gives a way of understanding behavior off of E_0, but it is natural to ask what can be said of behavior within E_0-classes. This question is significantly more delicate, and we handle only a special case. We say that a partial order $\preceq$ is an *assignment of linear orders to the classes of* E_0 if $\preceq$ is contained in E_0 and $x \; E_0 \; y \Longrightarrow x \preceq y$ or $y \preceq x$.

Before we canonize such assignments of linear orders, we must first prove a technical proposition.

Proposition 2.10. *Suppose that $\varphi : (2^\omega)^2 \to \omega$ is a Baire measurable function. Then there exists a function $f : (2^{<\omega})^2 \to \omega$ and a nonsmooth compact set $K \subseteq 2^\omega$ such that for all $n \in \omega$, $s, t \in 2^n$, and $x \in 2^\omega$,*

$$s^\frown x, t^\frown x \in K \Longrightarrow \varphi(s^\frown x, t^\frown x) = f(s, t).$$

Proof. We first recursively construct functions $g_n : 2^n \to 2^{<\omega}$ and sequences u_n in $2^{<\omega}$ such that for all $n \in \omega$, $i \in 2$, and $s, t \in 2^n$,

$$g_{n+1}(s^\frown i) = g_n(s)^\frown i^\frown u_n.$$

As we go, we will also build functions $f_n : (2^n)^2 \to \omega$ and sequences $(U_n^k)_{k \in \omega}$ of open dense subsets of 2^ω such that for all $x, y \in 2^\omega$, $n \in \omega$, and $s, t \in 2^n$,

$$\left(\exists z \; (x = g_n(s)^\frown z \wedge y = g_n(t)^\frown z) \wedge (x, y) \in \bigcap_{k \in \omega} U_n^k\right) \Longrightarrow \varphi(x, y) = f_n(s, t).$$

As φ is Baire measurable, we may find $u \in 2^{<\omega}$ such that the map $\varphi_\emptyset : x \mapsto \varphi(x, x)$ is constant on a set C_0 comeager in $\mathcal{N}_u$. Set $f_0(\emptyset)$ equal to this constant value, and fix open dense sets $U_0^k \subseteq 2^\omega$ such that

$$\mathcal{N}_u \cap \bigcap_{k \in \omega} U_0^k \subseteq C_0.$$

To finish this initial step, set $g_0(\emptyset) = u$.

Now suppose that we have completed the construction of g_n. Define for $s, t \in 2^{n+1}$ the map $\varphi_{s,t} : 2^\omega \to 2$ by $\varphi_{s,t}(x) = \varphi(g_n(s|n)^\frown s(n)^\frown x, g_n(t|n)^\frown t(n)^\frown x)$. Since each such map is Baire measurable, we may find $v_n \in 2^{<\omega}$ and C_{n+1} comeager in $\mathcal{N}_{v_n}$ such that each $\varphi_{s,t}$ is constant on C_{n+1}. Denote by $f_{n+1}(s,t)$ this constant, and fix open dense sets $U^k_{n+1} \subseteq 2^\omega$ such that

$$g_n(s|n)^\frown s(n)^\frown x \in \bigcap_{k \in \omega} U^k_{n+1} \Longrightarrow x \in C_{n+1}.$$

Then find $u_n \sqsupseteq v_n$ such that for all $s \in 2^{n+1}$,

$$\mathcal{N}_{g_n(s|n)^\frown s(n)^\frown u_n} \subseteq \bigcap_{j,k<n+1} U^k_j.$$

Predictably, set $g_{n+1}(s) = g_n(s|n)^\frown s(n)^\frown u_n$.

Let $g : 2^{<\omega} \to 2^{<\omega}$ be given by $g(s) = g_{|s|}(s)$, and let $f : (2^{<\omega})^2 \to \omega$ be such that for $s, t \in 2^n$, $f(g(s), g(t)) = f_n(s,t)$. Define $\gamma : 2^\omega \to 2^\omega$ by

$$\gamma(x) = \lim_{n \to \infty} g(x|n).$$

Since g is an aligned embedding, it follows that $K = \gamma[2^\omega]$ is nonsmooth. The remainder of the proof is devoted to showing that f and K satisfy the conclusion of the proposition.

Towards that end, fix $n \in \omega$, $s, t \in 2^n$, and $x \in 2^\omega$ such that $s^\frown x, t^\frown x \in K$. We may then find $m \leq n$, $s', t' \in 2^m$ and $x' \in 2^\omega$ such that $s^\frown x = \gamma(s'^\frown x')$ and $t^\frown x = \gamma(t'^\frown x')$. Then $s^\frown x \in \mathcal{N}_{g(s')} \cap \bigcap_{k\in\omega} U^k_m$ and $t^\frown x \in \mathcal{N}_{g(t')} \cap \bigcap_{k\in\omega} U^k_m$, so $\varphi(s^\frown x, t^\frown x) = f_m(s', t')$, which is what we require. $\dashv$

Remark 2.11. It is easy to check that if $f : (2^{<\omega})^2 \to \omega$ is as in the conclusion of Proposition 2.10, then for all $n \in \omega$, $s, t \in 2^n$, and $u \in 2^{<\omega}$, $f(s,t) = f(s^\frown u, t^\frown u)$.

For notational convenience, given a relation R, we say $s\ R^1\ t$ if $s\ R\ t$, and $s\ R^{-1}\ t$ if $t\ R\ s$. For $x, y \in 2^\omega$, we say $x \leq_0 y$ iff $x\ E_0\ y$ and either $x = y$ or $x(n) < y(n)$, where n is the *last* coordinate on which x and y differ. Then $\leq_0$ is a Borel assignment of linear orders to the classes of E_0 in which each class is ordered like $\mathbb{Z}$.

Theorem 2.12. *Suppose that $\preceq$ is a Baire measurable assignment of linear orders to the classes of E_0. Then there exists a nonsmooth compact set K such that*

$$\preceq | K \in \{(\leq_{\text{lex}} \cap E_0) \mid K, (\geq_{\text{lex}} \cap E_0) \mid K, \leq_0 \mid K, \geq_0 \mid K\}.$$

Proof. By Proposition 2.10, we may assume that $\preceq$ descends to $2^{<\omega}$, i.e., that the linear order given by $s \preceq t \Leftrightarrow s^\frown x \preceq t^\frown x$ is well defined for all $n \in \omega$ and $s, t \in 2^n$, independent of the choice of $x \in 2^\omega$.

The proof will bifurcate: either each rectangle contains a homogeneous rectangle, or some rectangle doesn't. If each rectangle does, we will find a nonsmooth K such that $\preceq | K$ agrees with one of $(\leq_{\text{lex}} \cap E_0) \mid K$ and $(\geq_{\text{lex}} \cap E_0) \mid K$. If, on the other hand, some rectangle does not, we will find a nonsmooth K such that $\preceq | K$ agrees with one of $\leq_0 \mid K$ and $\geq_0 \mid K$.

LEMMA 2.13. *Suppose that* $\preceq$ *is a linear order on* $2^{<\omega}$ *such that for all* $n \in \omega$, $s, t \in 2^n$, *and* $u \in 2^{<\omega}$, *we have* $s \preceq t \Leftrightarrow s^\frown u \preceq t^\frown u$. *Suppose further that for all* $n \in \omega$ *and distinct* $s_0, t_0 \in 2^n$ *there exist* $s_1 \sqsupseteq s_0$, $t_1 \sqsupseteq t_0$ *and* $a \in \{-1, 1\}$ *such that*

$$\forall s_2, t_2 \in 2^{<\omega}\ (|s_2| = |t_2| \text{ and } s_2 \sqsupseteq s_1 \text{ and } t_2 \sqsupseteq t_1) \Longrightarrow s_2 \preceq^a t_2.$$

Then there is an aligned embedding $f : 2^{<\omega} \to 2^{<\omega}$ *such that either*

$$\forall n \in \omega\ \forall s, t \in 2^n\ (f(s) \preceq f(t) \Leftrightarrow f(s) \leq_{\text{lex}} f(t)), \text{ or}$$
$$\forall n \in \omega\ \forall s, t \in 2^n\ (f(s) \preceq f(t) \Leftrightarrow f(s) \geq_{\text{lex}} f(t)).$$

PROOF. We recursively construct functions $g_n : 2^n \to 2^{<\omega}$, $c_n : 2^n \to \{-1, 1\}$, and sequences u_n^i in $2^{<\omega}$ such that for all $n \in \omega$, $i \in 2$, $s \in 2^n$, and $v, w \in 2^{<\omega}$,

1. $|u_n^0| = |u_n^1|$,
2. $u_n^i(0) = i$,
3. $g_{n+1}(s^\frown i) = g_n(s)^\frown u_n^i$,
4. $g_n(s)^\frown u_n^{0\frown} v \preceq^{c_n(s)} g_n(s)^\frown u_n^{1\frown} w$,

Begin by setting $g_0(\emptyset) = \emptyset$, and fix $u_0^0 \sqsupseteq 0$, $u_0^1 \sqsupseteq 1$, and $a \in \{-1, 1\}$ such that for all $t^0 \sqsupseteq u_0^0$ and $t^1 \sqsupseteq u_0^1$ of the same length, $t^0 \preceq^a t^1$. Set $c_0(\emptyset) = a$ and $g_1(i) = u_0^i$.

Now, suppose that we have constructed g_n. Again, repeatedly using the hypotheses of the lemma, we may find $u_n^0 \sqsupseteq 0$, $u_n^1 \sqsupseteq 1$, and $c_n : 2^n \to \{-1, 1\}$ such that for all $s \in 2^n$, $m > n$, and $t^0, t^1 \in 2^m$,

$$(t^0 \sqsupseteq g_n(s)^\frown u_n^0 \text{ and } t^1 \sqsupseteq g_n(s)^\frown u_n^1) \Longrightarrow t^0 \preceq^{c_n(s)} t^1.$$

To continue the construction, let $g_{n+1}(s^\frown i) = g_n(s)^\frown u_n^i$.

Define $g : 2^{<\omega} \to 2^{<\omega}$ by $g(s) = g_{|s|}(s)$, and similarly define $c : 2^{<\omega} \to \{-1, 1\}$ by $c(s) = c_{|s|}$. By Theorem 1.3, we may find an aligned embedding $h : 2^{<\omega} \to 2^{<\omega}$ and $a \in \{-1, 1\}$ such that for all $s \in 2^{<\omega}$, $c(h(s)) = a$. We define $f : 2^{<\omega} \to 2^{<\omega}$ by $f(s) = g(h(s))$. Unfolding the definitions, we see $f(s^\frown 0^\frown t^0) \preceq^a f(s^\frown 1^\frown t^1)$, which is exactly what we require. ⊣

LEMMA 2.14. *Suppose that* $\preceq$ *is a linear order on* $2^{<\omega}$ *such that for all* $n \in \omega$, $s, t \in 2^n$, *and* $u \in 2^{<\omega}$, *we have* $s \preceq t \Leftrightarrow s^\frown u \preceq t^\frown u$. *Suppose further that for some* $n \in \omega$ *there exist distinct* $s_0, t_0 \in 2^n$ *such that for all* $s_1 \sqsupseteq s_0$, $t_1 \sqsupseteq t_0$

$$\exists s_2, s_2' \sqsupseteq s_1\ \exists t_2, t_2' \sqsupseteq t_1\ (s_2 \prec t_2 \text{ and } t_2' \prec s_2').$$

Then there is an aligned embedding $f : 2^{<\omega} \to 2^{<\omega}$ *and* $a \in \{-1, 1\}$ *such that*

$$\forall n \in \omega\ \forall s, t \in 2^n\ (f(s) \preceq f(t) \Leftrightarrow f(s) \leq_0^a f(t)).$$

PROOF. Fix s_0 and t_0 as in the statement of the lemma, and assume without loss of generality that $s_0 <_{\text{lex}} t_0$. We recursively construct functions $g_n : 2^n \to 2^{<\omega}$ and sequences u_n^i in $2^{<\omega}$ such that for all $n \in \omega$, $i \in 2$, and $s, t \in 2^n$,

1. $|u_n^0| = |u_n^1|$,
2. $u_n^0 <_{\text{lex}} u_n^1$,
3. $g_{n+1}(s^\frown i) = g_n(s)^\frown u_n^i$,
4. $g_{n+1}(s^\frown 0) \prec g_{n+1}(t^\frown 1)$.

Begin by setting $g_0(\emptyset) = \emptyset$, and fixing $u_0^0 \sqsupseteq s_0$ and $u_0^1 \sqsupseteq t_0$ with $u_0^0 \preceq u_0^1$. Our hand is now forced: we must set $g_1(i) = u_0^i$.

Now, suppose that we have constructed g_n. Choose $s_{\min}, s_{\max} \in 2^n$ such that $g_n(s_{\min}) = \min_{\preceq} g_n[2^n]$ and $g_n(s_{\max}) = \max_{\preceq} g_n[2^n]$. Note that $g_n(0^\frown s') \prec g_n(1^\frown s')$ for all $s' \in 2^{n-1}$, so we must have $g_n(s_{\min}) \sqsupseteq u_0^0$ and $g_n(s_{\max}) \sqsupseteq u_0^1$. Thus, by the hypotheses of the lemma, we may find $u_n^0 <_{\text{lex}} u_n^1$ such that $g_n(s_{\max})^\frown u_n^0 \prec g_n(s_{\min})^\frown u_n^1$. We then, of course, define $g_{n+1}(s^\frown i) = g_n(s)^\frown u_n^i$. To check that condition 4 still holds at stage $n+1$, simply note that

$$g_{n+1}(s^\frown 0) \preceq g_{n+1}(s_{\max}{}^\frown 0) \prec g_{n+1}(s_{\min}{}^\frown 1) \preceq g_{n+1}(t^\frown 1).$$

We define a function $c : \omega \to \{-1, 1\}$ by

$$c(n) = 1 \Leftrightarrow u_n^0 <_0 u_n^1,$$
$$c(n) = -1 \Leftrightarrow u_n^1 <_0 u_n^0,$$

and choose an increasing sequence $(n_k)_{k\in\omega}$ such that for some $a \in 2$ and all $k \in \omega$, $c(n_k) = a$. Define $h : 2^{<\omega} \to 2^{<\omega}$ inductively by $h(\emptyset) = \emptyset$ and for $s \in 2^k$, $h(s^\frown i) = h(s)^\frown 0^{n_{k+1}-n_k-1\frown} i$. Finally, set $f(s) = g(h(s))$.

Suppose now that $n \in \omega$ and $s <_0 t$ are two elements of 2^n. We can find finite strings $s', t', u \in 2^{<\omega}$ such that

$$s = s'^\frown 0^\frown u, \text{ and}$$
$$t = t'^\frown 1^\frown u.$$

Set $k = |s'|$. Then there is some string $u' \in 2^{<\omega}$ such that

$$f(s) = f(s')^\frown u_{n_k}^0{}^\frown u', \text{ and}$$
$$f(t) = f(t')^\frown u_{n_k}^1{}^\frown u'.$$

By condition 4 of the construction, $f(s')^\frown u_{n_k}^0 \prec f(t')^\frown u_{n_k}^0$, thus $f(s) \prec f(t)$. On the other hand, since $c(n_k) = a$, we have $u_{n_k}^0 <_0^a u_{n_k}^1$, thus $f(s) <_0^a f(t)$. ⊣

It is clear that one of the two lemmas will always apply. As in the preceding arguments, the theorem then follows by considering the image of the limit of the aligned embedding thus created. ⊣

REFERENCES

[1] L.A. Harrington, A.S. Kechris, and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, ***Journal of the American Mathematical Society***, vol. 3 (1990), no. 4, pp. 903–928.

[2] A.S. Kechris, ***Classical descriptive set theory***, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995.

[3] B.D. Miller, *The Kanovei-Zapletal canonization*, ***Set theory and its applications*** (L. Babinkostova, A.E. Caicedo, S. Geschke, and M. Scheepers, editors), Contemporary Mathematics, vol. 533, American Mathematical Society, Providence, 2011, pp. 281–286.

[4] Stevo Todorcevic, ***Introduction to Ramsey spaces***, Annals of Mathematics Studies, vol. 174, Princeton University Press, Princeton, NJ, 2010.

KURT GÖDEL RESEARCH CENTER FOR MATHEMATICAL LOGIC
UNIVERSITY OF VIENNA
WÄHRINGER STRAßE 25, 1090 WIEN, AUSTRIA
Current address: Department of Mathematics, Cornell University, Ithaca, NY 14853, USA
E-mail: clintonc@math.cornell.edu