CHAPTER II

STRESS AND STABILITY

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Introduction

Our basic approach to tensegrity frameworks is through the notion of what we call a stress or self-stress in the framework. These concepts are explained in this chapter, and it ends with what we call the fundamental theorem of stress matrices and applications to some examples. This is the basic tool that can be used to show the stability of many of the tensegrites seen in art as well as the applications to geometry, engineering and physics.

2.1 Configurations and frameworks

Start with a collection of labeled points $p_1, \dots, p_n$ each thought of as a column vector in some Euclidean space $\mathbb{E}^d$, $d = 0, 1, 2, \dots$. We call these the *vertices* of the configuration. Note that two vertices with different labels $i \neq j$ for points $p_i = p_j$ could occupy the same point in $\mathbb{E}^d$. We identify any configuration with $p = [p_1, \dots, p_n]$, a single vector in $\mathbb{E}^{nd}$. To be even more specific, we even think of p as single 1 by dn matrix or column vector. But when we refer to p as a configuration, the underlying ambient space $\mathbb{E}^d$ is always part of its definition, and when we say that p is in $\mathbb{E}^d$, we mean that each p_i is in $\mathbb{E}^d$, for all $i = 1, \dots, n$. When we want to represent such a configuration graphically, we do that with a display of points on the page or try to suggest that the points as in $\mathbb{E}^3$ as in Figure 2.1 below.

We need to consider how to describe the constraints on configurations. One sort of constraint is that some of the the vertices of the configuration could be pinned or held fixed. These are denoted separately, and we use a special symbol to distinguish them as in Figure 2.2.

FIGURE 2.2. The symbol for a pinned vertex of a configuration

FIGURE 2.1. Vertices of a configuration

We wish to consider constraints on certain pairs of vertices of the configuration. The constraints can be described by an underlying finite graph G, without loops or multiple edges, whose vertices are the vertices of the configuration. Each edge of G is one of three types, which are called *cables*, *struts* or *bars*. Each edge of G, whether it is a cable, strut, or bar is called a *member*. We call such a graph, with these three types of edges and two types of vertices, a *tensegrity graph* or a *framework graph*. The set of vertices of G will be denoted by $V(G)$, and the set of members of G will be denoted by $M(G)$. Each bar, cable, and strut is denoted graphically as in Figure 2.3, as a solid line segment, a dashed line segment, and a double line segment, respectively.

The whole configuration p together with the graph G is denoted by $G(p)$ and is called a *framework*. Figure 2.3 shows several examples of such frameworks with various combinations of members and types of vertices.

FIGURE 2.3. Frameworks

When the framework consists only of bars, it is often called a *pin jointed framework* or a *bar-and-joint framework* in the engineering literature. When the framework consists of only cables and struts, or only cables and bars, it is often called a *tensegrity* following R.

Buckminster Fuller, who coined the term after seeing one shown to him by the sculptor Kenneth Snelson, who first discovered and appreciated these sturctures. The word was meant to imply that the structure had both "tension" and "integrity". We shall somewhat cavalierly use the words framework and tensegrity interchangeably, even though the integrity portion of tensegrity implies some sort of rigidity or stability, which may not be present in the structures discussed here. The determination of the rigidity and stability of a given framework is one of the major goals of this book. For a given $G(p)$, it may not be initially obvious whether it is rigid or stable, and so if we are restricted from using the word until we know, it seems somewhat restrictive. In any case, tensegrity in the mathematical literature is the same as a framework as defined above. Also in some places it is insisted that tensegrities not have any two struts or bars at a comnon vertex. Again, this is somewhat unnecessarily restrictive, and here such a restriction will not be enforced. In any case, what has been generally considered as a tensegrity, will be included as a tensegrity here.

2.2 Member and vertex constraints

The notion of a framework as defined in Section 2.1 would be of little interest by itself. Things become interesting when constraints are placed on frameworks relative to a given framework $G(p)$. These constraints are all relative to another configuration q, with the same graph G, in other words another framework $G(q)$. The following are the *standard constraints.*

Vertex constraints: For any pinned vertex i, $q_i = p_i$.

Member constaints:

Cable: For $\{ij\}$ a cable, $|q_i - q_j| \leq |p_i - p_j|$.
Strut: For $\{ij\}$ a strut, $|q_i - q_j| \geq |p_i - p_j|$.
Bar: For $\{ij\}$ a bar, $|q_i - q_j| = |p_i - p_j|$.

All the notation here is standard for vectors. All the vertices are vectors regarded as d-tuples in $\mathbb{E}^d$. The notation $|\dots|$ denotes the Euclidean length of a vector.

So the cable constraint allows the member to decrease in length or stay the same, but not increase in length going from the configuration p to the configuration q.

The strut constraint allows the member to increase in length or stay the same, but not decrease in length going from the configuration p to the configuration q.

The bar constraint forces the member to stay the same length going from the configuration p to the configuration q.

So each pair of vertices of G is either a cable, strut, bar, or a non-member, a non-member having no constraints.

Note also that the members are permitted to pass through each other as in some of the examples of Figure 2.3. When this happens, it is ignored in the constraints as described above. Of course, physically, such intersections or crossings cannot be ignored, but such crossing or intersecting constraints are beyond the scope of the analysis here.

In our graphic representation of frameworks, if two members do intersect but not at a vertex, there will be no vertex (shown as a small circle) at the intersection.

2.3 Rigidity and flexes

In order to state the fundamental concept of rigidity we start with the concept of a flex. Let $p(t) = [p_1(t), \ldots, p_n(t)]$, be a configuration in $\mathbb{E}^d$ for each $0 \leq t \leq 1$ for the vertices of a framework $G(p)$. If each $p(t)$ satisfies the vertex and member constraints of Section 2.2 for $q = p(t)$ and $p(0) = p$, we call $p(t)$ a *flex* of $G(p)$. Figure 2.4 graphically shows some examples of flexes a bar framework in the plane starting as a square.

FIGURE 2.4. Flexes of a framework

In the case when there are no pinned vertices, there are certain flexes that are always present. We say a flex $p(t)$ is *trivial*, if for each $0 \leq t \leq 1$, there is a rigid motion, an orthogonal linear transformation, $A_t : \mathbb{E}^d \to \mathbb{E}^d$, where A_0 is the identity, $A_t(p_i) + b_t = p_i(t)$ and $b_t \in \mathbb{E}^d$. The different, but equivalent, definitions of rigidity differ basically as to the differentiability assumptions of the flex $p(t)$. We say that a flex $p(t) = (p_1(t), \ldots, p_n(t))$ is *continuous* if each of the d coordinates of each of the vertices $p_i(t)$ is continuous in t. Similarly we define $p(t)$ to be *analytic* if each of its coordinates is an analytic function of t.

We say that a framework (or tensegrity) $G(t)$ with n vertices is *rigid* in $\mathbb{E}^d$ if any one of the following properties hold:

Definition 1: Each continuous flex $p(t)$ of $G(p)$ in $\mathbb{E}^d$ is trivial.
Definition 2: Each analytic flex $p(t)$ of $G(p)$ in $\mathbb{E}^d$ is trivial.
Definition 3: There is an $\epsilon > 0$ such that for every configuration of n labeled vertices $q = (q_1, \ldots, q_n)$ in $\mathbb{E}^d$ satisfying the constraints of Section 2.2 for the configuration p and $|p - q| < \epsilon$, then q is congruent to p.

REMARK. We will be concerned with two situations, when there are no pinned vertices in G and when there will be enough of them and the configuration is such that the vertices do not lie in a $(d-2)$-dimentional linear subspace. In this last case, the only trivial flex will be the identity flex that is constant on all the vertices. When there are no pinned vertices, then we must consider the full range of congruences that are possible.

THEOREM 2.1. *All three definitions of rigidity are equivalent.*

Appendix 1 contains a proof of this result, which in turn relies on some basic results from algebraic geometry.

So we can use whichever of these definitions is convenient. If a framework $G(p)$ in $\mathbb{E}^d$ is not rigid, then we say it is *flexible*. In the engineering literature, a flexible framework $G(p)$ is said to be a *finite mechanism*, and we will use that word from time to time as well. The engineering terminology is meant to contrast with the notion of an "infinitesimal" mechanism, which will be defined in Chapter III.

Later there will be several modifiers applied to the word "rigidity", such as infinitesimal, static, global, prestress, and second-order, and these are important notions in their own right. But we feel that this very basic concept is fundamental to all the others. A framework or tensegrity can feel quite "loose" or "floppy", and still be rigid by the definition here. But few people would say that a framework that is flexible by the definition here is "rigid".

None of these definitions explicitly help in determining whether a framework $G(p)$ is rigid or not. This is what makes the theory of these structures interesting and challenging. The analytic definition does allow the possibility of using power series expansions of each of the coordinates, and this can help in rigidity determination.

Notice that the ambient space $\mathbb{E}^d$ is part of the set-up and the consideration of whether the framework $G(p)$ is rigid. For example, the bar framework in Figure 2.5 is easily seen to be rigid in the plane, but it is flexible when it is considered as being in $\mathbb{E}^3$.

FIGURE 2.5. A framework rigid in the plane, but flexible in $\mathbb{E}^3$

2.4 Global rigidity

Although the concept of rigidity as defined in Section 2.2 is quite basic and fundamental, there is another concept that is even simpler and quite natural. We say that a framework (or tensegrity) $G(p)$ in $\mathbb{E}^d$ with n labeled vertices is *globally rigid in* $\mathbb{E}^d$ if for any configuration q in $\mathbb{E}^d$ with corresponding n labeled vertices such that the member constraints of Section 2.2 are satisfied, then the configuration q is congruent to the configuration p.

Some simple examples of frameworks that are globally rigid $\mathbb{E}^3$ are shown in the right of Figure 2.3. The example in Figure 2.6 sits in a 2-dimensional subspace of $\mathbb{E}^3$, but it is still globally rigid in $\mathbb{E}^3$.

FIGURE 2.6. A framework globally rigid in $\mathbb{E}^2$ and in $\mathbb{E}^3$

Note that the condition of global rigidity is considerably stronger than just rigidity. Global rigidity in $\mathbb{E}^d$ implies rigidity in $\mathbb{E}^d$, but there are many examples of frameworks

$G(p)$ that are rigid but not globally rigid in $\mathbb{E}^d$. For example, the bar framework of Figure 2.7 is globally rigid in $\mathbb{E}^2$ but not in $\mathbb{E}^3$. The proofs that these examples have the properties attributed to them are fairly easy, but when we have the tools from later sections, it will be quite a bit easier still.

FIGURE 2.7. A framework globally rigid in $\mathbb{E}^3$ but not in $\mathbb{E}^2$

Another related concept that also comes up in this context, is the following. If $G(p)$ is a framework or tensegrity in $\mathbb{E}^d$ and it is globally rigid in $\mathbb{E}^k \supset \mathbb{E}^d$ for all $k \geq d$, then we say $G(p)$ is *universally globally rigid*. For example, a bar triangle, or more generally when G has all bars between all pairs of vertices, $G(p)$ is univerally globally rigid for all configurations p. The Cauchy polygon tensegrities, Grünbaum polygon tensegrities, and many of the examples in the catalogue to be described later have the universally globally rigid property.

We will develop some tools that can be used to show global rigidity and universal global rigidity in later sections and later chapters.

2.5 Quadratic Energy

Consider a tensegrity graph G with n vertices as fixed, but regard all possible configurations q in $\mathbb{E}^d$ as a single variable in $\mathbb{E}^{nd}$. For each $1 \leq i \leq n$ and $1 \leq j \leq n$ consider a scalar ω_{ij}. We shall call ω_{ij} the *stress coefficient* for $\{ij\}$. This will be unambiguous since we shall always assume that $\omega_{ij} = \omega_{j,i}$ and $\omega_{ij} = 0$ for $\{ij\}$ a non-member of G.

We shall denote this whole collection of stress coefficients as one single vector $\omega = (\dots, \omega_{ij}, \dots)$ and refer to this as the *stress* ω. Note that if G has m members, there are m free choices for the values of a stress ω. So we can represent ω as a vector with m coordinates. This will be done later when we discuss the static theory in Chapter III. But it is quite helpful to consider the stress for non-members to be 0 and to allow both permutations of i and j in the indices.

Especially for tensegrities with cables or struts present, we say that a stress ω is a *proper stress* if

(1) $\omega_{ij} \geq 0$ when $\{ij\}$ is a cable.
(2) $\omega_{ij} \leq 0$ when $\{ij\}$ is a strut.

Notice that there is no condition for bars. The stress for a bar can be positive, negative or zero.

Given a stress ω for a framework graph G, we define the *stress-energy function* or simply the *potential function* $E_\omega : \mathbb{E}^{dn} \to \mathbb{E}^1$ to be

$$E_\omega(q) = \sum_{\{ij\} \in M(G)} \omega_{ij} |q_i - q_j|^2 \tag{2-1}$$

where $M(G)$ are the members of G. There is one term for each member of G. Notice that E_ω is a quadratic function in the coordinates of $\mathbb{E}^{nd}$, and in case there are no pinned vertices, is a quadratic form. In other words, as a function in its coordinates, E_ω is a polynomial, where each term is either a square, or the product of two different coordinates, of the configuration q multiplied by the constant stress coefficient. In case there are pinned vertices, their coordinates act as constants as well. The theory of quadratic forms comes into play in a very useful way in the discussion later.

2.6 Equilibrium stresses

The notion of a stress should rightfully be connected to some particular configuration in order for it to be useful. Since it is used to define a potential function on the space of configurations, it is natural to ask what congurations correspond to critical points. We will be particularly concerned with critical points that are minima of the stress-energy function, but we look at the general situation with any critical point first.

In order to test for critical points, consider a particular configuration p in $\mathbb{E}^d$ with n labeled points, and a particular stress ω defined for those n points. Let $p' = [p'_1, \dots, p'_n]$ denote another configuration in $\mathbb{E}^d$. We think of p' as a direction or velocity in the configuration space $\mathbb{E}^{nd}$, or equivalently, a set of directions one for each vertex of the configuration p. (This is a sort of discrete vector field and is closely related to infinitesimal flexes, which will come up in Chapter III.) We will evaluate the energy potential function on the line $p+tp'$, $t \in \mathbb{E}^1$. However, we note that when the tensegrity graph G has pinned vertices, we never consider configurations where the pinned vertices move. So we say that p' is a *permissable velocity* if $p'_i = 0$ for all pinned vertices i. When the first derivative of $E_\omega(p+tp')$ is zero at $t=0$ for all permissable $p' \in \mathbb{E}^n d$, then p is a *critical point* for E_ω.

Suppose that $\omega = (\dots, \omega_{ij}, \dots)$ is a stress corresponding to the tensegrity graph G. We say that ω is an *equilibrium stress* for the configuration p in $\mathbb{E}^d$ or the framework $G(p)$ if for each vertex i that is not pinned

$$\sum_j \omega_{ij}(p_j - p_i) = 0 \tag{2-2}$$

where the sum is taken over all vertices j, but due to our convention about non-edges having zero stress, it is equivalent to taking the sum over only vertices j of $V(G)$ that have an edge in common with the vertex i. Note that there is no equilibrium condition on the vertices that are pinned. Figure 2.8 graphically shows an equilibrium stress. Note that a negative stress reverses the directed line segment from p_i to p_j, and, of course, the value of each stress rescales the vector in the vector sum in (2-2).

The square in Figure 2.9 is shown with a proper equilibrium stress on all of its members. It turns out that all four cable stresses are $+1$ and the two strut stresses are -1.

THEOREM 2.2. *A stress $\omega = (\dots, \omega_{ij}, \dots)$ is an equilibrium stress for a configuration $p = (p_1, \dots, p_n)$ if and only if p is a critical point for the associated potential function E_ω.*

FIGURE 2.8. An equilibrium stress

FIGURE 2.9. An equilibrium stress in a square framework

PROOF. Let $p' = (p'_1, \dots, p'_n)$ be a configuration of n directions or velocities, each $p'_i \in \mathbb{E}^d$. Then we expand each term of $E_\omega(p + tp')$.

$$\begin{aligned} |p_i + tp'_i - (p_j - tp'_j)|^2 &= |(p_i - p_j) - t(p'_i - p'_j)|^2 \\ &= |p_i - p_j|^2 - 2t(p_i - p_j) \cdot (p'_i - p'_j) + t^2|p'_i - p'_j|^2 \end{aligned}$$

where we are using the standard dot product or inner product of vectors as d-tuples of reals. Then from (2-1) $E_\omega(p + tp')$ is

$$\sum_{\{ij\} \in M(G)} \omega_{ij} \left(|(p_i - p_j) - (p'_i - p'_j)|^2 + 2t(p_i - p_j) \cdot (p'_i - p'_j) + t^2|p'_i - p'_j|^2 \right).$$

From this it is clear that at $t = 0$ the first derivative of $E_\omega(p + tp')$ is 0 if and only if

$$\sum_{\{ij\} \in M(G)} \omega_{ij}(p_i - p_j) \cdot (p'_i - p'_j) = 0. \tag{2-3}$$

So p will be a critical point for $E_\omega(p+tp')$ if and only if (2-3) holds for all permissable p'.

Let j be any vertex of G that is not pinned, and consider only velocities p' where $p'_i = 0$ for all $i \neq j$. Then (2-3) reduces to

$$\sum_{i \in M(G)} \omega_{ij}(p_i - p_j) \cdot (-p'_j) = 0$$

and this must hold for all for all $p'_j \in \mathbb{E}^d$. But if we take $p'_j = -\sum_{i \in M(G)} \omega_{ij}(p_i - p_j)$, we see that the vector sum itself

$$\sum_{i \in M(G)} \omega_{ij}(p_i - p_j) = 0,$$

which shows that if p is a critical point for E_ω then it is an equilibrium configuration for the stress ω.

Conversly suppose that the equilibrium condition (2-2) holds for all vertices j of G that are not pinned. Define $\hat{p}'_j = (0, \dots, 0, p'_j, 0, \dots, 0)$, so $p' = \sum_j \hat{p}'_j$. Since the equilibrium condition (2-2) holds for each j, (2-3) holds for each $\hat{p}'_j$. But the expression on the left in (2-3) is linear in the coordinates of p'. This gives us the equality in (2-3) and finishes the proof.

Remark. We have concentrated on a very particular potential function that is very unusual from the standpoint of physics and engineering. But it is at the heart of all other more physically realistic energy functions. This will be shown carefully in Chapter III. The proof of Theorem 1 above relies on directional derivatives to provide the equilibrium condition. There are several other points of view that could be used just as well. For example, we could directly work with the coordinates of the configurations and calculate the Jacobian.

The conclusion of the Theorem that critical points have an equilibrium stress is quite fundamental and holds in a much greater generality. However, in this chapter we just need it for the quadratic case.

The question of just when a framework $G(p)$ has a non-zero equilibrium stress is important. In principle, the equations (2-2) can be solved if the configuration p is known, but there are other points of view. For example, in the plane there is a graphical method that can be quite useful. This will be explained later. For the following we will only need the definition as presented here.

2.7 The Principle of least work

One of the simplest ways to show that a tensegrity framework is rigid and especially to show that it is globally rigid is to use energy functions, the most basic of which was defined in Section 2.5. The basic principle that is used is often called the principle of least work or Castigliano's principle in the engineering literature. It has been used throughout mathematics at least since the advent of calculus. Here we start with a very special situation, that is nevertheless quite useful and representative of the more general cases that we will discuss later.

THEOREM 2.3. *Let ω be a proper stress for a tensegrity graph G (necessarily with pinned vertices) such that the configuration p in $\mathbb{E}^d$ is the unique minimum for the associated energy function E_ω. Then the tensegrity framework $G(p)$ is globally rigid in $\mathbb{E}^d$.*

PROOF. Suppose that the configuration $q = (\dots, q_i, \dots)$ satifies the tensegrity constraints for $G(p)$ of Section 2.2, where $p = (\dots, p_i, \dots)$. Since $\omega = (\dots, \omega_{ij}, \dots)$ is a proper stress for G, for all ij we have that

$$\omega_{ij}|q_i - q_j| \leq \omega_{ij}|p_i - p_j|.$$

Hence we have

$$E_\omega(q) = \sum_{\{ij\}\in M(G)} \omega_{ij}|q_i - q_j|^2 \leq \sum_{\{ij\}\in M(G)} \omega_{ij}|p_i - p_j|^2 = E_\omega(p).$$

Thus the configuration q is a minimum for the energy function E_ω. Since p is the unique minimum for E_ω, $p = q$, as desired.

We call such a stress ω, as in the hypothesis of Theorem 2.3, a *rigidfying stress* for $G(p)$ in $\mathbb{E}^d$. Note that a rigidifying stress is necessarily an equilibrium stress by Theorem 2.2.

The tensegrities in Figure 2.10 show examples of tensegrities that are globally rigid. The level lines for the positions of the middle vertex for the rigidifying stress of Figure 2.10a are shown. In Figure 2.10b there is a proper stress, where the configuration is a minimum, but it is not unique. The other configuration that satisfies the tensegrity constraints is congruent to the original.

FIGURE 2.10. Globally rigid tensegrity frameworks

One of the most natural applications of the principle of least work is to the case of spider webs. We say that a graph, usually containing some pinned vertices, is a *spider web* or a *spider web graph* if all its members are cables.

PROPOSITION 2.1. *Any proper stress ω for a spider web graph G that is non-zero (i.e. positive) on each member of G and such that every vertex is connected to a pinned vertex by members of G has a unique configuration p such that ω is an equilibrium stress for $G(p)$, and p is the minimum point for the associated energy function E_ω. Thus $G(p)$ is universally globally rigid.*

PROOF. Since each of the terms of E_ω are non-negative, it is clear that $E_\omega(q) \geq 0$ for all configurations q. Choose some fixed configuration $p(0)$. The connectivity condition insures that there a constant $C > 0$ such that when any vertex $|q_i| \geq C$, then $E_\omega(q) \geq E_\omega(p(0))$. Thus the function E_ω has at least one minimum point say at p.

Let q be any configuration that is a critical point for E_ω. Define $q' = p - q$. Then for $0 \leq t \leq 1$,

$$E_\omega(q + t(p-q)) = E_\omega(q + tq') =$$

$$E_\omega(q) + 2t\left[\sum_{\{ij\}\in M(G)} \omega_{ij}(q_i - q_j)\cdot(q'_i - q'_j)\right] + t^2\left[\sum_{\{ij\}\in M(G)} \omega_{ij}(q'_i - q'_j)^2\right].$$

But the middle term is 0, and the last term is strictly positive for $t > 0$, unless $q'_i = q'_j$ for all cables $\{ij\}$. So $p_i - q_i = p_j - q_j$, and $p_i - p_j = q_i - q_j$ for all cables $\{ij\}$. So by the connectivity hypothesis, we inductively show that $p_i = q_i$ for all vertices i. Thus $p = q$ as desired.

2.8 Examples of tensegrities and exercises

2.8.1 In Figures 2.11a and 2.11b the cable lines from the pinned vertices intersect in a point. Show that these tensegrities have a proper non-zero equilibrium stress that is non-zero on all the cables and thus is necessarily (uniformally) globally rigid.

2.8.2 Do the same for Figure 2.11c, which is symmetric about a horizontal line and a vertical line.

2.8.3 Suppose that a tensegrity $G(p)$ such as the one 2.11d, where the underlying graph G is a tree with the end points pinned, no zero length members, and each non-pinned vertex is contained in the relative interior of the convex hull of its neighboring vertices. Show that $G(p)$ has an equilibrium stress that is non-zero on all its cables and hence is universally globally rigid.

2.8.4 Apply the result of Exercise 2.8.3 to tensegrities such as Figure 2.11e, where the upper cable is a portion of a convex polygonal curve with its end points pinned, and a third pinned vertex is joined to all the non-pinned vertices on the polygonal curve. (Hint: The lower pinned vertex can be regarded as several overlapping pinned vertices, each connected to one of the incident cables.)

2.8.5 Determine which of the Figures 2.11f to 2.11l are universally globally rigid, even though they may not have an equilibrium stress that is non-zero on all the cables.

2.9 The stress matrix

Rigidity information related to spider webs is useful, but there is even more rigidity information to be found when there are struts as well as cables in the tensegrity. It is also useful to not allow pinned vertices. This insures that the stress energy function E_ω is a quadratic form. In other words, there are no linear or constant terms in E_ω. So in this case, we can compute the matrix of the quadratic form in terms of the coordinates of the vertices of the configuration $p = (p_1, \ldots, p_n)$.

Let $\{ij\}$ be a member of a tensegrity graph G, and define the stress $\omega(\{ij\}) = (0, \ldots, 0, 1, 0, \ldots, 0)$, where all the coefficients are 0, except for $w_{ij} = 1$. Then for any

FIGURE 2.11. Examples of spider webs in the plane

configuration p in $\mathbb{E}^d$,

$$E_{\omega(\{ij\})}(p) = |p_i - p_j|^2 = (x_i - x_j)^2 + (x_i - x_j)^2 + \dots,$$

where $p_i = (x_i, y_i, \dots)$ and $p_j = (x_j, y_j, \dots) \in \mathbb{E}^d$. Observe that

$$(x_i - x_j)^2 = x_i^2 - 2x_i x_j + x_j^2 = \begin{bmatrix} x_i & x_j \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix}.$$

So we define an n-by-n symmetric matrix Ω_{ij}, where all the entries are 0 except for the entries (i,i) and (j,j), which are 1, and the (ij) and (j,i) entries, which are -1. Then

$$[\,x_1 \quad \dots \quad x_n\,]\,\Omega_{ij} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = (x_i - x_j)^2,$$

and similarly for the other coordinates

$$[\,y_1 \quad \dots \quad y_n\,]\,\Omega_{ij} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = (y_i - y_j)^2,$$

etc. So for an arbitrary stress $\omega = (\dots, \omega_{ij}, \dots)$ we have

$$\begin{aligned} \sum_{\{ij\}\in M(G)} \omega_{ij}(x_i - x_j)^2 &= \sum_{\{ij\}\in M(G)} \omega_{ij}\,[\,x_1 \quad \dots \quad x_n\,]\,\Omega_{ij} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ &= [\,x_1 \quad \dots \quad x_n\,] \left(\sum_{\{ij\}\in M(G)} \omega_{ij}\Omega_{ij} \right) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \end{aligned}$$

and similarly for the other coordinates. We define

$$\Omega = \sum_{\{ij\}\in M(G)} \omega_{ij}\Omega_{ij}$$

as the *stress matrix* corresponding to the stress $\omega = (\dots, \omega_{ij}, \dots)$. It is easy to see that Ω is a symmetric n-by-n matrix and that

(1) When $i \neq j$, the (ij) entry of Ω is $-\omega_{ij}$.
(2) The sum of the row and column entries of Ω is 0.

Note that these conditions determine Ω directly, and for $p = (p_1, \dots, p_n)$ where $p_i = (x_i, y_i \dots)$, $i = 1, \dots, n$,

$$E_\omega(p) = [\,x_1 \quad \dots \quad x_n\,]\,\Omega \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + [\,y_1 \quad \dots \quad y_n\,]\,\Omega \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} + \dots.$$

To compactify the notation even further from linear algebra, if $A = (a_{ij})$ and $B = (b_{ij})$ are two matrices, the *Kronecker product* of A and B is defined as

$$A \otimes B = \begin{bmatrix} a_{1,1}B & a_{1,2}B & \dots \\ a_{2,1}B & a_{2,2}B & \\ \vdots & & \ddots \end{bmatrix}$$

in terms of block matrices. See XX for a discussion of this operation on matrices.

So if we regard a configuration of n points in $\mathbb{E}^d$ as a single dn column vector,

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix},$$

then

$$E_\omega(p) = p^T \Omega \otimes I^d p, \tag{2-4}$$

where I^d is the d-by-d identity matrix and $(\dots)^T$ represents the transpose operation on matrices. Note that $\Omega \otimes I^d$ and $I^d \otimes \Omega$ differ only in a permutations of the rows and columns.

In the case of a spider web with pinned vertices, the energy function had a strict minimum when each vertex was connected to a pinned vertex. But in this case, with no pinned vertices, that cannot happen. For example, condition (2) for a stress matrix implies Ω has the vector of all 1's in its kernel in addition to the zero vector. But if there is a configuration vector $q \in \mathbb{E}^{nd}$ such that $E_\omega(q) = q^T \Omega \otimes I^d q < 0$, then $\lambda q^T \Omega \otimes I^d \lambda q = \lambda^2 q^T \Omega \otimes I^d q \to \infty$ as $\lambda \to \infty$, and there is no minimum for E_ω. In the language of quadratic forms E_ω is not positive semi-definite. Recall that E_ω is *positive semi-definite* if for all $p \in \mathbb{E}^{nd}$, $E_\omega(p) \geq 0$. In any case, it can never turn out that any E_ω is positive definite, since there is always the vector of all 1's in the kernel of Ω.

2.10 Affine transformations

If we wish to show that a tensegrity is rigid and there no pinned vertices, there will always be translations, rotations, and any sort of rigid congruence of $\mathbb{E}^d$ restricted to the vertices of the configuration.

Suppose that a configuration p in $\mathbb{E}^d$ is a critical point for an energy function E_ω, for some stress ω. Then by Theorem 2.2, ω is an equilibrium stress for p.

PROPOSITION 2.1. *A stress ω is an equilibrium stress for the configuration p if and only if*

$$\Omega \otimes I^d p = 0.$$

PROOF. For any symmetric matrix Q, the gradient of quadratic form $p \to p^T Q p$ is the function $p \to 2Qp$. So the result follows from Theorem 2.2.

Alternatively, we calculate

$$\Omega_{ij} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x_i - x_j \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and then

$$\Omega \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \vdots \\ \sum_{\{ij\}\in M(G)} \omega_{ij}(x_i - x_j) \\ \vdots \end{bmatrix}$$

and this implies the result.

Recall that an affine transformation of $\mathbb{E}^d$ is given by

$$p_i \to Ap_i + b, \tag{2-5}$$

where $p \in \mathbb{E}^d$, $b \in \mathbb{E}^d$ is fixed, and A is a d-by-d matrix.

PROPOSITION 2.2. *If $\omega = (\dots, \omega_{ij}, \dots)$ is an equilibrium stress for the configuration $p = (p_1, \dots, p_n)$ in $\mathbb{E}^d$, then ω is also an equilibrium stress for any affine image of p.*

PROOF 1. The condition for an equilibrium stress is equation (2-2) for each vertex i, $\sum_j \omega_{ij}(p_j - p_i) = 0$. Let the affine transformation be given by (2-5). Then for each i we calculate the equilibrium condition for the affine image as

$$\sum_j \omega_{ij}(Ap_j + b - [Ap_i + b]) = \sum_j \omega_{ij}(Ap_j - Ap_i) = A\sum_j \omega_{ij}(p_j - p_i) = 0.$$

PROOF 2. Here we use the stress matrix notation from Proposition 2.1. This is $\Omega \otimes I^d p = 0$. But the affine image can be seen as

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \to \begin{bmatrix} Ap_1 + b \\ Ap_2 + b \\ \vdots \\ Ap_n + b \end{bmatrix} = \begin{bmatrix} A & 0 & & \\ 0 & A & & \\ & & \ddots & \\ & & & A \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} + \begin{bmatrix} b \\ b \\ \vdots \\ b \end{bmatrix} = I^n \otimes Ap + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \otimes b.$$

So when we calculate the equilibrium condition,

$$(\Omega \otimes I^d)(I^n \otimes Ap + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \otimes b) = (I^n \otimes A)(\Omega \otimes I^d)p + \Omega \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \otimes b = 0,$$

where we have used the property of Kronecker products, $(X \otimes Y)(W \otimes Z) = (XW \otimes YZ)$, for matrices X, Y, W, Z, such that the dimensions of the matrices are appropriate for the multiplications indicated.

Notice that orthogonal projections are included as a special case of affine images, and the equilibrium equation in $\mathbb{E}^d$ is equivalent to the equilibrium equation for each of its projections onto the coordinate axes.

2.11 The configuration matrix

It is very helpful, for the calculations to come, to be able to rewrite the equilibrium equations directly in terms of the the stress matrix Ω without having to use the Kronecker product. For any configuration $p = (p_1, \dots, p_n)$, we regard each point $p_i \in \mathbb{E}^d$ as a column vector. We then assemble these into a single d-by-n matrix

$$P = [\, p_1 \quad p_2 \quad \dots \quad p_n \,],$$

which we call the *configuration matrix* P. Furthermore it is convenient to define the following $(d+1)$-by-n matrix

$$\hat{P} = \begin{bmatrix} p_1 & p_2 & \dots & p_n \\ 1 & 1 & \dots & 1 \end{bmatrix}, \tag{2-6}$$

which we call the *augmented configuration matrix* $\hat{P}$. The first row of P and $\hat{P}$ is the row of x-coodinates of the points of the configuration; the second row of P and $\hat{P}$ is the row of y-coodinates, etc. The only difference between P and $\hat{P}$ is the additional row of ones in in $\hat{P}$. With this notation it is clear that if Ω is the stress matrix corresponding to a stress ω, then $P\Omega = 0$ and $\Omega P^T = 0$ are equivalent to the equilibrium conditions. Similarly $\hat{P}\Omega = 0$ and $\Omega\hat{P}^T = 0$ are equivalent to the equilibrium conditions as well. Indeed any n-by-n symmetric matrix Ω that satisfies $\hat{P}\Omega = 0$ for some augmented configuraton matrix will correspond to an equilibrium stress for the corresponding configuration p. Of course, we are often interested in the case when certain of the off-diagonal entries of Ω are 0, not to mention when the sign of other entries are determined.

We next investigate the relation between the affine properties of the configuration p and the augmented configuration matrix P. Recall that the *affine span* of vectors $p_1, \dots, p_n$ is

$$< p_1, \dots, p_n >= \{p_0 \mid p_0 = \lambda_1 p_1 + \dots \lambda_n p_n, \lambda_1 + \dots \lambda_n = 1\}$$

From this the following is clear.

PROPOSITION 2.3. *The affine span of the vertices of the configuration $p = (p_1, \dots, p_n)$ is the same as the linear span of the columns of the augmented configuration matrix $\hat{P}$ intersected with those vectors whose last coordinate is* 1*, thus the dim* $(< p_1, \dots, p_n >) + 1 = rank(\hat{P})$.

With the notation for the configuration matrix, it is possible to understand the effect of an affine transformation. Suppose that A is a d-by-d matrix and $b \in \mathbb{E}^d$, so that an affine transformation is defined as in (2-5). Then the augmented configuration matrix of the affine image is given by

$$\begin{bmatrix} Ap_1 + b & Ap_2 + b & \dots & Ap_n + b \\ 1 & 1 & \dots & 1 \end{bmatrix} = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & \dots & p_n \\ 1 & 1 & \dots & 1 \end{bmatrix}. \tag{2-7}$$

PROPOSITION 2.4. *Suppose that the columns of the $(d+1)$-by-n augmented configuration matrix $\hat{P}$ (2-7) span $\mathbb{E}^{d+1}$. Let q be any configuration whose augmented configuration matrix $\hat{Q}$ is such that the row span of $\hat{Q}$ is contained in the row span of $\hat{P}$. Then $\hat{Q}$ is given by (2-7).*

PROOF. It is clear that $\hat{Q}$ is given by

$$\begin{bmatrix} A & b \\ c & d \end{bmatrix} \begin{bmatrix} p_1 & p_2 & \dots & p_n \\ 1 & 1 & \dots & 1 \end{bmatrix} = \hat{Q},$$

where c is a 1-by-d row, d is a 1-by-1 scalar, and A and b are as in (2-7). But for the last coodinate of of each column of $\hat{Q}$ to be 1 we must have for $i = 1, \dots, n$

$$[\, c \quad d\,] \begin{bmatrix} p_i \\ 1 \end{bmatrix} = [\,1\,]. \tag{2-8}$$

Since the vectors $\begin{bmatrix} p_i \\ 1 \end{bmatrix}$ span $\mathbb{E}^{d+1}$, the only solution to $[\, c \quad d\,] \begin{bmatrix} p_i \\ 1 \end{bmatrix} = [\,0\,]$ is the zero vector. So the only solution to (2-8) is when $[\, c \quad d\,] = [\,0 \quad 1\,]$. This is what is to be proved.

Notice that equation (2-7) gives yet another proof of Proposition 2.2 that an affine map preserves the equilibrium condition for stresses.

2.12 Universal configurations exist

We are now in a position to show how to find universal configurations and to determine when a configuration is universal with respect a given stress ω. Recall that a configuration p is universal with respect to a stress ω if any configuration q that is in equilibrium with respect to ω is an affine image of p.

THEOREM 2.4. *Let Ω be an n-by-n stress matrix such that the configuration $p = (p_1, \dots, p_n)$ in $\mathbb{E}^d$ is in equilibrium with respect to the corresponding stress ω, and the affine span of p is all of $\mathbb{E}^d$. Then*

$$rank(\Omega) \leq n - d - 1, \tag{2-9}$$

and p is universal with respespect to ω if and only if (2-9) holds with equality. Furthermore, there always is a universal configuration $\tilde{p}$ in $\mathbb{E}^k \supset \mathbb{E}^d, k \geq d$ which projects orthogonally onto p.

PROOF. Let $\hat{P} = \begin{bmatrix} p_1 & p_2 & \dots & p_n \\ 1 & 1 & \dots & 1 \end{bmatrix}$ be the augmented configuration matrix corresponding to the configuration p. The rows of $\hat{P}$ are in the co-kernel of Ω, by the equilibrium condition. In other words, $[\, x_1 \quad \dots \quad x_n\,]\,\Omega = 0$ for the x-coordinates of the configuration, and similarly for the other coordinates and the row of 1's. Since the affine span of the vertices of the configuration is all of $\mathbb{E}^d$, by Proposition 2.3, the rank of $\hat{P}$ is

$d+1$ and the rows are independent. Thus the dimension of the co-kernel of Ω is at least $d+1$. This implies the inequality (2-9).

It is also clear that the co-kernel of Ω is the linear span of the entire row space of $\hat{P}$ if and only if (2-9) is an equality. When the row space of $\hat{P}$ is the whole co-kernel of Ω, then the row space of any other augmented configuration matrix, $\hat{Q}$ corresponding to a configuration q in equilibrium with respect to ω, will be contained in the row space of $\hat{P}$. Then Proposition 2.4 implies that q is an affine image of p. Hence p is universal.

If p is not universal, then the rows of $\hat{P}$ do not span the co-kernel of Ω, but it is always possible to add rows to $\hat{P}$ so that (2-9) does hold. This corresponds to the universal configuration $\tilde{p}$ which projects orthogonally onto p.

Theorem 2.4 is almost the complete story for what we want to show about global rigidity. However, the Theorem leaves open the question what to do with configuratins that might be affine images of the starting configuration p. This will be addressed later.

But even if we do have configurations that are affine images of p to contend with, or even if the configuration is not universal with respect to the stress ω, we can still get a lot of information when the stress matrix is positive semi-definite.

We say that a tensegrity $G(p)$, $p = (p_1, \dots, p_n)$ is *unyielding* if any other configuration $q = (q_1, \dots, q_n)$, satisfying the constraints of Section 2.2, must have all those constraints satisfied as equalities. In other words, for all members $\{ij\}$ of G, (not just the bars), $|q_i - q_j| = |p_i - p_j|$.

THEOREM 2.5. *If a tensegrity $G(p), (p_1, \dots, p_n)$ has an equilibrium stress ω with a positive semi-definite stress matrix Ω and all the members G with a non-zero stress, then $G(p)$ is unyielding.*

PROOF. We use a variation of the principle of least work. If the configuration $q = (q_1, \dots, q_n)$ satisfies the tensegrity constraints of Section 2.2, then

$$E_\omega(q) = \sum_{\{ij\} \in M(G)} \omega_{ij} |q_i - q_j|^2 \leq \sum_{\{ij\} \in M(G)} \omega_{ij} |p_i - p_j|^2 = E_\omega(p).$$

where the inequality is strict if any of the tensegrity constraints are strict. But since Ω, and therefore E_ω, is positive semi-definite, we see that $G(p)$ is unyielding, as desired.

2.13 Unyielding and globally rigid examples

We apply the results of the previous sections to some illustrative examples. Consider the configuration of Figure 2.12. We label the vertices as below, and use the stresses as indicated.

Then the stress matrix Ω is the following:

$$\Omega = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

Since $n = 4$, and $d = 3$, Theorem 2.4 implies that the rank of Ω is at most 1. But clearly Ω is not the 0 matrix. So Ω has only one non-zero eigenvector. The trace of Ω, the sum

FIGURE 2.12. An unyielding, globally rigid tensegrity

of its diagonal values, is 4, which is positive. Thus the only non-zero eigenvalue of Ω is 4. Thus Ω is positive semi-definite, and Theorem 2.5 implies that $G(p)$ is unyielding. Since every pair of vertices of G has a cable or strut between them, $G(p)$ is universally globally rigid.

Note also that $G(p)$ is universal with respect to the stress ω. But Theorem 2.4, by itself, does not preclude the possibility that there might be another configuration that is a non-congruent affine image of p with corresponding members of equal length. This is easy to see anyway, but affine transformations are often not a problem; universality of the configuration p and Ω being positive semi-definite are usually the most relevant considerations.

It is possible that there can be a tensegrity that is unyielding, and yet it is not globally rigid. Consider the tensegrity of Figure 2.13, which is two square tensegrities of Figure 2.13, the overlapping vertices are combined into one.

FIGURE 2.13. An unyielding tensegrity,not universal, not globally rigid

We can add the equilibrium stresses, each with its positive semi-definite stress matrix on each side such that we get an equilibrium stress whose stress matrix is positive semi-definite. So this tensegrity is unyielding, but it is also clear that it is not globally rigid, since it is possible to fold one of squares on top of the other in the plane. This folding map is not the restriction of an affine transformation in the plane, so we see that the configuration must not be universal with respect to the positive definite stress indicated (or any other equilibrium stress). But what is the universal configuration? Each square is determined up to congruence by the first example, so the only possiblity is that the universal configuration for the whole tensegrity is when the affine span of the two squares is three dimensional. Such a configuration is shown in Figure 2.14.

FIGURE 2.14. An unyielding tensegrity, universal, not globally rigid

Note that although this framework is universal and the stress matrix is positive semi-definite, it is still not globally rigid. It is not even rigid. There is a flex, where one plane rotates about the other in three-space, that is an affine motion.

It is also possible to combine tensegrities and appropriately chosen equilibrium stresses for each in such a way that on one (or more) of the members the sum of the stresses vanish, and yet still obtain an unyielding tensegrity. For example, the tensegrity of Figure 2.15 combines the equilibrium stress for a square and a rhombus (which is an affine image of a square and thus also has a corresponding proper, equilibrium stress) to get an equilibrium stress with a positive semi-definite stress matrix. Since the stress on a strut of one tensegrity is -1 and the corresponding stress on the cable of the other tensegrity is $+1$, the stresses cancel. Thus we can remove that member and we will still have an unyielding tensegrity for the combination. Note that this combined tensegrity is still not universal for any stress, since its affine span is not three-dimensional.

FIGURE 2.15. An unyielding tensegrity with a missing member

2.14 Small unyielding tensegrities

There are a whole class of unyielding tensegrities that are quite helpful in many situations and seem to come up quite often. These are examples of tensegrities where there are just $d+2$ vertices in $\mathbb{E}^d$. Recall that an n-dimensional *simplex* σ is the convex hull of $n+1$ points $p_1, \dots, p_n$in $\mathbb{E}^d$such that they are affine independent. In other words, no $k+2$ of the points lie in a k-dimensional hyperplane in $\mathbb{E}^d$.

PROPOSITION 2.4. *Suppose that an a-dimensional simplex σ_1 and a b-dimensional simplex σ_2 have a point that is in the relative interior of both simplices. Create a configuration consisting of the vertices of σ_1 and σ_2, and a tensegrity graph G consisting of struts corresponding to all the edges of σ_1 and all the edges of σ_2 and cables connecting each vertex of σ_1 to each of σ_2. This tensegrity $G(p)$ has a proper stress, non-zero on each cable and strut, such that p is a minimum point for the associated quadratic form for the corresponding stress matrix Ω. Thus $G(p)$ is unyielding.*

PROOF. Let $(p_1, \dots, p_{a+1})$ be the vertices of σ_1, and let $(p_{(a+1)+1}, \dots, p_{(a+1)+(b+1)})$ be the vertices of σ_2. Since they share a point each in their relative interiors, there are scalars, all positive, $\lambda_1, \lambda_2, \dots, \lambda_{(a+1)+(b+1)}$ such that

$$\sum_{i=1}^{a+1} \lambda_i p_i = \sum_{i=(a+1)+1}^{(a+1)+(b+1)} \lambda_i p_i, \tag{2-10}$$

and

$$\sum_{i=1}^{a+1} \lambda_i = 1 = \sum_{i=(a+1)+1}^{(a+1)+(b+1)} \lambda_i .$$

The configuration of the vertices of both simplices is $p = (p_1, \dots, p_{(a+1)+(b+1)})$. Define a stress matrix as

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{a+1} \\ -\lambda_{(a+1)+1} \\ \vdots \\ -\lambda_{(a+1)+(b+1)} \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_{a+1} & -\lambda_{(a+1)+1} & \dots & -\lambda_{(a+1)+(b+1)} \end{bmatrix} = \Omega.$$

From this we see that the quadratic form corresponding to Ω is positive semi-definite and that the stress coefficients $\omega_{ij} = \lambda_i \lambda_j$ if i and j are vertices of the same simplex, and $\omega_{ij} = -\lambda_i \lambda_j$ if i and j are vertices of different simplices. (Recall that the off diagonal entries of Ω are the corresponding stress coefficients but with the opposite sign by Condition (1) in Section 2.9.) It is clear that Ω is a symmetric matrix, so by condition (2) for a stress matrix, we only need to check that row and column sums are 0. The row sum can be calculated by multipying Ω on the right by column vector of all one's. But

$$\begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_{a+1} & -\lambda_{(a+1)+1} & \dots & -\lambda_{(a+1)+(b+1)} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$
$$= \sum_{i=1}^{a+1} \lambda_i - \sum_{i=(a+1)+1}^{(a+1)+(b+1)} \lambda_i = 1 - 1 = 0.$$

Hence Ω is a stress matrix for ω by conditions (1) and (2) in Section 2.9. To show that p is a minimum point for E_ω, again we calculate $E_\omega(p)$. We already know that the quadratic form for Ω is positive semi-definite, and thus E_ω is positive semi-definite. Hence p is a minimum point for E_ω if and only if $E_\omega(p) = 0$. Let $(x_1, x_2, \dots, x_{a+b+2})$ be the first coordinates of each point of p. Then from (2-10) we have

$$\begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_{a+1} & -\lambda_{(a+1)+1} & \dots & -\lambda_{(a+1)+(b+1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_{a+b+2} \end{bmatrix}$$
$$= \sum_{i=1}^{a+1} \lambda_i x_i - \sum_{i=(a+1)+1}^{(a+1)+(b+1)} \lambda_i x_i = 0.$$

Applying a similar argument to the other coordinates we see that $E_\omega(p) = 0$. Thus p is a minimum point for E_ω, and $G(p)$ is unyielding by Theorem 2.5.

With Proposition 2.4 in mind we say that a tensegrity, constructed as above with an a-dimensional simplex and a b-dimensional simplex intersecting in their relative interiors, is an (a,b)-*tensesegrity*. Note that any (a,b)-tensegrity is globally rigid since all of the members are either a cable or strut and it is unyielding. Figure 2.16 shows some examples of such unyielding (a,b)-tensegrities in dimensions one, two and three with the (a,b) designation for each.

2.15 Affine motions revisited

We have still not completely dealt with affine motions that arise even when we have a universal configuration for a positive semidefinite stress. We need to understand the nature of affine motions with regard to which pairs of distances are increasing or decreasing. Suppose that we have an affine transformation of $\mathbb{E}^d$given by $p_i \to Ap_i + b$, where A is a d-by-d matrix and $b \in \mathbb{E}^d$ for each $p_i \in \mathbb{E}^d$. We want to determine when the distance between p_i and p_j increases, decreases or stays the same under such a transformation. We do this by calculating the squares of the distances involved and subtracting.

$$\begin{aligned} |(Ap_i + b - (Ap_j + b)|^2 - |p_i - p_j|^2 &= (Ap_i - Ap_j)^2 - (p_i - p_j)^2 \\ &= [A(p_i - p_j)]^2 - (p_i - p_j)^2 \\ &= (p_i - p_j)^T A^T A (p_i - p_j) - (p_i - p_j)^T I^d (p_i - p_j) \\ &= (p_i - p_j)^T [A^T A - I^d](p_i - p_j), \end{aligned} \tag{2-11}$$

FIGURE 2.16. Unyielding globally rigid (a,b) tensegrities

where I^d denotes the d-by-d identity matrix, the squaring operation refers to the dot product and $(\dots)^T$ is the transpose operation. From this calculation we see that the symmetric matrix $A^T A - I^d$ and its associated quadratic form determine when distances increase, decrease or stay the same.

It is quite natural and helpful, if we think in terms of the projective plane (or projective space in dimensions greater than three) that is defined in terms of lines through the origin in $\mathbb{E}^d$. So if we have a configuration $p_1, \dots, p_n$in $\mathbb{E}^d$we say that a *member direction* for the member $\{ij\}$ is the equivalence class determined by $p_i - p_j$, where two directions are equivalent if they are scalar multiples of each other. If $p_i = p_j$ we will say that they do not determine a direction.

In the case $d = 3$, the directions are a standard model for the points of the real projective plane. Let Q be any d-by-d non-zero symmetric matrix. We say that the set of directions defined by

$$C = \{v \in \mathbb{E}^d \mid v^T Q v = 0\}$$

is a conic at infinity. It is clear that C is well-defined since scalar multiples of a vector satisfy the same quadratic equation defining C. So we can say whether a direction or a set of directions lies on a conic at infinity. Note that when Q is definite (positive or negative) the corresponding conic at infinity is empty. It is also possible that Q could determine a single plane through the origin, which is regarded as a projective line in the projective plane of directions, or it could determine two distinct planes through the origin, which is regarded as two projective lines in the projective plane of directions. But generally one would expect that C would be the set of lines from the origin to the points of an ellipse, say, in some plane not through the origin. These cases, of course, are descriptions of the situation in three-space.

For an affine transformation given by $p_i \to Ap_i + b$, we say that a direction given by a vector $v \in \mathbb{E}^d$ is *length preserving* if the terms in equation (2-11) are 0 with $p_i - p_j = v$.

PROPOSITION 2.5. *Suppose that D is a set of directions in $\mathbb{E}^d$. There is an affine transformation that is not a congruences and preserves lengths in all the directions in D if and only if the directions in D lie on a conic at infinity. Furthermore when the directions D do lie on a conic at infinity, there is a continuous flex of all of $\mathbb{E}^d$ that preserves the directions D.*

PROOF. The equation (2-11) shows that an affine transformation determines a conic where $Q = A^T A - I^d$. It is clear that $Q = 0$ if and only if $A^T A = I^d$ which holds if and only if A is an orthogonal matrix. So for an affine transformation that is not a congruence, the directions that are length preserving lie on a conic at infinity.

Conversly suppose that the non-zero symmetric matrix Q determines a conic at infinity. Then for $0 < t < \epsilon$ for some ϵ, tQ determines the same conic at infinity. Then if ϵ is small enough, $I^d + tQ$ is a symmetric matrix and is positive definite. This means that there is a d-by-d matrix A_t such that $A_t^T A_t = I^d + tQ$. See the Appendix for information about quadratic forms of the sort that is needed here. Reading equation (2-11) from the other direction we see that A_t provides the affine transformation.

An example when Proposition 2.5 applies is shown in Figure 2.14. All the members lie in two planes, so the directions of the members lie on two lines at infinity, a conic. This accounts for the affine flex in that case.

A more interesting example is shown in Figure 2.17. Start with a single line segment parallel to the y-z-plane but which intersects the unit circle in the x-y plane at the point $(1, 0, 0)$. Rotate this segement about the z-axis to get several other disjoint line segements as in Figure 2.14a. These segmants will be struts with vertices at their end points. Reflect all the struts about the x-z plane to get another set of struts. Each strut in one family intersects many of the struts in the other family. (These struts are part of two rulings of lines on a hyperboloid of revolution.) Put a vertex at each point of intersection. Connect all the vertices that lie on a strut with cables to the end vertices of the strut. So the lengths of each of the struts and cables are fixed. In other words, this tensegrity is unyielding. But it is possible for two rotate around their common vertex. But notice that the stressed directions of this tensegrity lie on a circle at infinity. So it flexes keeping all the struts and cables at a fixed length. (This example also is described in the book, Geometry and the Imagination, David Hilbert and Cohn-Vosson.)

FIGURE 2.17. The hyperboloid flexible tensegrity

2.16 The fundamental theorem of tensegrity structures

We now put the information we have together to state the basic theorem that allows us to determine rigidity and global rigidity of tensegrities.

THEOREM 2.6. *Suppose a tensegrity $G(p)$, $(p_1, \dots, p_n)$ in $\mathbb{E}^d$, with no pinned vertices, has affine span $\mathbb{E}^d$ and has an equilibrium stress ω with a stress matrix Ω such that*

1. *Each vertex is a vertex of at least one member with a non-zero stress.*
2. *The configuration $(p_1, \dots, p_n)$ is universal with respect to ω. In other words, the rank of Ω is $n - d - 1$.*
3. *The matrix Ω is positive semidefinite.*
4. *The directions of the members that have a non-zero stress do not lie on a conic at infinity.*

Then $G(p)$ universally globally rigid.

PROOF. We may assume without loss of generality that all the members of $G(p)$ have a non-zero stress. By Theorem 2.5, $G(p)$ is unyielding. Since by condition (2), p is universal with respect to ω, if any other configuration $q = (q_1, \dots, q_n)$ is such that $G(q)$ satisfies the tensegrity constraints for p, then q must by an affine image of p. But by Proposition 2.5 every affine image of p that satisfies the equality distance constraints, must be a congruence. Thus $G(p)$ is globally rigid in all dimensions, as was to be shown.

The conditions (1), (2), (3), (4) are so important that we say that any tensegrity $G(p)$ that satisfies them is called *super stable*, a word coined by Alex Tsow, who was a student at Cornell.

We can apply Theorem 2.6 to show that several tensegrities are super stable. For example, we can combine tensegrities of Section 2.14, which are super stable themselves, to get several others that are superstable. Figure 2.18 shows how to combine $(1,1)$ tensegrities to get a tensegrity on five vertices that is super stable.

FIGURE 2.18. Adding tensegrities and stresses to get super stable tensegrities

2.17 Applications of the fundamental theorem: Cauchy polygons

Suppose that $(p_1, \dots, p_n)$ are the vertices in cyclic order of a convex polygon in the plane. Let $\{i, i+1\}$, $i = 1, \dots, n$, indices modulo n, be the cables, and let $\{i, i+2\}$, $i = 1, \dots, n-2$ be the struts. We call this tensegrity a *Cauchy polygon*, $C_n(p)$.

PROPOSITION 2.6. *All Cauchy polygons are super stable.*

PROOF. We proceed by induction on the number of vertices n in the configuration $(p_1, \dots, p_n)$ of the Cauchy polygon, starting with $n = 4$, $C_4(p)$. When $n = 4$ the Cauchy polygon $C_4(p)$ is a $(1,1)$ polygon, and from the discussion in Section 2.4, it has an equilibrium stress with positive semidefinite stress matrix of rank $1 = 4 - 2 - 1$. In the plane, a conic at infinity is just one or two points, and the $(1,1)$ polygon has at least four distinct stressed directions. Thus a $(1,1)$ polygon, which is the same as a Cauchy polygon $C_4(p)$, is super stable.

We now assume that any Cauchy polygon $C_n(p)$, for some $n \geq 4$, is super stable, and we wish to show that any Cauchy polygon $C_{n+1}(p)$ is super stable. Recall $p = (p_1, \dots, p_{n+1})$. Remove p_n from p to get $q = (p_1, \dots, p_{n-1}, p_{n+1})$, and apply the inductive hypothesis to this Cauchy polygon $C_n(q)$. Let $\omega(C_n) = (\dots, \omega_{ij}(C_n), \dots)$, be a proper, non-zero, equilibrium stress for $C_n(q)$. Let $C_4(r)$ be a Cauchy polygon on the four vertices $r = (p_1, p_{n+1}, p_n, p_{n-1})$, and let $\omega(C_4) = (\dots, \omega_{ij}(C_4), \dots)$ be the corresponding proper, non-zero, equilibrium stress for $C_4(r)$. Note that $\{n-2, n+1\}$ is a strut in C_n and it is a cable in C_4. So we can rescale one of the stresses to assume that $\omega_{n-2,n+1}(C_n) = -\omega_{n-2,n+1}(C_4)$. We add these two stresses to get an equilibrium stress $\omega(C_{n+1})$, where $\omega_{ij}(C_{n+1}) = \omega_{ij}(C_n) + \omega_{ij}(C_4)$, when the member $\{ij\}$ lies in both C_n and C_4. When $\{ij\}$ lies in just one of C_n or C_4, then $\omega_{ij}(C_{n+1})$ is just the stress for the graph it lies in.

We see that the quadratic form corresponding to the stress matrix $\Omega(C_{n+1})$ corresponding to the stress $\omega(C_{n+1})$ is clearly positive semidefinite, since it is the sum of two other semidefinite forms corresponding to the stress matrices corresponding to the stresses C_n and C_4.

We now show that the rank of $\Omega(C_{n+1})$ is $n+1-3 = n+2$. In other words we need to show that a universal configuration for $\omega(C_{n+1})$ is $C_n(p)$. Since the quadratic form for $\Omega(C_{n+1})$ is the sum of two positive semidefinite quadratic forms, each of the those forms must themselves be 0 for the configuration p. Thus both universal configurations for $\omega(C_n)$ and $\omega(C_4)$ must have a 3-dimensional affine span. The points of the configurations C_n and C_4 overlap on the points, $p_{n-2}, p_{n-1}, p_{n+1}$, whose affine span is 3-dimensional. Thus the span of the whole universal configuration corresponding to ω is 3-dimensional.

Lastly, we must show that the equilibrium stress ω is proper. In other words the sign of the stresses on the members must be positive for the cables, negative for the struts, and 0 elsewhere. We have arranged that $\omega_{n-2,n+1} = 0$, as desired. All the other members of C_{n+1} are the sum of stresses of the same sign as desired, except for $\omega_{n-1,n+1}$, which is the sum of a positive stress $\omega_{n-1,n+1}(C_n)$ and a negative stress $\omega_{n-1,n+1}(C_4)$. But notice that we have equilibrium at the point p_{n+1} and there are only 3 members incident to the vertex p_{n+1}, coming from the vertices p_1, p_{n-1}, and p_n. But the members $\{1, n+1\}$ and $\{n, n+1\}$ are cables and have a positive stress. If $\omega_{n-1,n+1} \geq 0$, by the convexity of the Cauchy polygon at p_{n+1}, equilibrium could not hold. Thus $\omega_{n-1,n+1} < 0$, as desired. This finishes the proof that Cauchy polygons are super stable.

See Figure 2.19 for a graphic example of this process.

It is especially interesting to specialize to the case when we add further restrictions. For example, if a cable is replaced by a bar, the cable constraint still remains, but often it turns out that although the additional constraints are not needed for the property of being globally rigid, indeed the property of being globally rigid is still quite non-trivial and useful. For example, with a Cauchy polygon as above, replace all the cables by bars, except the one $\{1, n\}$ which remains a cable. If p is the original convex configuration, and q is another configuration that satisfies all the bar constriants, then the strut constaints can be regarded as saying that the internal angle at each point q_i is no smaller than the internal angle at p_i for $i = 2, \dots, n-1$. So if q is not congruent to p, then the cable constriant must be violated. In other words the distance from p_1 to p_n must be less than the distance from q_1 to q_n. This is the infamous 'opening arm' lemma of Cauchy in 1813.

FIGURE 2.19. Adding tensegrities and stresses to get a super stable Cauchy polygon tensegrity