CLASSIFICATION OF FIRST-ORDER FLEXIBLE REGULAR BICYCLE POLYGONS

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Abstract. A bicycle (n,k)-gon is an equilateral n-gon whose k-diagonals are equal. S. Tabachnikov proved that a regular n-gon is first-order flexible as a bicycle (n,k)-gon if and only if there is an integer $2 \leq r \leq n-2$ such that $\tan(\pi/n)\tan(kr\pi/n) = \tan(k\pi/n)\tan(r\pi/n)$. In the present paper, we solve this trigonometric diophantine equation. In particular, we describe the family of first order flexible regular bicycle polygons.

1. Introduction

Sergei Tabachnikov studied in [4] diverse questions related to the problem of how to determine the direction of a bicycle motion from the tracks of the wheels. He calls a closed smooth curve Γ a *bicycle curve*, if there is another closed curve γ, possibly with cusp singularities, such that the information that Γ and γ are the tracks of the front and rear wheels of a bicycle, respectively, is not enough to determine which way the bicycle went. Bicycle curves can be characterized by the property that a pair of points x, y can move around Γ in such a way that the length of the chord $[x,y]$ and the distance of x and y along the curve Γ do not change. Bicycle polygons are discrete analogues of bicycle curves. A *bicycle* (n,k)*-gon* is an n-gon having equal sides, and equal k-diagonals. (A k-diagonal is a diagonal connecting a vertex to its kth neighbor.)

Stanislav Ulam asked whether spheres are the only solids of uniform density which will float in water in equilibrium in any position (problem 19 in "The Scottish Book" [3]). The question, known as the *floating body problem* makes sense in any dimension $d \geq 2$. Bicycle curves are closely related to the two-dimensional floating body problem. Namely, a closed *convex* planar curve is a bicycle curve if and only if its convex hull D is a solution of the floatation problem with a certain density ([1]). Since k-diagonals of a convex bicycle (n,k)-gon cut off from the polygonal domain pieces of equal area, convex bicycle polygons seem to be proper discrete analogues of the solutions of the planar floating body problem as well.

Regular n-gons are bicycle (n,k)-gons for any $2 \leq k \leq n-2$. One can try to find non-regular bicycle polygons by deforming the regular one within the family of bicycle (n,k)-gons. Such a deformation is known to exist in the following two cases:

(1) n is even, k is odd;
(2) $n = 2k$ (see [4] for details).

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S. Tabachnikov (see [4]) also gave the following algebraic characterization of infinitesimally flexible regular bicycle polygons:

Theorem. *A regular n-gon is first-order flexible as a bicycle (n,k)-gon if and only if the equation*

$$\tan(k\pi/n)\tan(r\pi/n) = \tan(\pi/n)\tan(kr\pi/n) \tag{1}$$

has an integer solution r belonging to the range $2 \le r \le n-2$.

We shall refer to equation (1) shortly as the *flex equation*. The aim of the present paper is to describe all integer solutions of the flex equation. To formulate our main result, we start with some simple observations.

Observation 1. Although k and r play different role in the original geometric problem, their role is symmetric in the flex equation. If (n,k,r) is a solution, then $(n, n\pm k, r)$ and (n,r,k) are also solutions, so it is enough to find solutions of the equation which satisfy the inequality $0 \le k \le r \le n/2$.

Observation 2. $(n,0,r)$ and $(n,1,r)$ are solutions for any n and r, call them the *trivial solutions*. Trivial solutions are related to the obvious flexibility of regular n-gons as bicycle $(n,0)$-gons or $(n,1)$-gons. $[(n=2k,k,r),\ r$ is odd$]$ is a *singular solution* (both sides are ∞). Singular solutions correspond to the two classes of deformable regular bicycle (n,k)-gons mentioned above. Trivial and singular solutions can be excluded assuming $1 < k \le r < n/2$.

Observation 3. If $n = 2(k+r)$ and $n|(k-1)(r-1)$, then (n,k,r) is a solution. Indeed, in this case

$$\tan(k\pi/n) = \tan(\pi/2 - r\pi/n) = \cot(r\pi/n), \tag{2}$$

$$\tan(kr\pi/n) = \tan(\pi/2 - \pi/n) = \cot(\pi/n), \tag{3}$$

so for these solutions, both sides of the flex equation are equal to 1.

Theorem 1. *The only integer solutions of the flex equation are the ones listed in the above observations. Equivalently, if $1 < k \le r < n/2$ is an integer solution of the flex equation, then $k + r = n/2$ and n divides $(k-1)(r-1)$.*

2. Proof of Theorem 1

Suppose throuhgout this section that $1 < k \le r < m = n/2$ is a solution of the flex equation (1). Set $\phi = \pi/n$.

2.1. **Verification of Theorem 1 for $n \le 200$.** Let ξ denote the nth primitive root of unity $e^{2\pi i/n}$. Since

$$\tan(s\pi/n) = \frac{(\xi^s - 1)}{i(\xi^s + 1)},$$

(n,k,r) is a solution of (1) if and only if ξ is a root of the polynomial equation

$$(x^{kr}-1)(x-1)(x^k+1)(x^r+1) = (x^{kr}+1)(x+1)(x^k-1)(x^r-1). \tag{4}$$

After simplification (4) reduces to the equivalent equation

$$G(x) = (x^{kr+1}+1)(x^k+x^r) - (x^{kr}+x)(x^{k+r}+1) = 0. \tag{5}$$

Let Φ_n be the nth cyclotomic polynomial. Φ_n is the minimal polynomial of ξ, therefore, ξ is a root of G if and only if Φ_n divides G.

Thus, the question whether a given triple (n,k,r) satifies the flex equation or not is equivalent to a divisibility question for some explicitly computable polynomials with integer coefficients, therefore, it can be answered exactly with the help of a computer algebra system like Maple, Mathematica or MuPAD. For example, the following MuPAD session finds all solutions of the flex equation satisfying $1 < k \leq r < n/2$, $n \leq 200$ and writes the results into the file RESULTS.

```
//MuPAD-SESSION
FILE:=fopen("RESULTS",Write,Text):
for n from 5 to 200 do
  for r from 2 to (n-1)/2 do
    for k from 2 to r do
      Phi:=polylib::cyclotomic(n,x):
      G:=(x^(k*r+1)+1)*(x^k+x^r)-(x^(k*r)+x)*(x^(k+r)+1):
      if divide(G,Phi(x),Rem)=0 then
        fprint(Unquoted,FILE,n,",",k,",",r)
      end_if:
    end_for
  end_for
end_for:
fclose(FILE):
//EVAL
//INPUT
```

Looking at the solutions given in Table 1 we can see that Theorem 1 holds for $n \leq 200$.

2.2. **Reduction to the equation** $\cos\omega_1 + \cos\omega_2 + \cos\omega_3 + \cos\omega_4 = 0$**.** Let l be the residue of kr to base n. Since

$$\tan(l\phi) = \tan(k\phi)\tan(r\phi)/\tan(\phi) > 0,$$

we have $1 \leq l < m$.

Let

$$0 < \alpha \leq \beta \leq \gamma \leq \delta < \pi/2 \tag{6}$$

be the angles $\alpha = \phi$, $l\phi$, $(m-k)\phi$, $(m-r)\phi$ arranged in increasing order. Then these angles satisfy the equation

$$\tan(\alpha)\tan(\beta)\tan(\gamma)\tan(\delta) = 1 \tag{7}$$

TABLE 1. Solutions (n, k, r) of the flex equation satisfying $1 < k \leq r < n/2$, $n \leq 200$.

(24,5,7)	(78,14,25)	(114,20,37)	(144,17,55)	(174,28,59)
(30,4,11)	(80,9,31)	(120,29,31)	(150,26,49)	(176,23,65)
(40,9,11)	(84,13,29)	(120,19,41)	(152,37,39)	(180,19,71)
(42,8,13)	(88,21,23)	(120,11,49)	(154,34,43)	(182,27,64)
(48,7,17)	(90,19,26)	(126,8,55)	(156,25,53)	(184,45,47)
(56,13,15)	(96,17,31)	(130,14,51)	(160,31,49)	(186,32,61)
(60,11,19)	(102,16,35)	(132,23,43)	(168,41,43)	(190,39,56)
(66,10,23)	(104,25,27)	(136,33,35)	(168,29,55)	(192,31,65)
(70,6,29)	(110,21,34)	(138,22,47)	(168,13,71)	(198,10,89)
(72,17,19)	(112,15,41)	(140,29,41)	(170,16,69)	(200,49,51)

Using the identity $\tan(x)\tan(y) = \frac{\cos(x-y)-\cos(x+y)}{\cos(x-y)+\cos(x+y)}$, (7) can be written as

$$\frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{\cos(\alpha-\beta)+\cos(\alpha+\beta)} \cdot \frac{\cos(\gamma-\delta)-\cos(\gamma+\delta)}{\cos(\gamma-\delta)+\cos(\gamma+\delta)} = 1. \tag{8}$$

After simplification this equation turns out to be equivalent to

$$\cos(\alpha+\beta)\cos(\gamma-\delta) + \cos(\alpha-\beta)\cos(\gamma+\delta) = 0, \tag{9}$$

and also to

$$\begin{aligned} &\cos(\alpha+\beta+\gamma-\delta) + \cos(\alpha+\beta-\gamma+\delta) + \\ &\qquad + \cos(\alpha-\beta+\gamma+\delta) + \cos(-\alpha+\beta+\gamma+\delta) = 0 \end{aligned} \tag{10}$$

in virtue of the identity $2\cos(x)\cos(y) = \cos(x+y) + \cos(x-y)$. Set

$$\begin{aligned} \omega_1 &= |\alpha+\beta+\gamma-\delta|, & \omega_2 &= \alpha+\beta-\gamma+\delta, \\ \omega_3 &= \alpha-\beta+\gamma+\delta, & \omega_4 &= \pi - |\pi - (-\alpha+\beta+\gamma+\delta)|. \end{aligned} \tag{11}$$

By (6) and (10), these angles satisfy

$$0 \le \omega_1 \le \omega_2 \le \omega_3 \le \omega_4 \le \pi, \tag{12}$$

and

$$\cos\omega_1 + \cos\omega_2 + \cos\omega_3 + \cos\omega_4 = 0. \tag{13}$$

Equation (13) in ω_i commensurable with π has been completely solved by Ł. Włodarski in [5]. We recall his result.

Theorem 2 (Ł. Włodarski [5])**.** *If the angles $0 \le \omega_i \le \pi$ are commensurable with π and satisfy (13), then $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ either belongs to the infinite families of solutions*

$$\{\lambda, \mu, \pi-\mu, \pi-\lambda\}, \quad (0 \le \lambda \le \mu \le \pi/2), \tag{14}$$

or

$$\left\{\nu, \frac{2}{3}\pi - \nu, \frac{2}{3}\pi + \nu, \frac{1}{2}\pi\right\}, \quad (0 \le \nu \le \pi/3), \tag{15}$$

or is one of the quadruples

$$\left\{\frac{2}{5}\pi, \frac{1}{2}\pi, \frac{4}{5}\pi, \frac{1}{3}\pi\right\}, \qquad \left\{\frac{1}{5}\pi, \frac{1}{2}\pi, \frac{3}{5}\pi, \frac{2}{3}\pi\right\}, \tag{16}$$

$$\left\{0, \frac{1}{5}\pi, \frac{3}{5}\pi, \frac{1}{3}\pi\right\}, \qquad \left\{0, \frac{2}{5}\pi, \frac{4}{5}\pi, \frac{2}{3}\pi\right\}, \tag{17}$$

$$\left\{\frac{2}{5}\pi, \frac{7}{15}\pi, \frac{13}{15}\pi, \frac{1}{3}\pi\right\}, \qquad \left\{\frac{2}{15}\pi, \frac{8}{15}\pi, \frac{3}{5}\pi, \frac{2}{3}\pi\right\}, \tag{18}$$

$$\left\{\frac{1}{15}\pi, \frac{11}{15}\pi, \frac{4}{5}\pi, \frac{1}{3}\pi\right\}, \qquad \left\{\frac{1}{5}\pi, \frac{4}{15}\pi, \frac{14}{15}\pi, \frac{2}{3}\pi\right\}, \tag{19}$$

$$\left\{\frac{2}{7}\pi, \frac{4}{7}\pi, \frac{6}{7}\pi, \frac{1}{3}\pi\right\}, \qquad \left\{\frac{1}{7}\pi, \frac{3}{7}\pi, \frac{5}{7}\pi, \frac{2}{3}\pi\right\}. \tag{20}$$

Since the numbers ω_i defined in (11) satisfy the cosine equation (13), they coincide with one of the quadruples in Theorem 2. We distinguish three main cases. First we study the case, when the quadruple $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ belongs to the family

(14). Then we consider the case when it is a member of the second infinite family (15) and finally we deal with the case when it is one of the sporadic solutions (16-20).

2.3. **Solutions associated to the family (14).** If $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ belongs to the family (14), then $\omega_2 + \omega_3 = 2(\alpha + \delta) = \pi$. In this case, we have

$$\tan(\alpha)\tan(\delta) = \tan(\beta)\tan(\gamma) = 1,$$

and therefore, $\alpha+\delta = \beta+\gamma = \pi/2$. Since $\phi < (m-r)\phi \le (m-k)\phi$, δ equals either $l\phi$ or $(m-k)\phi$. If $\delta = l\phi$, then $\beta+\gamma = \pi/2$ implies $k+r = m$; $\alpha+\delta = \pi/2$ implies $kr + 1 \equiv m = k + r \pmod n$ and we are done. Case $\delta = (m-k)\phi$ is not possible, since then $\alpha + \delta = \pi/2$ would imply $k = 1$.

2.4. **Solutions associated to the family (15).** If $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ is equal to the quadruple (15), then $\omega_1 = \nu$, $\omega_4 = \nu + 2\pi/3$ and $\{\omega_2, \omega_3\} = \{\pi/2, 2\pi/3 - \nu\}$. If we know the values of the numbers ω_i, then we can determine the values of $\alpha, \beta, \gamma, \delta$ using the system of equations (11), which becomes a linear system of equation if we know also the signs of the two quantities $\alpha+\beta+\gamma-\delta$ and $\pi-(-\alpha+\beta+\gamma+\delta)$ in the absolute values. We consider four subcases depending on these signs.

Case 2.4.1. If both $\alpha+\beta+\gamma-\delta$ and $\pi-(-\alpha+\beta+\gamma+\delta)$ are non-negative, then solving (11) gives $\alpha = \pi/8 - \nu/4 \ge \pi/8 - \pi/12 = \pi/24$. Since $\alpha = \pi/n$, we see that $n \le 24$ in this case, however, the theorem has already been verified for $n \le 200$.

Case 2.4.2. If $\alpha+\beta+\gamma-\delta < 0 \le \pi-(-\alpha+\beta+\gamma+\delta)$, then we have $\alpha = \pi/8-3\nu/4$

$$|\beta - \gamma| = |\omega_2 - \omega_3|/2 = |(2\pi/3\nu) - \pi/2|/2 = |2\alpha/3|.$$

However, this is not possible, since $|\beta-\gamma|$ must be an integer multiple of $\alpha = \pi/n$.

Case 2.4.3. If $\alpha + \beta + \gamma - \delta \ge 0 > \pi - (-\alpha + \beta + \gamma + \delta)$, then

$$\alpha = \frac{\nu}{4} - \frac{\pi}{24}, \quad \{\beta,\gamma\} = \left\{\frac{\nu}{4} + \frac{7\pi}{24}, \frac{3\pi}{8} - \frac{\nu}{4},\right\}, \quad \delta = \frac{5\pi}{8} - \frac{3\nu}{4}. \tag{21}$$

Since $\alpha = \pi/n$, ν can be expressed as $\nu = (1/6 + 4/n)\pi$. Substituting this value into (21) we obtain

$$\alpha = \frac{\pi}{n}, \quad \{\beta,\gamma\} = \left\{\frac{\pi}{3} + \frac{\pi}{n}, \frac{\pi}{3} - \frac{\pi}{n},\right\}, \quad \delta = \frac{\pi}{2} - \frac{3\pi}{n}. \tag{22}$$

Since $n\beta/\pi$, $n\gamma/\pi$ and $n\delta/\pi$ are integers, n must be divisible by 6. We may assume that $n = 6s$ with $s > 2$, since the theorem has already been proved for $n \le 200$.Then we have that

$$\{n\beta/\pi, n\gamma/\pi, n\delta/\pi\} = \{m-k, m-r, l\} = \{2s+1, 2s-1, 3s-3\}.$$

In consequence, the numbers $-k$, $-r$ and kr are congruent to a permutation of the numbers 1, -1 and -3 modulo s, however, this is not possible, since $(-k)(-r) = kr$, but no two of the numbers $\{1, -1, -3\}$ has a product congruent to the third one modulo s.

Case 2.4.4. If both $\alpha + \beta + \gamma - \delta$ and $\pi - (-\alpha + \beta + \gamma + \delta)$ are negative, then (11) yields

$$2(\delta - \alpha) = (-\alpha + \beta + \gamma + \delta) - (\alpha + \beta + \gamma - \delta) = (4\pi/3 - \nu) - (-\nu) = 4\pi/3.$$

This equation contradicts $2(\delta - \alpha) < 2\delta < \pi$.

We conclude that there is no solution of the flex equation associated to the second family (15) of the cosine equation (13).

2.5. **Solutions associated to a sporadic quadruple (16-20).** Any of the sporadic solutions of the cosine equation (13) can be written in the form

$$\left(\frac{p_1}{q}\pi, \frac{p_2}{q}\pi, \frac{p_3}{q}\pi, \frac{p_4}{q}\pi\right), \tag{23}$$

where p_1, p_2, p_3, p_4 and q are non-negative integers, and the greatest common divisor of the numerators p_1, p_2, p_3, p_4 is 1. For any of the 10 sporadic solutions, q belongs to the set $\{15\,,\,21\,,\,30\}$. If the numbers $(\omega_1, \omega_2, \omega_3, \omega_4)$ derived from the solution (n, k, r) of the flex equation coincide with the sporadic solution (23) of the cosine equation, then the system of equations (11) takes the form

$$\alpha+\beta+\gamma-\delta = \frac{\hat{p}_1}{q}\pi, \qquad \alpha+\beta-\gamma+\delta = \frac{p_2}{q}\pi,$$
$$\alpha-\beta+\gamma+\delta = \frac{p_3}{q}\pi, \qquad -\alpha+\beta+\gamma+\delta = \frac{\hat{p}_4}{q}\pi,$$

where $\hat{p}_1$ and $\hat{p}_4$ are certain integers. Solving for α gives

$$\frac{\pi}{n} = \alpha = \frac{\hat{p}_1 + p_2 + p_3 - \hat{p}_4}{4q}\pi,$$

which implies, that n must divide $4q \le 120$. As the theorem has already been proved for $n \le 200$, this case cannot lead to solutions of the flex equation not listed in the theorem.

3. Miscellaneous

Proposition 1. *Let $n = 2m \ge 4$ be a fixed natural number. Identify two solutions (n, k_1, r_1) and (n, k_2, r_2) of the flex equation if and only if $k_1 \equiv k_2$ and $r_1 \equiv r_2$ modulo n, and denote by G_n the set of equivalence classes of those solutions for which $k + r \equiv m$ (mod n). Then G_n is an Abelian group with respect to the multiplication $(n, k_1, r_1) * (n, k_2, r_2) = (n, k_1k_2, m - k_1k_2)$. G_n is isomorphic to the direct product of some cyclic groups of order 2. In particular, the order of G_n is a power of 2.*

Proof. Since r can be expressed from the assumption $k+r \equiv m$, a solution $(n, k, r) \in G_n$ is uniquely determined by k. In terms of k, $(n, k, m-k)$ belongs to G_n if and only if $k^2 = ms + 1$ for an integer s, for which $k + s$ is odd. Thus, all we have to check is that if k_1 and k_2 have these properties, then their product k_1k_2 has them as well.

Case 1. If m *is odd,* then the parity condition on s is fulfilled automatically as $k \equiv k^2 = ms + 1 \equiv s + 1 \pmod 2$. On the other hand, if $k_1^2 \equiv k_2^2 \equiv 1 \pmod m$, then $(k_1k_2)^2 \equiv 1 \pmod m$ as well.

Case 2. If m *is even,* then k must be odd, and s must be even. If $k_1^2 = ms_1 + 1$ and $k_2^2 = ms_2 + 1$ are odd numbers with s_1 and s_2 even, then $(k_1k_2)^2 = ms_3 + 1$, where $s_3 = ms_1s_2 + s_1 + s_2$, k_1k_2 is odd and s_3 is even. □

Remark. There is a general result of J.H. Conway and A.J. Jones ([2]) about trigonometric diophantine equations of the form

$$R(r_1, \ldots, r_k, \operatorname{trig}_1(\pi\theta_1(r_1, \ldots, r_k)), \ldots, \operatorname{trig}_n(\pi\theta_n(r_1, \ldots, r_k))) = 0,$$

where $R, \theta_1, \ldots, \theta_n$ are rational functions with rational coefficients, $\mathrm{trig}_1, \ldots, \mathrm{trig}_n$ are standard trigonometric functions from the set $\{\cos, \sin, \tan, \cot, \sec, \mathrm{cosec}\}$. The equation is to be solved in integers for some of the r_i and rational numbers for the others. They prove that any such trigonometric diophantine equation is equivalent to an ordinary diophantine equation, which is an equation of the same type, but with no trigonometric functions involved. In general, the solution of the resulting ordinary diophantine equation can be arbitrarily hard. From this viewpoint, a Conway-Jones-type reduction of the flex equation with the help of the result of Ł. Włodarski was given in subsection 2.2. All the other subsections were devoted to the solution of the obtained system of ordinary diophantine equations.

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