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Chapter 2: Basic Concepts

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1. Introduction:

We present here some of the fundamental ideas used in the analysis of "rigid structures". For us the structures that we will focus on are what have come to be called frameworks in Euclidean space, which can be thought of as a collection of rigid bars connected at their end points with joints that allow the bars to make any angle with each other.

The joints are mathematical points, and the bars are simply certain of the pairs of the joints. Our point of view treats frameworks as mathematical geometrical objects, and our results are theorems in Geometry. However much of the motivation and inspiration has come from Structural Engineering and Physics. Indeed, one can reinterpret many of the results in terms of forces being the basic object as in Physics and Mechanics. We regard the the treatment here as being more fundamental, and we derive forces as mathematical objects, starting only from basic Euclidean Geometry and Linear Algebra.

It turns out that there are several notions of the rigidity of frameworks. The basic notion of rigidity is that the framework does not flex. That is, there is no motion of the joints that preserve the lenths of the bars other than the restriction of a rigid congruence of all of Euclidian space.

Another important kind of rigidity is infinitesimal rigidity. An infinitesimal flex is a vector assigned to each joint in such a way that if the joints move with velocities equal to the given vectors, then the first derivative of the (square of the) bar lengths is zero. A framework is called infinitesimally rigid if the only infinitesimal flexes are those that extend to an infinitesimal flex of the entire Euclidean space. Roughly speaking, infinitesimal rigidity is the linearized form of rigidity.

A basic goal of this chapter is to give three proofs that an infinitesimally rigid framework is rigid. (The converse is false.)

Yet another form of rigidity is the notion of static rigidity, a form of rigidity fimiliar to structural engineers. If vectors are assigned to the joints of a framework in such a way that their sum is zero and such that there is no net angular velocity, then we say that this set of vectors, now thought of as forces, is in equilibrium. Such a set of equilibrium forces is said to be resolved if there are scalars that can be assigned to each bar such that at each joint the vector sum of the scalars times the bars is equal to the given force at that joint. The framework is called staticly rigid if every set of equilibrium forces can be resolved. Our next fundamental result is that static and infinitesimal rigidity are duals, and thus they are equivalent.

Throughout it also helpful to extend the notion of a framework to include cables and (what have come to to be called) struts between the joints. Cables allow the corresponding joints to decrease or stay the same distance apart, but they do not allow the joints to increase their distance. Struts do not allow their joints to decrease their distance. Must of the ideas for bar frameworks extend to these frameworks, which are called tensegrity frameworks, and we have included them here so that we can compare the proofs of infinitesimal rigidity implies rigidity.

We start with a discussion of the classification of the congruences of Euclidean space as well as basic linear algebra to set the notaion as well as to motivate and to calculate the notion infinitesimal congruences. These serve to define the trivial infinitesimal flexes that extend to all of Euclidean space.

2. Linear Algebra: Basic Definitions

-May governments, linear algebra, and the following notation stay out of your way but provide the greatest possible service.-

We use the following notation, where points in real d-dimensional space are written as column vectors.

$$\mathbb{R}^d = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix} \mid x_i \text{ real},\ i = 1,\ldots,d \right\}.$$

For $p, q \in \mathbb{R}^d$, $p = \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix}$, $q = \begin{bmatrix} y_1 \\ \vdots \\ y_d \end{bmatrix}$ define

$$p \cdot q = x_1y_1 + \ldots + x_dy_d \equiv \text{the (standard) \underline{inner product}},$$

$$p \pm q = \begin{bmatrix} x_1 \pm y_1 \\ \vdots \\ x_d \pm y_n \end{bmatrix}, \quad tp = \begin{bmatrix} tx_1 \\ \vdots \\ tx_d \end{bmatrix}, \quad \text{for } t \in \mathbb{R} = \mathbb{R}^1 = \text{reals}.$$

$$|p'| = \sqrt{p \cdot p} \equiv \text{the \underline{norm} of } p \cdot p,$$

$$|p - q| = \sqrt{(p - q)\cdot(p - q)} \equiv \underline{\text{distance from}}\ p\ \underline{\text{to}}\ q$$

$$= \sqrt{(x_1 - y_1)^2 + \ldots + (x_d - y_d)^2}\ .$$

We abbreviate $p \cdot p \equiv p^2$.

Exercise 2.1: Show that the inner product can be written in terms of the norm as follows:

$$p \cdot q = \tfrac{1}{2}(|p|^2 + |q|^2 - |p - q|^2).$$

We call this the <u>polarization identity</u>.

$L: \mathbb{R}^d \to \mathbb{R}^n$ is called a linear map (or linear function or linear transformation) if for all $p, q \in \mathbb{R}^d$, $t \in \mathbb{R}^1$

$$L(tp + q) = tL(p) + L(q).$$

L is called an affine map (or affine linear function or affine transformation) if for all $p, q \in \mathbb{R}^d$, $t \in \mathbb{R}^1$,

$$L(tp + (1-t)q) = tL(p) + (1-t)L(q).$$

Proposition 2:1: $L: \mathbb{R}^d \to \mathbb{R}^n$ is an affine map if and only if there is a linear map $\hat{L}: \mathbb{R}^d \to \mathbb{R}^n$ such that for all $p \in \mathbb{R}^d$

$$L(p) = \hat{L}(p) + L(0).$$

The proof is left as an exercise. Note the vector $0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

3. Matrices

A matrix A is an n by d rectangular array of real numbers, which we denote by $A = (a_{ij})$ and

$$A = A_{n \times d} = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1d} \\ \vdots & & \\ a_{n1} & a_{n2} \cdots & a_{nd} \end{bmatrix}.$$

If two matrices $A_{n \times d}$ and $B_{d \times m}$ are such that the number of columns of A is the same as the number of rows of B, then we define the matrix product $C = (c_{ij})$, denoted by AB, by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{id}b_{dj} ,$$

where $C = C_{n\times m}$ is an n by m matrix.

Note that we can now regard our column vectors in $\mathbb{R}^d$ as d by 1 matrices.

For any matrix $A = (a_{ij})$ define the <u>transpose</u> of $A = A_{n\times d}$, written A^T, by

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ \vdots & & & \\ a_{1d} & \dots & & a_{nd} \end{bmatrix} .$$

Thus using this notation we can write the inner product as a matrix product. I.e., for $p, q \in \mathbb{R}^d$

$$p \cdot q = p^T q.$$

It is easy to check that if $A_{n\times d}$ and $B_{d\times m}$ are matrices then

$$(AB)^T = B^T A^T ,$$

and if A and B have the same number of rows and columns (respectively),

$$A+B = (a_{ij}+b_{ij}),$$

and

$$(A+B)^T = A^T+B^T .$$

The following is a standard result in linear algebra.

<u>Proposition 2.2</u>: Any linear map $L: \mathbb{R}^d \to \mathbb{R}^n$ can be written uniquely as

$$L(p) = Ap$$

for all $p \in \mathbb{R}^d$, where A is an n by d matrix.

<u>Corollary 2.3</u>: For matrices $A_{n\times d}$, $B_{d\times m}$, $C_{m\times r}$,

$$(AB)C = A(BC).$$

4. Congruences

Basic to the study of rigidity is the notion of a <u>congruence</u>. A function $h: \mathbb{R}^d \to \mathbb{R}^d$ is called a <u>congruence</u> if distances between points are preserved. This means that for all $p, q \in \mathbb{R}^d$

$$|h(p) - h(q)| = |p - q|.$$

For example, for any fixed $p_0 \in \mathbb{R}^d$, the following function, called a <u>translation</u> is a congruence.

$$h(p) = p + p_0.$$

A square matrix $A_{d\times d}$ is called <u>orthogonal</u> if

$$A^T A = AA^T = I,$$

where $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots \\ & & & 0 \\ 0 & & 0 & 1 \end{bmatrix}$ is the d by d <u>identity matrix</u>. Note for $p, q \in \mathbb{R}^d$

$$(Ap)\cdot(Aq) = (Ap)^T Aq = p^T A^T Aq = p^T Iq = p^T q = p\cdot q.$$

Thus orthogonal matrices "preserve" the inner product.

Let A be any orthogonal matrix, and let $p_0 \in \mathbb{R}^d$ be any fixed vector. Define an affine linear function $L: \mathbb{R}^d \to \mathbb{R}^d$ by

$$L(p) = Ap + p_0, \text{ for all } p \in \mathbb{R}^d.$$

Then

$$\begin{aligned} |L(p) - L(q)|^2 &= ([Ap + p_0] - [Aq + p_0])^2 \\ &= (Ap - Aq)^2 \\ &= (A[p - q])^2 \\ &= (p - q)^2 \\ &= |p - q|^2. \end{aligned}$$

Thus $|L(p) - L(q)| = |p - q|$ and L is a congruence. We will show that such functions are the only congruences.

Suppose $L: \mathbb{R}^d \to \mathbb{R}^d$ is a linear function such that $L(p)^2 = p^2$, for all $p \in \mathbb{R}^d$. Note that by the polarization identity for all $p, q \in \mathbb{R}^d$

$$L(p)\cdot L(q) = \tfrac{1}{2}(L(p)^2 + L(q)^2 - L(p - q)^2) = p\cdot q.$$

By Proposition 2.2 $L(p) = Ap$ for some d by d matrix A. A is orthogonal, since for all $p, q \in \mathbb{R}^d$

$$p \cdot q = Ap \cdot Aq = p^T A^T Aq = p^T Iq, \text{ or } p^T (A^T A - I)q = 0.$$

But if for any matrix B, for all $p, q \in \mathbb{R}^d$, $p^T Bq = 0$, then $B = 0$. (E.g. take p, q to be vectors with all zeros except in the i-th and j-th position respectively. Then $p^T Bq$ is the i,j-th entry of B, which must be 0.) Thus $A^T A = I$. Since A is a square matrix $A^T = A^{-1}$, the inverse of A, and thus $AA^T = I$.

<u>Proposition 2.4</u>: Let $h: \mathbb{R}^d \to \mathbb{R}^d$ be any congruence. Then there is an orthogonal matrix A such that for all $p \in \mathbb{R}^d$

$$h(p) = Ap + h(0).$$

<u>Proof</u>: Define the function $L: \mathbb{R}^d \to \mathbb{R}^d$ by

$$L(p) = h(p) - h(0).$$

Then for all $p, q \in \mathbb{R}^d$

$$L(p) \cdot L(q) = \tfrac{1}{2}[L(p)^2 + L(q)^2 - (L(p) - L(q))^2]$$

$$= \tfrac{1}{2}[(h(p) - h(0))^2 + (h(q) - h(0))^2 - (h(p) - h(q))^2]$$

$$= \tfrac{1}{2}[p^2 + q^2 - (p - q)^2]$$

$$= p \cdot q.$$

Thus *if* L were linear it would correspond to an orthogonal matrix and we would be done. To show L is linear consider any $p, q \in \mathbb{R}^d$, $t \in \mathbb{R}^1$.

$$[tL(p) + L(q) - L(tp + q)]^2$$

$$= t^2L(p)^2 + L(q)^2 + L(tp + q)^2 + 2tL(p)\cdot L(q) - 2tL(p)\cdot L(tp + q) - 2L(p)\cdot L(tp + q)$$

$$= t^2p^2 + q^2 + (tp + q)^2 + 2tp\cdot q - 2tp\cdot (tp + q) - 2p\cdot (tp + q)$$

$$= [tp + q - (tp + q)]^2$$

$$= 0.$$

Thus L is linear and corresponds to an orthogonal matrix as desired.

Remark 2.5: It is clear, without going into detail about the computation above, that the polarization identity and the inner product preserving nature of L allow one to remove L from the bracket above.

See Elmer G. Rees, Notes on Geometry, Springer Verlag (1983), pages 1-11 for another treatment of this subject.

5. Subspaces

Recall the usual definition a linear subspace $X \subset \mathbb{R}^d$.

X is a <u>linear subspace</u> if for all $p, q \in X$, $t \in \mathbb{R}^1$, $tp + q \in X$.

X is an <u>affine subspace</u> if for all $p, q \in X$, $t \in \mathbb{R}^1$, $tp + (1-t)q \in X$.

The following is an easy result from linear algebra.

<u>Proposition 2.6</u>: $X \subset \mathbb{R}^d$ is an affine subspace if and only if $X = \hat{X} + p_0 = \{ p + p_0 \mid p \in \hat{X} \}$ for a linear subspace $\hat{X}$ of $\mathbb{R}^d$.

The <u>affine span</u> of a set S, denoted by $< S >$, is defined by

$$< S > = \{ t_1 s_1 + \ldots + t_n s_n \mid s_i \in S,\ i = 1, \ldots, n,\ t_1 + \ldots + t_n = 1 \}.$$

Exercise 2.2: Show that $< S >$ is an affine subspace of $\mathbb{R}^d$.

<u>Proposition 2.7</u>: Let $S \subset \mathbb{R}^d$ be any set. Let $h: S \to \mathbb{R}^d$ be any function such that for all $p, q \in S$

$$|h(p) - h(q)| = |p - q|.$$

Then h extends uniquely to an affine linear function $\hat{h}: < S > \to \mathbb{R}^d$ such that for all $p, q \in < S >$

$$|\hat{h}(p) - \hat{h}(q)| = |p - q|.$$

Furthermore, $\hat{h}$ extends to a congruence of $\mathbb{R}^d$, but not uniquely if $< S > \neq \mathbb{R}^d$.

Proof: Let $p_0 \in S$. Applying Proposition 2.4 we see that $< S > - p_0 = \{s - p_0 \mid s \in S\}$ is a linear subspace of $\mathbb{R}^d$. Let $p_1 - p_0, \ldots, p_k - p_0$ be a basis for $< S > - p_0$. See Halmos (xx), for the definition of a basis and linear independence. For any $s \in < S >$, we have $s - p_0 = t_1(p_1 - p_0) + \ldots + t_k(p_k - p_0)$, for some $t_1, \ldots, t_k \in \mathbb{R}^1$. Then

$$s = (1 - [t_1 + t_2 + \ldots])p_0 + t_1 p_1 + \ldots + t_k p_k \,.$$

The coordinates $t_1, \ldots, t_k$, and $t_0 = 1 - (t_1 + \ldots t_k)$ are unique since $p_1 - p_0, \ldots, p_k - p_0$ are independent.

For $s \in < S >$ as above, define

$$\hat{h}(s) = t_1 h(p_1) + t_2 h(p_2) + \ldots + t_k h(p_k).$$

We check that $\hat{h}$ is a congruence. Let $p, q \in < S >$ be arbitrary. Write $p = t_0 p_0 + t_1 p_1 + \ldots + t_k p_k$, $q = u_0 p_0 + u_1 p_1 + \ldots + u_k p_k$, where $t_0 + t_1 + \ldots + t_k = u_0 + u_1 + \ldots + u_k = 1$.

$$[\hat{h}(p) - \hat{h}(q)]^2$$
$$= [(t_0 - u_0)h(p_0) + (t_1 - u_1)h(p_1) + \ldots + (t_k - u_k)h(p_k)]^2$$

$$= [(t_1 - u_1)(h(p_1) - h(p_0)) + \ldots + (t_k - u_k)(h(p_k) - h(p_0))]^2$$

$$= [(t_1 - u_1)(p_1 - p_0) + \ldots + (t_k - u_k)(p_k - p_0)]^2$$

$$= [(t_0 - u_0)p_0 + (t_1 - u_1)p_1 + \ldots + (t_k - u_k)p_k]^2$$

$$= (p - q)^2,$$

where the h's are removed by the polarization identity again.

$\hat{h}$ is unique and is an affine map by an argument similar to the one used in the proof of Proposition 2.3. By extending an orthonormal basis for $\langle S \rangle - p_0$ to all of $\mathbb{R}^d$ it is easy to extend $\hat{h}$ to a congruence of $\mathbb{R}^d$.

Problems:

Problem 2.2: Let p_1, p_2, p_3, p_4 be four distinct point in $\mathbb{R}^3$ such that $|p_1 - p_2| = |p_3 - p_4|$, $|p_2 - p_3| = |p_1 - p_4|$.

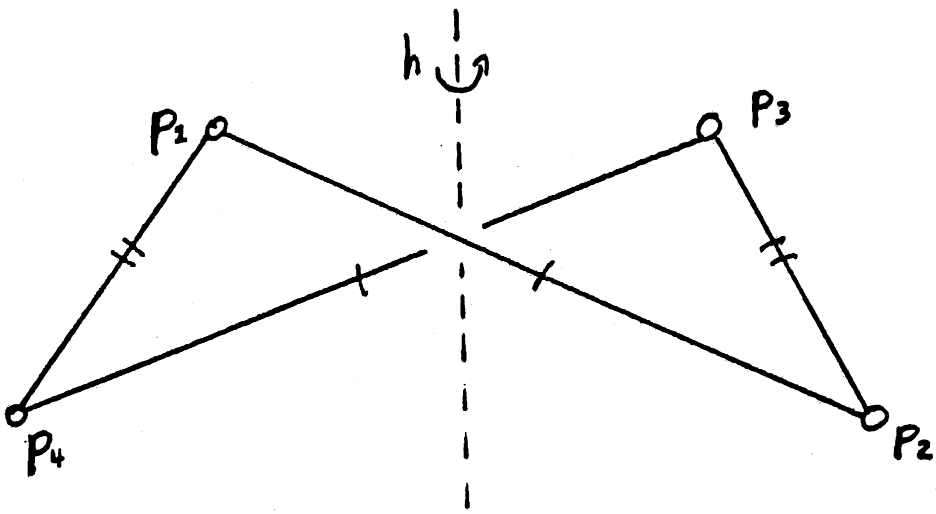


Figure 2.1

(a) Show that there is a congruence $h: \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(p_1) = p_3$ and $h(p_2) = p_4$.

(b) If $\langle p_1, p_2, p_3, p_4 \rangle = \mathbb{R}^3$ show that h fixes an (affine) line in $\mathbb{R}^3$.

(c) If $\langle p_1, p_2, p_3, p_4 \rangle$ is an affine plane, show that h can be chosen to fix all the points on a line.

(d) In both cases (a) and (b) above, show h is 180° rotation about a fixed line.

Problem 2.3: Let $p = (p_1, p_2, p_3, p_4, p_5)$ be five points, a *pentagon*, in $\mathbb{R}^3$, such that $|p_i - p_{i+1}| = |p_{i-1} - p_i|$ and $|p_i - p_{i+2}| = |p_{i-2} - p_i|$ for $i = 1,\ldots,5$, indices modulo 5. Show that p is a planar pentagon, one of the two examples in Figure 2.2.

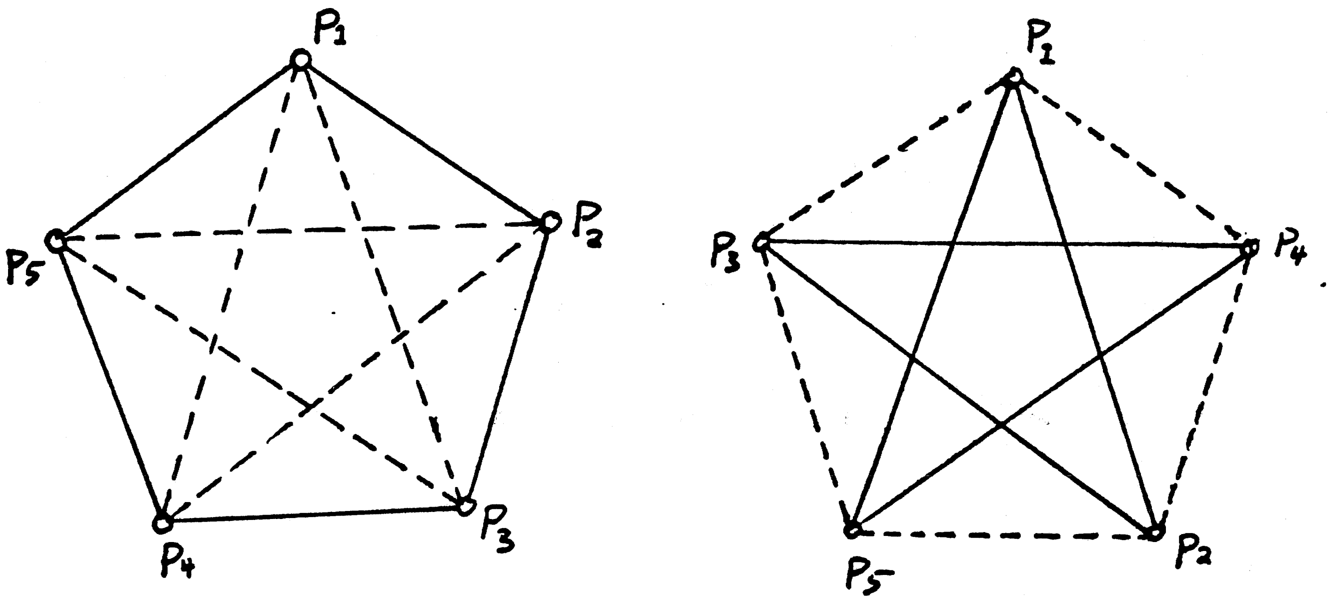


Figure 2.2

Problem 2.5: Let $h: \mathbb{R}^2 \to \mathbb{R}^2$ a function such that if $p, q \in \mathbb{R}^2$ and $|p - q| = 1$, then $|h(p) - h(q)| = |p - q| = 1$. Show h is a congruence.

Problem 2.6: Consider six positive real numbers $d_{12} = d_{21}$, $d_{13} = d_{31}$, $d_{14} = d_{41}$, $d_{23} = d_{32}$, $d_{34} = d_{43}$. Suppose the following version of the triangle inequality holds for every three of the d_{ij}'s with exactly three distinct indices $i \neq j \neq k \neq i$, $i, j, k \in \{ 1, 2, 3, 4 \}$,

$$d_{ij} + d_{jk} \geq d_{ik} .$$

Is it necessarily true that there are four points $p_1, p_2, p_3, p_4 \in \mathbb{R}^3$ such that $|p_i - p_j| = d_{ij}$ for all $i \neq j$, $i, j \in \{ 1, 2, 3, 4 \}$.

Problem 2.7: Let $p_1, p_2, p_3, q_1, q_2, q_3$ be six distinct points in $\mathbb{R}^3$. We regard these as the six points of a not necessarily convex, possibly self-intersecting octahedron. Suppose opposite edges are of equal length. I.e.,

$$|p_i - p_j| = |q_i - q_j|, \ i \neq j, \ i, j \in \{ 1, 2, 3 \}$$

and

$$|p_i - q_j| = |q_i - p_j|, \ i \neq j, \ i, j \in \{ 1, 2, 3 \}.$$

Show that the octahedron is symmetric about a point, line or plane, and find non-trivial examples of each.

6. Congruent Motions

Proposition 2.3 classified the congruences of $\mathbb{R}^n$. We now wish to classify the derivatives of "congruent motions". These derivatives, thought of as vector fields defined on all of $\mathbb{R}^n$, are infinitesimal motions that every body has, and we need to know what they are in order to make a proper definition of infinitesimal rigidity later.

For every $t \in [0,1]$ suppose $h_t: \mathbb{R}^d \to \mathbb{R}^d$ is a congruence, and $(p,t) \mapsto h_t(p)$ is continuously differentiable (abbreviated as C^1) simultaneously in p and t. Also for all $p \in \mathbb{R}^d$, $h_0(p) = p$. We call such an h_t a <u>congruent motion</u>.

Let $V_p = \frac{d}{dt}h_t(p)\big|_{t=0}$, the derivative with respect to t of $h_t(p)$, evaluated when $t = 0$. We call such a function $p \mapsto V_p$ a <u>trivial vector field</u> or an <u>infinitesimal congruence</u>.

<u>Example 2.8</u>: For $d = 2$, define

$$h_t\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1\cos t + x_2\sin t \\ -x_1\sin t + x_2\cos t \end{bmatrix}$$

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} .$$

Then

$$V_{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} .$$

We can view this "infinitesimal rotation" in the plane as in Figure 2.3.

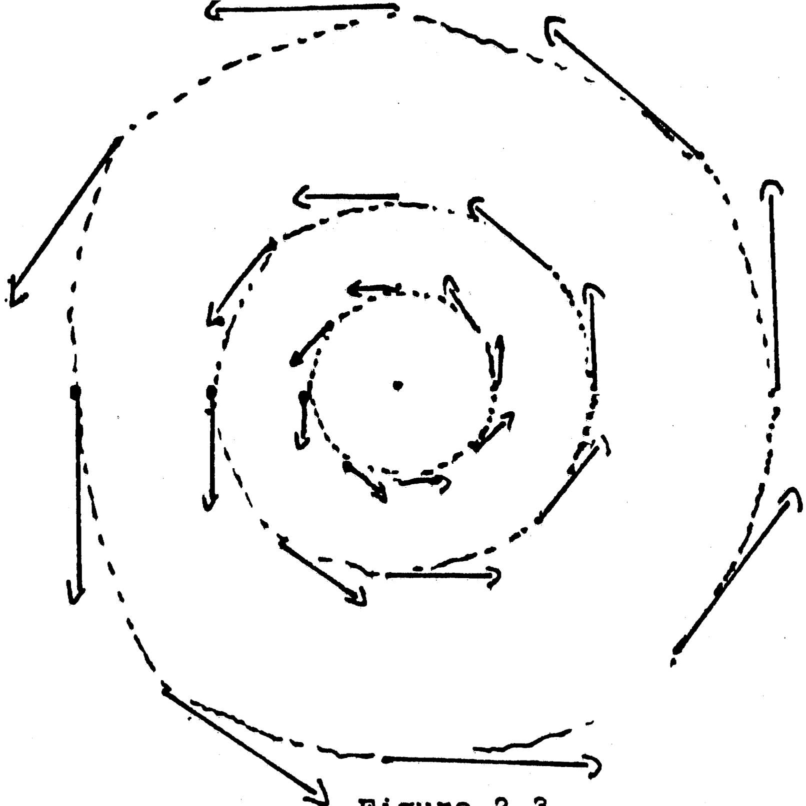


Figure 2.3

<u>Proposition 2.9</u>: Let V_p be an infinitesimal congruence. Let $p_1, p_2 \in \mathbb{R}^d$. Then

$$(p_1 - p_2)\cdot(V_{p_1} - V_{p_2}) = 0.$$

<u>Proof</u>: Let h_t be the congruent motion defining V_p. Then $|h_t(p_1) - h_t(p_2)| = |p_1 - p_2|$, and we differentiate the following inner product with respect to t.

$$[h_t(p_1) - h_t(p_2)]\cdot[h_t(p_1) - h_t(p_2)] = \text{constant}.$$

We get

$$2[h_t(p_1) - h_t(p_2)]\cdot[\tfrac{d}{dt}h_t(p_1) - \tfrac{d}{dt}h_t(p_2)] = 0.$$

Evaluating this when $t = 0$ gives the result.

7. Explicit Form of Infinitesimal Congruences

Recall that for a congruent motion $h_t(p) = A_t p + h_t(0)$, where A_t is an orthogonal matrix, and $h_t(0)$ is a C^1 function. Thus each coordinate of A_t is a C^1 function. We can then differentiate the relation $A_t^T A = I$ with respect to t. We get

$$0 = \frac{d}{dt}(A_t^T A_t) = A_t^T(\frac{d}{dt}A_t) + (\frac{d}{dt}A_t^T)A_t.$$

Since $h_0(p) = p = A_0 p + h_0(0)$ for all $p \in \mathbb{R}^d$, $0 = h_0(0)$ and hence $p = A_0 p$ for all p. Thus $A_0 = I$. Let $S = \frac{d}{dt}A_t|_{t=0}$. Evaluating our formula above at $t = 0$, we get

$$0 = IS + S^T I = S + S^T.$$

In other words S is skew symmetric, $S^T = -S$.

Proposition 2.10: A vector field V_p ($V_p \in \mathbb{R}^d$ is a vector for each $p \in \mathbb{R}^d$) is an infinitesimal congruence if and only if

$$V_p = Sp + p_0',$$

where $S^T = -S$ is a skew symmetric matrix and $p_0' \in \mathbb{R}^d$.

Proof: We have already seen that any such infinitesimal congruence has the above form.

Let $S = -S^{T}$ be any skew symmetric matrix, and let $p_0 \in \mathbb{R}^d$. Define an orthogonal matrix A_t by

$$A_t = I + tS + \frac{t^2}{2!} S^2 + \ldots + \frac{t^n}{n!} S^n + \ldots \equiv e^{tS} .$$

It is easy to check that the above infinite series of matrices converges for all $t \in \mathbb{R}^1$. It is also easy to check that (using the above notation) if X and Y are two d by d matrices that commute, i.e., $XY = YX$, then $e^X e^Y = e^{X+Y}$. Thus since $SS^{T} = -S^2 = S^{T}S$, $A_t A_t^{T} = e^{tS} e^{tS^{T}} = e^{tS+tS^{T}} = e^0 = I$. Thus A_t is orthogonal for all t. Clearly

$$\frac{d}{dt} A_t \Big|_{t=0} = Se^{tS} \Big|_{t=0} = S.$$

Thus we define

$$h_t(p) = A_t p + tp_0.$$

Then

$$\frac{d}{dt} h_t(p) \Big|_{t=0} = Sp + p_0.$$

as desired.

Note that the collection of all infinitesimal congruences is a vector space. We compute its dimension.

<u>Corollary 2.11</u>: The dimension of the space of infinitesimal congruences of $\mathbb{R}^d$ is $\frac{d(d+1)}{2}$.

<u>Proof</u>: There is a vector space isomorphism between $\{ (S, p_0') \mid S_{d\times d}$ skew symmetric matrices, $p_0' \in \mathbb{R}^d \}$, and infinitesimal congruences. The dimension of the skew symmetric matrices is $d(d-1)/2$, and the dimension of $\mathbb{R}^d$ is d. So the total dimension is $d(d-1)/2 + d = d(d+1)/2$.

Example 2.12: Let $d = 3$, and write a skew symmetric matrix S as

$$S = \begin{bmatrix} 0 & s_{12} & s_{13} \\ -s_{12} & 0 & s_{23} \\ -s_{13} & -s_{23} & 0 \end{bmatrix} .$$

Let

$$r = \begin{bmatrix} -s_{23} \\ s_{13} \\ -s_{12} \end{bmatrix} .$$

Then for $p \in \mathbb{R}^3$, $Sp = r \times p$, the usual cross product. Thus

$$V_p = r \times p + p_0'$$

is a general infinitesimal congruence in $\mathbb{R}^3$.

Example 2.13: Let $d = 2$.

$$S = \begin{bmatrix} 0 & s_{12} \\ -s_{12} & 0 \end{bmatrix} = s_{12}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

Define

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} ,$$

which corresponds to rotation by 90° counterclockwise. Thus $V_p = s_{12}Jp + p_0'$.

Suppose we translate our coordinate system so that the origin is at $p_1 = -\frac{1}{s_{12}} J^{-1}p_0'$, where we assume $s_{12} \neq 0$. Note $J^{-1} = -J$. We calculate the vector field at p_1

$$V_{p_1} = s_{12}J(-\frac{1}{s_{12}} J^{-1}p_0') + p_0' = 0.$$

Thus in our new coordinate system the new p_0' is 0. Thus the vector field is the "infinitesimal rotation" as in Figure 2.3.

If $s_{12} = 0$, then $V_p = p_0'$ an infinitesimal translation. This gives a geometric picture of any infinitesimal congruence for dimension two.

In dimension three we cannot always completely eliminate p_0', but we can translate the origin so that p_0' and r (as in the example above) are parallel. This is sometimes called a "screw motion". Trajectories of the vector field (curves whose tangent at each point is the evaluation of the vector field) are helices about a central line, as in Figure 2.4. Of course, infinitesimal translations are also possible.

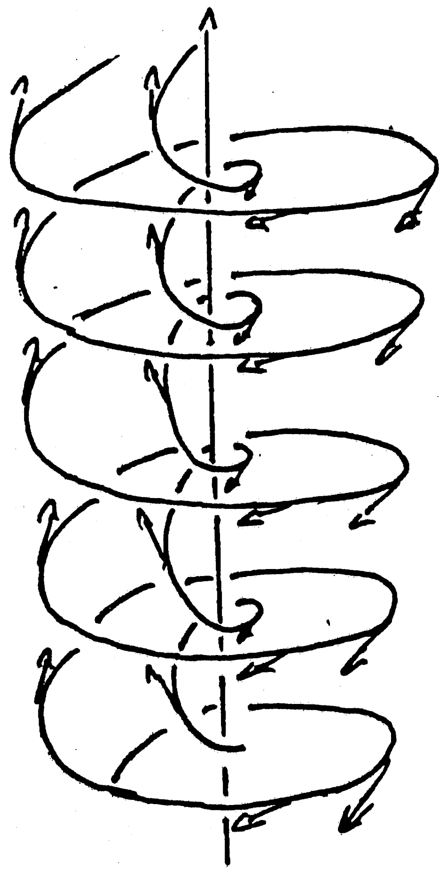


Figure 2.4

Problems:

Problem 2.8: (a) If $S_{d\times d}$ is a skew symmetric matrix and $A_{d\times d}$ is an orthogonal matrix, show that $A^{-1}SA$ is skew symmetric.

(b) Let V be a finite dimensional inner product space. (I.e., for $p, q \in V$, p, q is defined and obeys the usual properties for the inner product. See Halmos (xx) for complete definitions.) Show that the notion of a liner function $L: V \to V$ being skew symmetric is independent at the orthonormal basis of V used to define the skew symmetry.

Problem 2.9: Another approach to infinitesimal congruences is to simply define them as vector fields that satisfy the equation of Proposition 2.9.

(a) Show that the set of vector fields that satisfy the equation of Proposition 2.9 is a vector space, where vector addition is given by $(V + W)_p = V_p + W_p$, where V_p, W_p are two vector fields.

(b) If V_p is a C^1 vector field, show that we can "integrate" V_p to find $h_t(p)$ such that $\frac{d}{dt} h_t(p) = V_{h_t(p)}$, $h_0(p) = p$. If V_p satisfies the formula of Proposition 2.6 show that $h_t(p)$ is a congruence.

(c) If V_p is any vector field satisfying the formula of Proposition 2.6, show that there is a skew symmetric matrix S and $p_0' \in \mathbb{R}^d$ such that

$$V_p = Sp + p_0', \text{ for } p = 0, e_1, \ldots, e_d,$$

where e_i is the vector with 1 in the i-th coordinate and 0 elsewhere.

(d) If V_p satisfies the formula of Proposition 2.9 and $V_p = 0$ for $p - 0, e_1, \ldots, e_d$, then show that $V_p = 0$ for all $p \in \mathbb{R}^d$.

(e) Show that any V_p that satisfies the formula of Proposition 2.9 is of the form $V_p = Sp + p_0'$ for all $p \in \mathbb{R}^d$, where S is a skew symmetric matrix.

Problem 2.10: For J defined as in the exmaple above for $d = 2$, show that

$$e^{tJ} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} .$$

Problem 2.11: For J as above, and $V_p = Jp$, show (directly) that the integral of this vector field is $h_t(p) = e^{tJ}p$.

8. Frameworks and Configurations

We shall often consider a finite sequence of v points $p_1, \ldots, p_v \in \mathbb{R}^d$. We call $p = (p_1, \ldots, p_v)$ a configuration of v points in $\mathbb{R}^d$. We regard p as being in $\mathbb{R}^{vd}$, the configuration space. We say two configurations $p, q \in \mathbb{R}^{vd}$ are congruent and write $p \sim q$ if there is a congruence $h: \mathbb{R}^d \to \mathbb{R}^d$ such that

$$(h(p_1), \ldots, h(p_v)) = (h(q_1), \ldots, h(q_v)).$$

Note that congruence is an equivalence relation.

Let $G = (V,E)$ be a finite undirected graph without loops or multiple edges. V is the set of vertices, $V = (1, \ldots, v)$, and E is the set of edges, which are regarded as a subset of the set of all unordered pairs of distinct vertices. We will denote an edge by $\{i,j\}$, by i,j, or simply by ij, where $i,j \in V$. We will let e denote the number of edges. An edge is sometimes called a member. If $p = (p_1,\ldots,p_v)$ is a configuration of v points in $\mathbb{R}^d$, the pair G, p, written $G(p)$, is called a bar framework, a rod framework, a bar and joint framework, or simply a framework in $\mathbb{R}^d$. Thus $G(p)$ is an assignment of a point $p_i \in \mathbb{R}^d$ for each vertex $i = 1,\ldots, v \in V$. We also call p_i a joint of $G(p)$, and we say $G(p)$ is a realization of the configuration p.

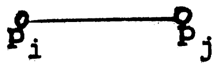
Suppose $G(p)$ and $G(q)$ are two realizations of G. We say $G(p)$ is equivalent to $G(q)$, and write $G(p) \sim G(q)$, if for all $\{i,j\}$ bars of G, $|p_i - p_j| = |q_i - q_j|$.

We say $G(p)$ is globally rigid, or uniquely realized, if when $G(p) \sim G(q)$, then $p \sim q$. In other words when $G(p)$ and $G(q)$ have corresponding bars of the same length, then p is congruent to q.

Example 2.14: If the configuration $p = (p_1,\ldots, p_v)$ is affinely independent (i.e., $p_v - p_1$, $p_{v-1} - p_1,\ldots, p_2 - p_1$ is linearly independent) and G has all $v(v + 1)/2$ rods, then $G(p)$ is called a $(v-1)$-dimensional bar simplex. (Note $v \leq d+1$.) By Proposition 2.7 every simplex is globally rigid. In fact every realization of G is globally rigid, not just the ones that have p as affinely independent. However, the notion of a simplex will be important later.

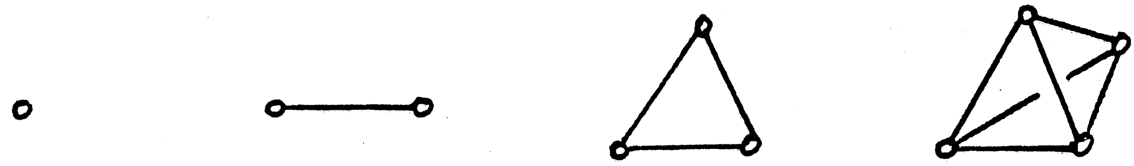
In order to graphically describe rod frameworks we will use the following notation.

o P_i represents a joint.



represents a bar (or rod) between P_i and P_j.

Figure 2.5 shows 0, 1, 2, and 3 dimensional bar simplices in $\mathbb{R}^3$.



0-simplex 1-simplex 2-simplex 3-simplex

Figure 2.5

Figure 2.6 shows other bar frameworks that happen to be globally rigid in $\mathbb{R}^2$ (as well as in $\mathbb{R}^3$).

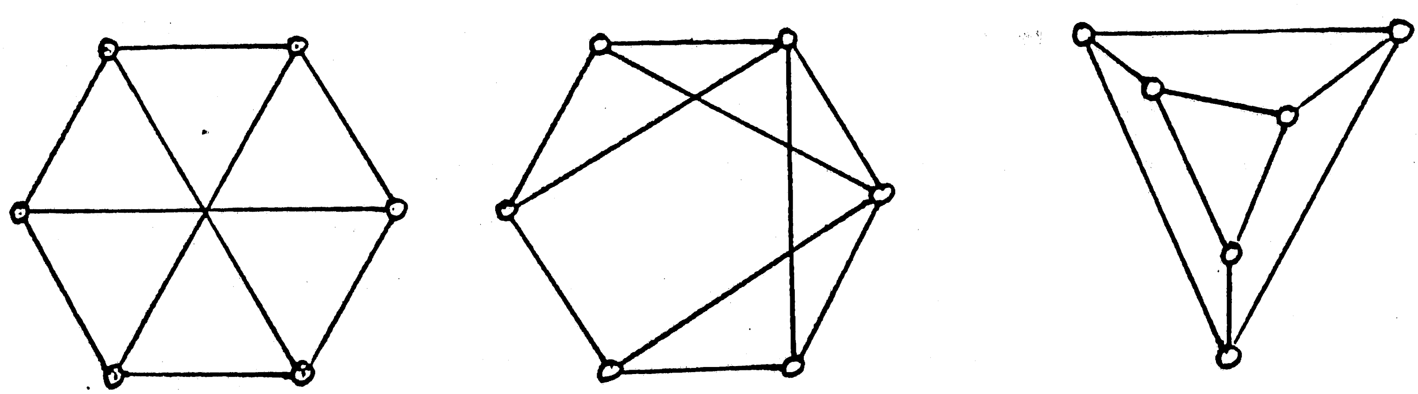


Figure 2.6

See Connelly (19xx) for a proof that these frameworks are globally rigid (even in $\mathbb{R}^3$).

Note that bars may cross (i.e., intersect) as in the first two examples in Figure 2.6, but this is ignored for our definitions.

Figure 2.7 shows some examples of bar frameworks that are not globally rigid in $\mathbb{R}^2$.

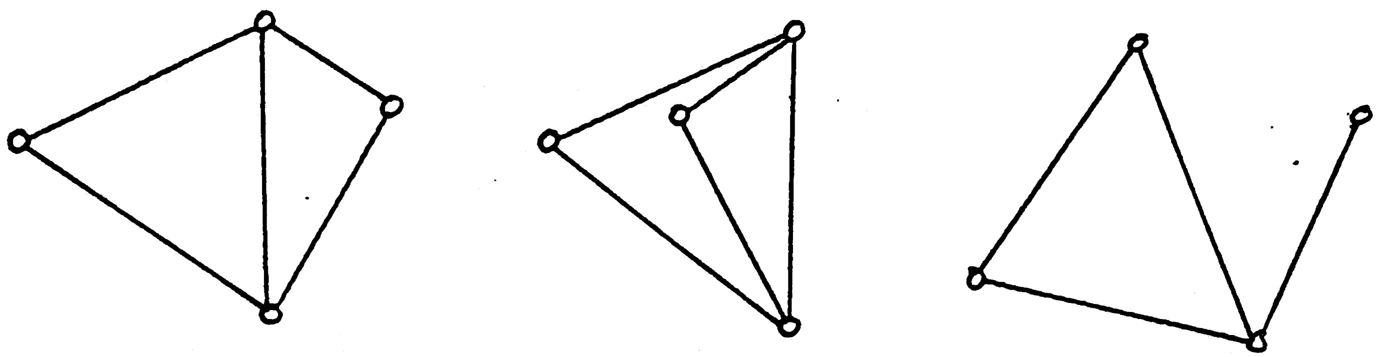


Figure 2.7

Note that the first two frameworks are equivalent to each other, but they are the only two equivalence classes modulo congruence of the configurations. The last framework has the "dangling" bar move continuously through infinitely many equivalence classes.

Figure 2.8 shows a bar framework that is globally rigid in $\mathbb{R}^1$, but not in $\mathbb{R}^2$ (when regarded as a subset $\mathbb{R}^2$ instead).

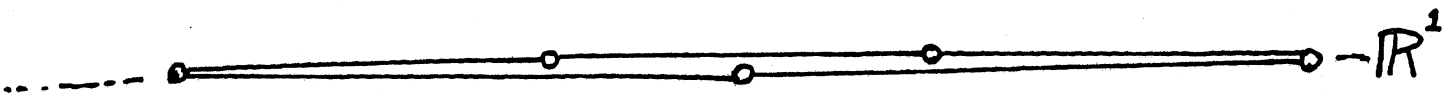


Figure 2.8

9. Tensegrity Frameworks

It is very helpful to expand our notion of a framework. In particular, for a graph $G = (V,E)$, we partition the set of edges E called members into three disjoint sets E_-, E_0, E_+ called cables, bars (or rods), and struts respectively. We call $G = (V;E_-,E_0,E_+)$ a signed graph, and if $V = (1,\ldots,v)$, $(p_1,\ldots,p_v) = p \in \mathbb{R}^{vd}$, we call the pair G, p, dentoed as $G(p)$, a tensegrity framework in $\mathbb{R}^d$, or simply a framework in $\mathbb{R}^d$. Note that if $E = E_0$, then we have our previous notion of a bar framework.

We can now define a partial ordering on the set of configurations $p \in \mathbb{R}^{vd}$. For two configurations $p, q \in \mathbb{R}^d$, we say $G(p)$ dominates $G(q)$ or $G(p)$ is a dominant of $G(q)$, and write $G(p) \geq G(q)$ if

$$|p_i - p_j| \geq |q_i - q_j| \text{ for } \{i, j\} \in E_-,$$
$$|p_i - p_j| = |q_i - q_j| \text{ for } \{i, j\} \in E_0, \text{ and}$$
$$|p_i - p_j| \leq |q_i - q_j| \text{ for } \{i, j\} \in E_+.$$

Thus cables are permitted to shorten or stay the same length, struts permitted to lengthen or stay the same length, and bars are only permitted to stay the same length. Note that if $G(p) \geq G(q)$ and $G(q) \geq G(p)$, then $G(p) \sim G(q)$, where all members are regarded as bars.

Note also that cables (as well as struts) are regarded as stretched "taut" (or compressed in the case of struts) and not "slack". This is because we are considering here only the local behavior of the framework. Distance constraints which are not sharp might as well not be part of the definition.

We say $G(p)$ is <u>globally rigid</u> if for every configuration $q \in \mathbb{R}^{vd}$, $G(p) \geq G(q)$ implies $p \sim q$. In other words $G(p)$ is globally rigid if every other non-congruent realization of G has a longer cable or bar, or a shorter strut or bar.

<u>Example 2.15</u>: We extend the notation for bar frameworks as follows:

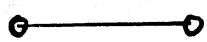
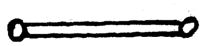
	represents a point (or joint) as before
	represents a bar (or rod) as before
	represents a cable
	represents a strut

Figure 2.9 shows some examples of tensegrity frameworks that are globally rigid in $\mathbb{R}^2$ (in increasing order of difficulty of showing they are indeed globally rigid).

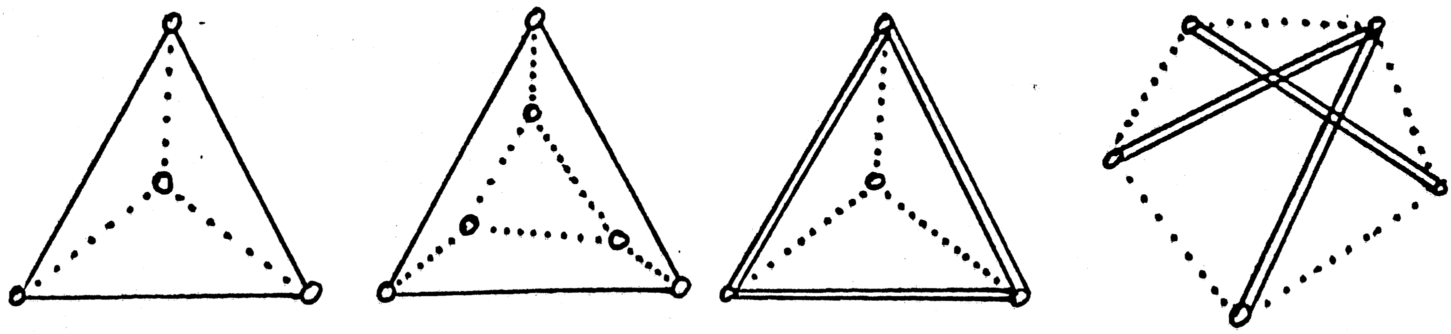


Figure 2.9

The second triangle must have the lines through the three cables, between the two triangles, go through a point. See later chapters on energy methods for proofs that these frameworks are globally rigid.

Figure 2.10 shows examples of globally rigid tensegrity frameworks in $\mathbb{R}^3$ which will be discussed in later chapters.

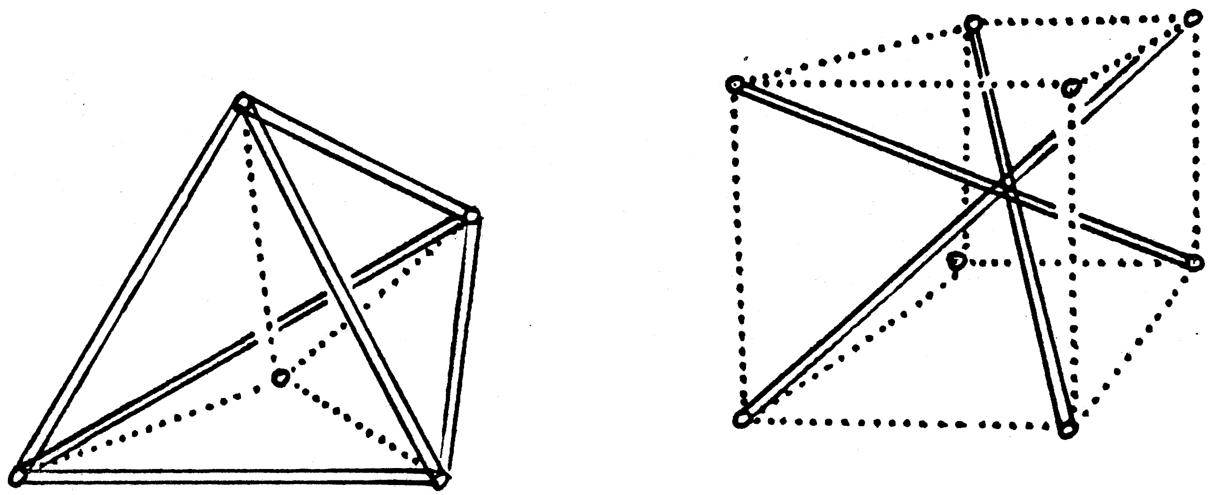


Figure 2.10

Note that for all of the tensegrity frameworks of Figure 2.9 and Figure 2.10, if cables are changed to struts and struts changed to cables, then they are not globally rigid.

Problems:

Problem 2.12: Let G be any bar graph such that G remains connected upon removing 0 or 1 vertices. Show that there is a realization of G, $G(p)$, in $\mathbb{R}^1$ with each p_i distinct, such that $G(p)$ is globally rigid in $\mathbb{R}^1$.

Problem 2.13: Find an example of a bar framework in $\mathbb{R}^1$ that is globally rigid in $\mathbb{R}^1$ and $\mathbb{R}^2$, but not in $\mathbb{R}^3$.

10. Rigidity

Let $G(p)$ be any tensegrity or bar framework in $\mathbb{R}^d$. We now face the question of what it means for a framework to be rigid. One natural idea is to say that there is no continuous motion of the joints — keeping all the cable, bar, and strut constraints — except for the trivial motions coming from congruences of all of $\mathbb{R}^d$. This is Definition 2 below. In Definition 1 we do not even assume that there is a motion, but only that there are realizations of G arbitrarily close to $G(p)$ — keeping all the cable bar and strut constraints. The following are adapted from Gluck (1974).

<u>Definition 1 (Topological)</u>: $G(\mathbf{p})$ is <u>rigid</u> in $\mathbb{R}^d$ if there is an $\epsilon > 0$ such that if $G(\mathbf{p}) \geq G(\mathbf{q})$ and $|\mathbf{p} - \mathbf{q}| < \epsilon$, then $\mathbf{p} \sim \mathbf{q}$.

<u>Definition 2 (Continuous)</u>: $G(p)$ is <u>rigid</u> in $\mathbb{R}^d$ if for every continuous path $p(t) \in \mathbb{R}^{vd}$, $p(0) = p$, such that $G(p) \geq G(p(t))$ for all $0 \leq t \leq 1$, then $p(t) \sim p$, for all $0 \leq t \leq 1$. We call $p(t)$ a <u>continuous flex</u> of $G(p)$.

<u>Definition 3 (Analytic)</u>: $G(p)$ is <u>rigid</u> in $\mathbb{R}^d$ if for every analytic path $p(t) \in \mathbb{R}^{vd}$, $p(0) = p$, such that $G(p) \geq G(p(t))$ for all $0 \leq t \leq 1$, then $p(t) \sim p$, for all $0 \leq t \leq 1$. We call $p(t)$ an <u>analytic flex</u> of $G(p)$.

If $G(p)$ is rigid by Definition 1, and if $\mathbf{p}(t)$ is any continuous flex of $G(\mathbf{p})$, then there is an $\delta > 0$ such that if $0 < t < \delta$, then $|p(t) - p(0)| < \epsilon$, and thus $p(t)$ is congruent to $p(0)$. Applying this argument to any interval and using a standard argument with the connectedness or compactness of the interval [0,1], one can show that G(p) is rigid by Definition 2. Clearly if G(p) is rigid by Definition 2, then G(p) is rigid by Definition 3.

Definition 1 is the easiest to verify, and Definition 3 is often convenient to use. The analytic flex is useful for finding many derivatives, and later it will be convenient to know that knowledge of all the derivatives determines the flex. Thus the folowing Proposition can be very useful, and shows that we do not have be concerned about which definition to use.

Proposition 2.16: All three definitions of rigidity are equivalent.

Proof: By the comment above, we need only show that if G(p) is rigid by Definition 3, then G(p) is rigid by Definition 1.

Suppose G(p) is not rigid by Definition 1. Define

$$M_p = \{ q \in \mathbb{R}^{vd} \mid G(p) \geq G(q) \}.$$

It is easy to see that M_p is a semi-algebraic set in the sense of algebraic geometry. See Milnor (xx), page xx for a definition. When the above conditions are written explicitly in terms of coordinates, we see that M_p is defined by a system of polynomial inequalities.

Define the orbit of p as

$$H_p = \{ q \mid q \sim p \}.$$

Thus $H_p \subset M_p$ is another semi-algebraic set. Since $G(p)$ is not rigid by Definition 1, by the Curve Selection Theorem, cf. Milnor (xx), page xx, there is an analytic path $p(t)$, $p(0) = p$, $0 \leq t \leq 1$, such that for $0 < t \leq 1$, $p(t) \in M_p \setminus H_p$. This contradicts the assumption that $G(p)$ is rigid by Definition 3.

Example 2.17: Figure 2.11 shows some examples of rigid but not globally rigid frameworks in $\mathbb{R}^2$.

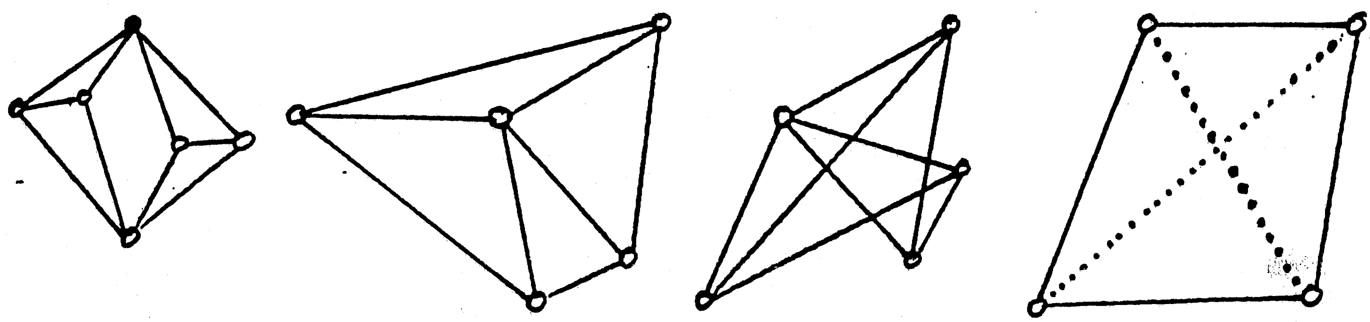


Figure 2.11

Figure 2.12 shows some examples of rigid but not globally rigid frameworks in $\mathbb{R}^3$

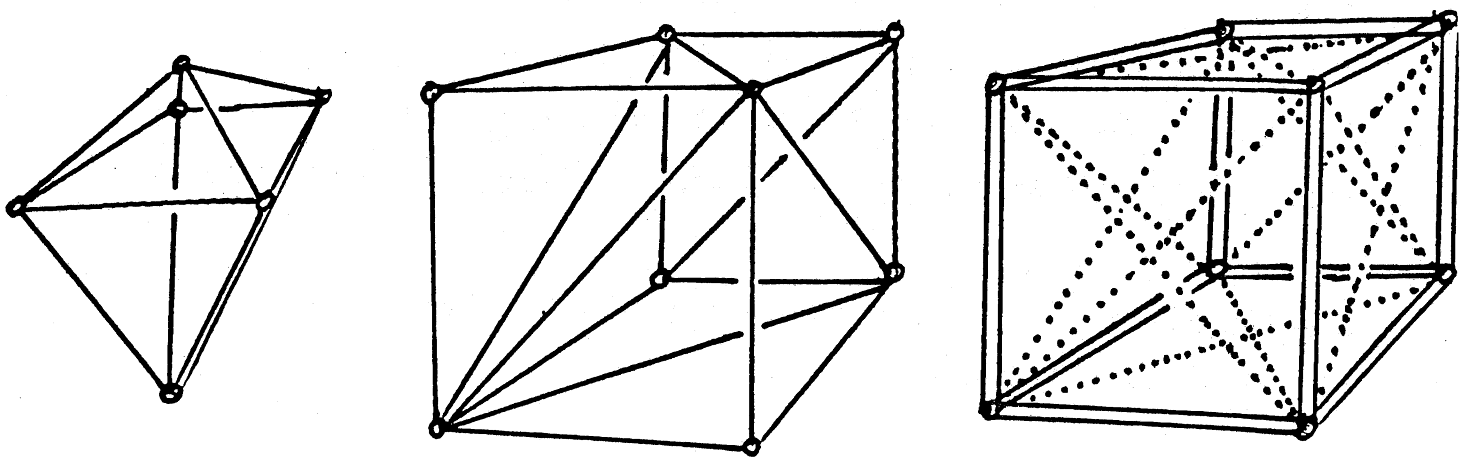


Figure 2.12

If $G(p)$ is not rigid, we say $G(p)$ is _flexible_, a _mechanism_, or a _linkage_, and as in Definition 2 or Definition 3, $p(t)$ is a _flex_ or _finite motion_ of $G(p)$.

Figure 2.13 shows some examples of (non-trivial) flexible frameworks in $\mathbb{R}^2$.

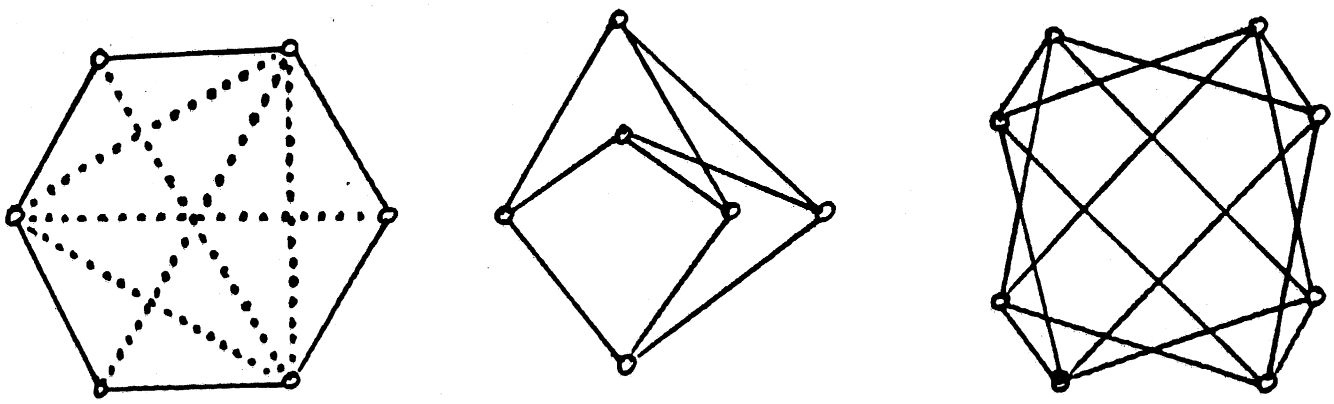


Figure 2.13

See Grünbaum and Shephard (xx) for examples such as the one on the left.

In the middle example two sets of three points are colinear on perpendicular lines. The example on the right is due to Bottema (xx) (called Bottema's 16 bar linkage. See Wunderlich (xx), as well, for other interesting properties of this mechanism.

Figure 2.14 shows an example of a flexible framework in $\mathbb{R}^3$.

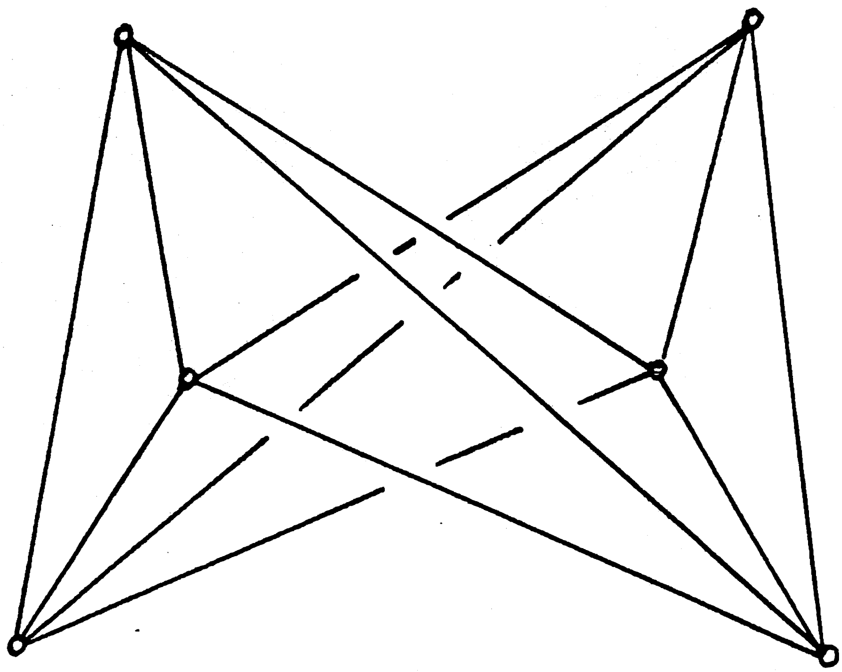
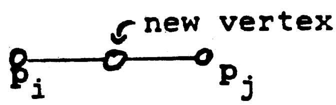


Figure 2.14

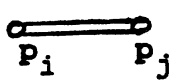
This example is due to Bricard (1898). See Connelly (1979) as well. The whole framework is symmetric about a line.

<u>Remark 2.18</u>: One can formally "reduce" tensegrity frameworks to the special case of bar frameworks. Namely replace every

cable 

with 

and replace every

strut 

with 

.

The rigidity of this altered framework is equivalent to the rigidity of the original framework, but (because of the added joints) we shall see later that other properties of interest, such as infinitesimal rigidity defined shortly, will not be preserved by such alterations. Cables and struts are important objects to keep in mind when they arise.

11. Infinitesimal Rigidity

Although some special techniques can be used to prove certain frameworks are rigid, we need a method that is widely applicable. Thus we linearize the notions of rigidity. This kind of rigidity can be checked by or related to standard techniques from linear algebra.

The idea behind the following definition is to differentiate the cable, bar, and strut conditions on the lengths of the members. For example, for $\{i,j\}$ a bar, during any flex $p(t)$, the square of the bar length, is $|p_i(t) - p_j(t)|^2$, which must remain constant. The derivative of this at $t = 0$ must be zero, but also can be calculated as $2(p_i - p_j)\cdot(p_i' - p_j') = 0$, where p_i' and p_j' are the derivatives of $p_i(t)$ and $p_j(t)$, respectively, at $t = 0$. Thus we are led to the following definition.

Let $G(p)$ be any tensegrity or bar framework in $\mathbb{R}^d$. Let $(p_1',\ldots,p_v') = p' \in \mathbb{R}^{vd}$. We say p' is an <u>infinitesimal flex</u> of $G(p)$ if for all members $\{i,j\}$ of $E = E_- \cup E_0 \cup E_+$

$$
\begin{aligned}
(p_i - p_j)\cdot(p_i' - p_j') &\le 0 \quad \text{for} \quad \{i,j\} \in E_- \\
(p_i - p_j)\cdot(p_i' - p_j') &= 0 \quad \text{for} \quad \{i,j\} \in E_0 \\
(p_i - p_j)\cdot(p_i' - p_j') &\ge 0 \quad \text{for} \quad \{i,j\} \in E_+.
\end{aligned}
\tag{2.1}
$$

We say p' is a trival infinitesimal flex if there is an infinitesimal congruence V_p in $\mathbb{R}^d$ such that $p_i' = V_{p_i}$ for all $i = 1,\ldots,v$. We say $G(p)$ is infinitesimally rigid in $\mathbb{R}^d$ if every infinitesimal flex of $G(p)$ is trivial.

It is not immediately obvious that an infinitesimally rigid framework is rigid. Nevertheless, it is still true, and this will be the object of the next several sections.

Meanwhile, we need to build a collection of frameworks that we know are infinitesimally rigid as well as some basic extension lemmas.

We observe that there is an analogue of Proposition 2.7 for infinitesimal flexes of bar simplices. Recall that a bar simplex is a bar framework $G(p)$, where $E = E_0$ consists of all possible bars, and $p = (p_1,\ldots,p_v)$ is an affine independent set.

Lemma 2.19: Let $G(p)$ be a bar simplex in $\mathbb{R}^d$. Let $(p_1',\ldots,p_k')$, $1 \le k \le v$, be an infinitesimal flex of the face $(p_1,\ldots,p_k)$, i.e., $(p_1',\ldots,p_k')$ is an infinitesimal flex of the sub-framework on the joints $(p_1,\ldots,p_k)$. Then $(p_1',\ldots,p_k')$ extends to $p' = (p_1',\ldots,p_k',\ldots,p_v')$, which is an infinitesimal flex of $G(p)$. Furthermore if $k = d+1$ or $k = d$, the extension is unique.

Proof: We show how to define p_{k+1}'. Then we can inductively define $p_{k+2}',\ldots,p_v'$. We need

$$(p_{k+1}' - p_i')\cdot(p_{k+1} - p_i) = 0, \text{ for } i = 1,\ldots,k.$$

But this is the same as

$$p'_{k+1} \cdot (p_{k+1} - p_i) = p'_i (p_{k+1} - p_i), \text{ for } i = 1,\ldots,k.$$

This is a system of k linear equations in d unknowns, the coordinates at p'_{k+1}. Since $p_1 - p_{k+1}, p_2 - p_{k+1},\ldots,p_k - p_{k+1}$ is a set of linearly independent vectors, the system always has a solution and is unique for $k = d$. (There is nothing to define for $k = d+1$.)

<u>Proposition 2.20</u>: Let G(p) be a bar simplex in $\mathbb{R}^d$. Then G(p) is infinitesimally rigid.

<u>Proof</u>: First consider the case when $v = d+1$, where v is the number of vertices of G, as usual. Let $p' = (p'_1,\ldots, p'_{d+1})$ be an infinitesimal flex of G(p). $(p'_{d+1}, p'_{d+1},\ldots, p'_{d+1})$ is a trivial infinitesimal flex, an infinitesimal translation,, of G(p), and thus $(p'_1 - p'_{d+1}, p'_2 - p'_{d+1},\ldots, p'_d - p'_{d+1}, 0)$ is a trivial infinitesimal flex if and only if p' is a trivial infinitesimal flex. (Recall that the trivial infinitesimal flexes form a vector space.) So without loss of generality, we assume $p'_{d+1} = 0$.

By replacing each p_i with $p_i - p_{d+1}$ we may also assume that $p_{d+1} = 0$. (p' is still an infinitesimal flex of $(p_1 - p_{d+1},\ldots, p_d - p_{d+1}, 0)$ and it is easy to check that p' is trivial for this translated configuration if and only if p' is trivial for the original configuration.) Thus $p_1,\ldots, p_d$, is a basis for $\mathbb{R}^d$.

Now define a matrix S by $Sp_1 = p_1'$, $Sp_2 = p_2', \ldots, Sp_d = p_d'$. Using (2.1) when one vertex is $d+1$, we observe that $(p_i - p_{d+1})\cdot(p_i' - p_{d+1}') = p_i \cdot p_i' = p_i \cdot Sp_i = 0$ for $i = 1,\ldots,d$. Using (2.1) again for $i \neq j$, $i,j \in \{1,\ldots,d\}$ we get

$$(p_i - p_j)\cdot(p_i' - p_j') = 0,$$

$$p_i \cdot p_i' + p_j \cdot p_j' - p_j \cdot p_i' - p_i \cdot p_j' = 0,$$

$$p_i \cdot p_j' = - p_j \cdot p_i',$$

$$p_i \cdot Sp_j = - p_j \cdot Sp_i.$$

Thus $p_i \cdot Sp_j = - p_j \cdot Sp_i$ for all $i,j \in \{1,2,\ldots,d\}$. Thus S is skew symmetric, and p' is a trivial infinitesimal flex. (The bilinear form defined by vSw, $v,w \in \mathbb{R}^d$, satisfies $v^T Sw = -w^T Sv$ if this equation holds for all v, w in a basis for $\mathbb{R}^d$.)

If $v \leq d$ extend $p_1, p_2, \ldots, p_v$ to an affine basis $p_1,\ldots,p_v, p_{v+1},\ldots, p_{d+1}$ of $\mathbb{R}^d$. If p' is an infinitesimal flex of $G(p)$, we can extend p' to $(p_1',\ldots, p_v',\ldots, p_{d+1}')$ an infinitesimal flex of the simplex defined by $p_1,\ldots, p_{d+1}$, by Lemma 2.19. By the case when $v = d+1$, $(p_1',\ldots, p_{d+1}')$ is a trivial infinitesimal flex at $(p_1,\ldots, p_{d+1})$. Thus $(p_1',\ldots, p_v')$ is a trivial infinitesimal flex at $(p_1,\ldots, p_v)$.

We can now apply Lemma 2.19 and Proposition 2.20 to build many examples of infinitesimally rigid bar frameworks.

Proposition 2.21: Let $G_0(p_1,\ldots,p_{v-1})$ be an infinitesimally rigid framework in $\mathbb{R}^d$. Suppose for some $\{i_1,\ldots,i_d\} \subset \{1,\ldots,v-1\}$, $p_{i_1},\ldots, p_{i_d}$ are affine independent in $\mathbb{R}^d$. Define a new graph G by adding one new vertex v, and d bars, $\{i_1,v\}, \{i_2,v\},\ldots, \{i_d,v\}$. Let $p_v \in \mathbb{R}^d$ be any point not in the affine span of $p_{i_1},\ldots, p_{i_d}$. Then $G(p_1,\ldots, p_{v-1}, p_v)$ is infinitesimally rigid in $\mathbb{R}^d$.

Proof: Let $p' = (p_1',\ldots, p_v')$ be any infinitesimal flex of $G(p)$. Since $G_0(p_1,\ldots, p_{v-1})$ is infinitesimally rigid in $\mathbb{R}^d$, $(p_1',\ldots, p_{v-1}')$ is a trivial infinitesimal flex at $(p_1,\ldots, p_v)$. By Lemma 2.19 there is a unique p_v' extending to $(p_1',\ldots, p_v')$. Clearly the trivial extension is one possible extension. Thus $(p_1',\ldots, p_v')$ is a trivial infinitesimal flex at $(p_1,\ldots, p_v)$.

Figure 2.15 shows some applications of Proposition 2.21 starting with a simplex. The numbering of the joints indicates the order in which the joints have been introduced by Proposition 2.21. The first d are the joints of the starting simplex. Such a framework is called a <u>simple</u> framework, and we call the sequence a <u>simple</u> ordering.

Example 2.22:

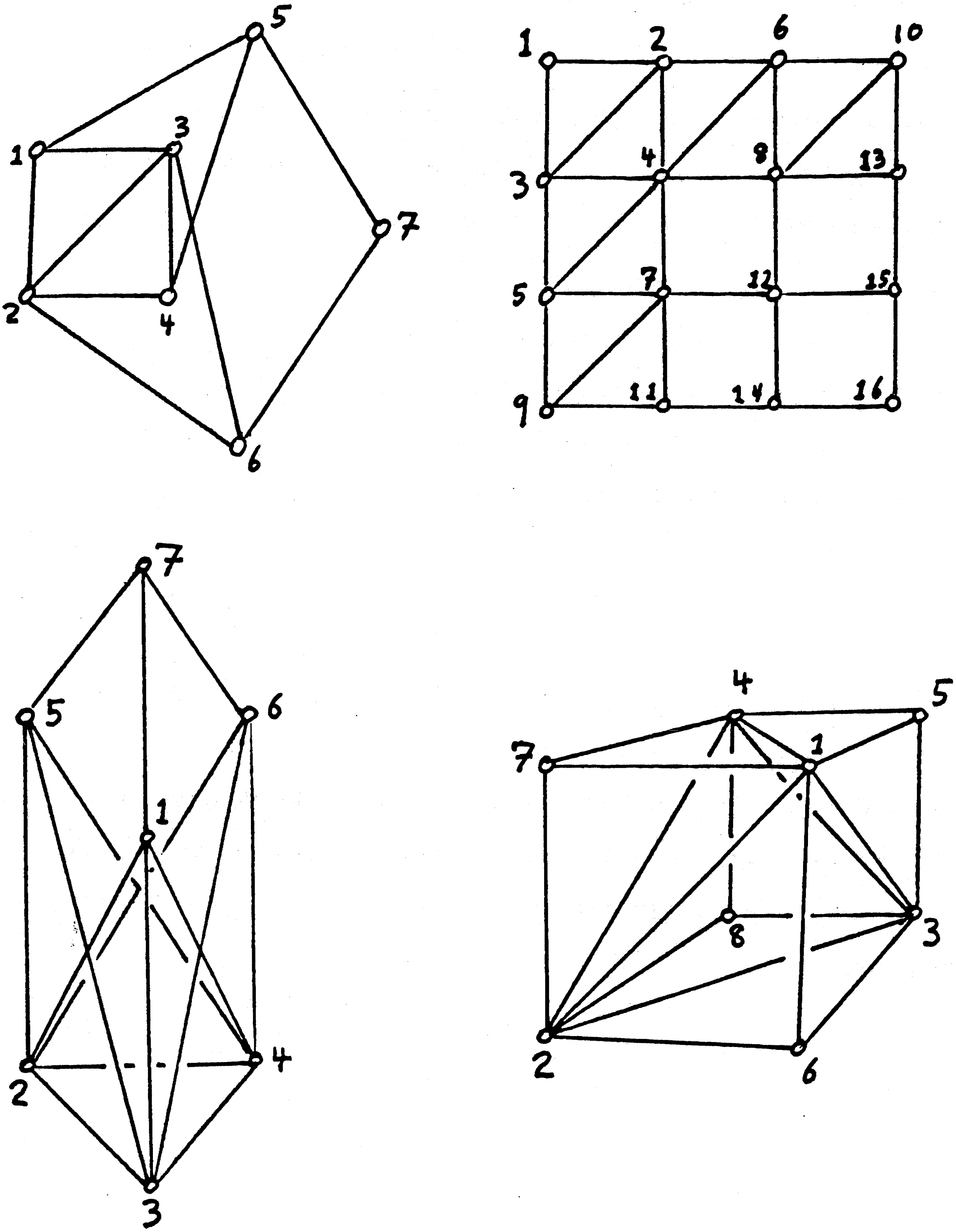


Figure 2.15

The first row is in $\mathbb{R}$ ... bottom right framework is a triangulation of the surface of a cube.

Note that for the above method to work, there must be a d-simplex in G somewhere for the starting point. Figure 2.16 shows an example in $\mathbb{R}^2$ where $G(p)$ is infinitesimally rigid, but G has no triangles.

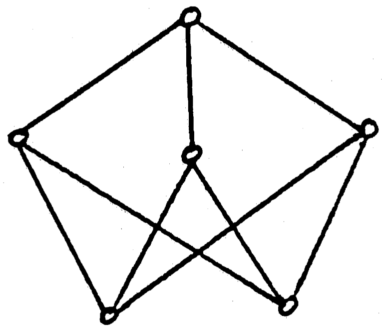


Figure 2.16

G is the graph $K_{3,3}$, the complete bipartite graph, and we must insist that the joints do not lie on any conic in $\mathbb{R}^2$. See Chapter xx for a discussion of this kind of example.

<u>Remark 2.23</u>: Note that if $p(t)$ is a analytic flex of a bar framework $G(p)$ in $\mathbb{R}^d$, then $\frac{d}{dt} p(t) = p'(t)$ is an infinitesimal flex of $G(p(t))$. This follows from differentiating the relation $|p_i(t) - p_j(t)|^2 = \text{constant}$, for $\{i,j\}$ a bar of G. However, even if $p(t_1)$ is not congruent to $p(t_2)$ for $0 \leq t_1 < t_2 \leq 1$, it may turn out that $p'(0)$ is trivial. For instance, we can replace any flex $p(t)$ with $p(t^2)$ for $t > 0$. However, for <u>some</u> $0 \leq t \leq 1$, $p'(t)$ must be a non-trivial infinitesimal flex of $G(p(t))$, since otherwise the derivative of

all the lengths between pairs of points would be 0, and $p(t)$ would be a congruence. This can provide us with many examples of non-trivial infinitesimal flexes.

On the other hand, Figure 2.17 shows an example of framework that is rigid, yet it still has a non-trivial infinitesimal flex and hence is not infinitesimally rigid.

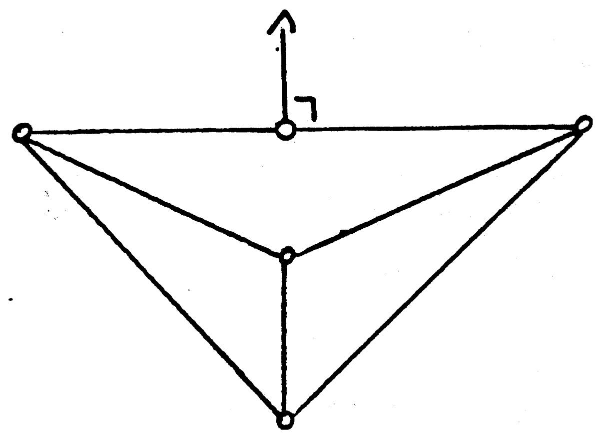


Figure 2.17

We indicate an infinitesimal flex p' by representing p_i' as an arrow at p_i. If there is no arrow at p_i, then it is understood that $p_i' = 0$. In the example above, the point with the arrow must lie on the line through the two points it has a common edge with. This example also shows that the condition, that p_i not be in the affine span of $p_{i_1},\ldots,p_{i_d}$ is necessary in Proposition 2.21.

The following simple consequence of Proposition 2.21 is very useful.

Corollary 2.24: Let $p = (p_1,\ldots, p_v)$ be a configuration of v points in $\mathbb{R}^d$ whose affine span is all of $\mathbb{R}^d$. Then there is a bar graph H on v vertices such that $H(p)$ is infinitesimally rigid.

Proof: Start with d+1 of the points of p whose affine span is $\mathbb{R}^d$. Say these points are $p_1, \ldots, p_{d+1}$. Put bars between each pair of these first d+1 points. For each of the other $p_i \in \mathbb{R}^d$, $i = d+2, \ldots, v$, p_i must not be in the affine span of some subset of d of the points $p_1, \ldots, p_{d+1}$. Put a bar from p_i to each of the d corresponding points to construct the bar graph H. Thus by applying Proposition 2.11, the bar simplex defined on the first d + 1 points is infinitesimally rigid. By Proposition 2.12, the framework H(p) is infinitesimally rigid.

Remark 2.25: H has exactly $\frac{d(d+1)}{2} + [v - (d+1)]d = vd - \frac{d(d+1)}{2}$ bars. This turns out to be the minimum needed. Of course for the Corollary H *could* be the complete graph connecting *all* pairs of the v vertices, but at this point the extra bars do not help in proving the needed infinitesimal rigidity.

12. Pinned Vertices

It is sometimes helpful to add more to the definition of a framework, as when we defined tensegrity frameworks. For a signed graph $G = (V; E_-, E_0, E_+)$ we designate a subset $V_0 \subset V$ of the set of vertices. A vertex in V_0 is called a *pinned* vertex. These vertices are regarded as having their corresponding joints pinned or fixed to the underlying $\mathbb{R}^d$. In each type of rigidity - global rigidity, rigidity, and infinitesimal rigidity - we alter the definition of dominance (or equivalence of frameworks), flex, and infinitesimal flex in the natural way to take account of the pinned vertices. In the case of infinitesimal rigidty, and infinitesimal flex is defined as a vector $p' \in \mathbb{R}^{vd}$ that satisfies (2.1) as before, but require that $p' = 0$ for all $i \in V_0$, in addition. We discuss here only the case of pinned vertices for infinitesimal rigidity.

Next, we must decide what the trivial infinitesimal flexes of pinned framworks are. This is most naturally determined by the affine span of the pinned joints $\langle p_i \rangle_{i \in V_0} = X_0$.

The congruences of $\mathbb{R}^d$ fixing X_0 are the orthogonal functions (linear maps corresponding to orthogonal matrices), regarding $0 \in X_0$, which fix each point in X_0. These functions in turn are determined by the orthogonal functions of the orthogonal complement of X_0 in $\mathbb{R}^d$. We define these orthogonal functions as the congruences modulo X_0, and they are clearly a subgroup of all congruences of $\mathbb{R}^d$.

As in section 5 we define the infinitesimal congruences modulo X_0 or the trivial infinitesimal flexes modulo X_0 as those infinitesimal congruences coming from congruences modulo X_0.

Thus, as before, we say a tenseguity framework $G(p)$, with pinned vertices, is infinitesimally rigid if the only infinitesimal flexes p', with $p_i' = 0$ for all $i \in V_0$, of $G(p)$ are trivial infinitesimal flexes modulo X_0.

The basic application of the above ideas is when X_0 is d or $(d-1)$-dimensional. Then the only trivial infinitesimal flex of $G(p)$ modulo X_0 is the 0 infinitesimal flex.

Our graphic description of frameworks will be augmented so that

● or  represents a pinned joint.

○ represents an unpinned joint.

The second notation for pinned joints is popular in structural engineering texts.

Example 2.26: Figure 2.18 represents some infinitesimally rigid pinned frameworks in $\mathbb{R}^2$.

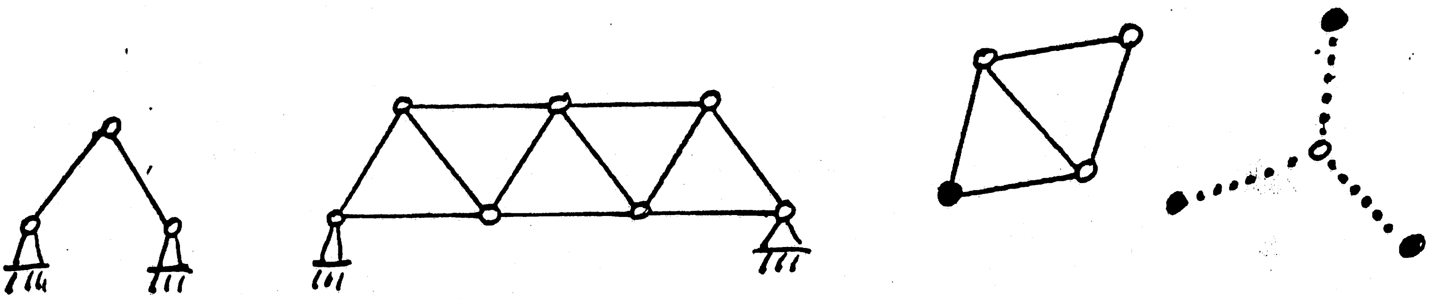


Figure 2.18

Notice that members between pinned vertices are superflous and pinning only one vertex does not change the infinitesimal rigidity of $G(p)$.

Figure 2.19 shows some examples of pinned frameworks in $\mathbb{R}^2$ that are not infinitesimally rigid. The (non-trivial) infinitesimal flex is indicated as before with arrows.

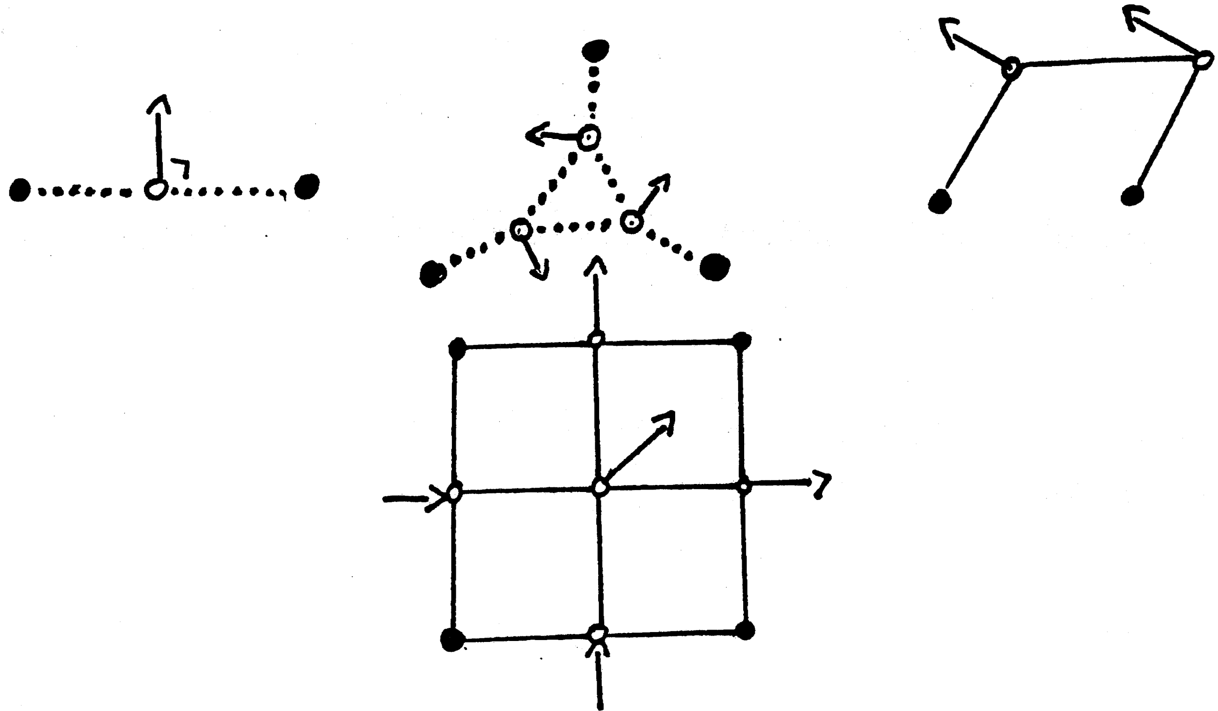


Figure 2.19

We have included this description of pinned vertices mostly for its convenience. The infinitesimal rigidity of pinned frameworks can be "reduced" to the usual infinitesimal rigidity of unpinned frameworks. Connect the pinned vertices by some infinitesimally rigid framework that contains the pinned vertices. Figure 2.20 shows an example.

The infinitesimal rigidity of:

is equivalent to the infinitesimal rigidity of:

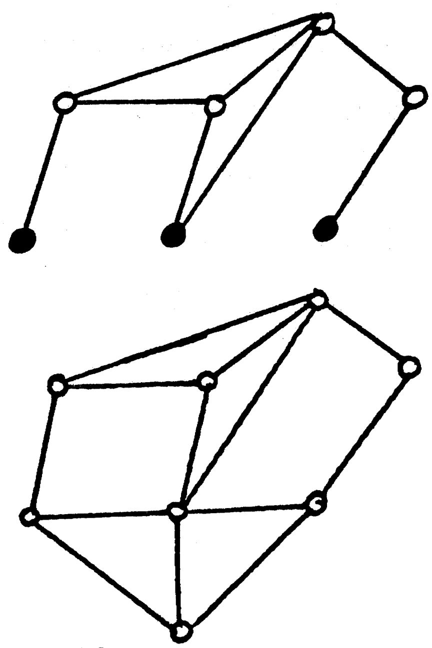
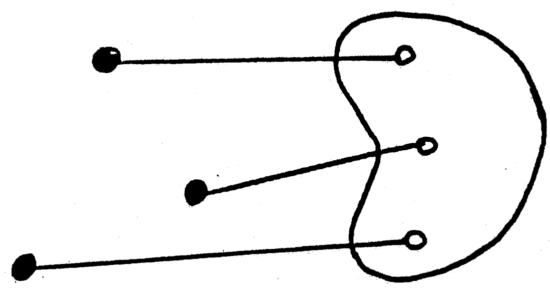


Figure 2.20

If the pinned joints have affine span all of $\mathbb{R}^d$ (or are affine independent), unlike the example in Figure 2.20, then the added framework can be chosen to consist of only bars added among the old pinned vertices. By the Remark 2.25 this can be accomplished by adding $v_0 d - \frac{d(d+1)}{2}$ bars, where v_0 is the number of the original pinned vertices.

<u>Example 2.27</u>: Figure 2.21 shows how the above can be done.

The infinitesimal rigidity of



is equivalent to the infinitesimal rigidity of

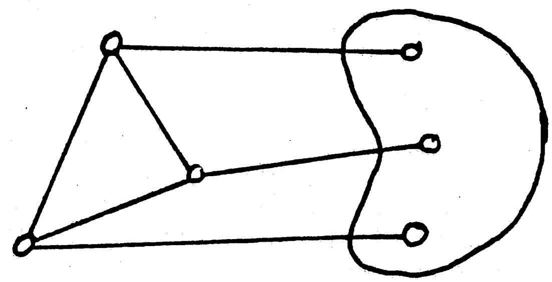
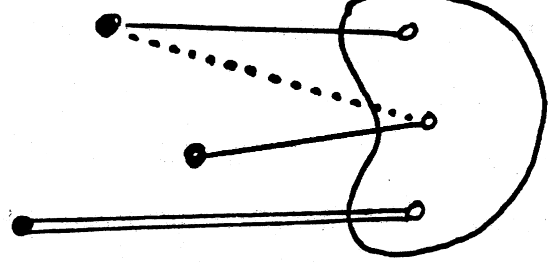


Figure 2.21

One advantage of having pinned joints is that it is possible to "move" a joint along a member to create other pinned frameworks which have the same set of infinitesimal flexes as the original. This might create new vertices as in the example in Figure 2.22.

The infinitesimal rigidity of



is equal to the infinitesimal rigidity of

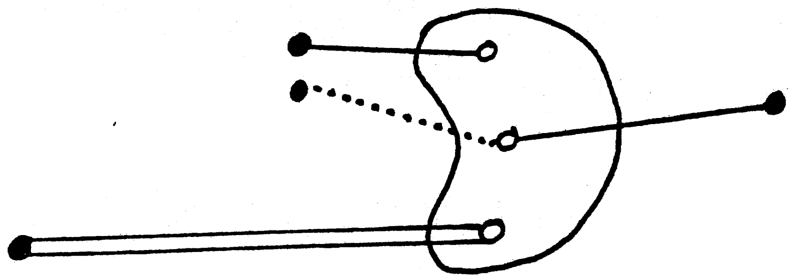


Figure 2.22

Note that for cables and struts, no fixed joint can be moved to the other side of the non-fixed joint on that member, but for bars the fixed joint can be moved to any point on the line containing the member except to the non-fixed joint.

We must be careful, however, because two frameworks $G_1(p)$, $G_2(q)$ with the same set of variable joints and the same set of infinitesimal flexes are not always both infinitesimally rigid or both not infinitesimally rigid. For example, if $G_1(p)$ has only one fixed joint and $G_2(q)$ has two distinct fixed joints, then the notion of what the trivial infinitesimal flexes are, are different for $G_1(p)$ and $G_2(p)$.

One disadvantage of trying to create frameworks with fixed vertices - even for bar frameworks - is that there may be no (d - 1)-dimensional simplex (for instance) in $G(p)$. (For a bar framework for $d = 2$, of course, there always is.) Without such a simplex it is hard to tell what to pin, a priori, to create a pinned framework whose infinitesimal rigidity is equivalent to the original unpinned framework. There are many examples of infinitesimally rigid bar frameworks in $\mathbb{R}^3$ whose underlying graph is $K_{m,n}$, the complete bipartite graph of order m, n. Such graphs have no triangles. See Bolker, Roth (198xx). ($K_{m,n}$ is defined as m vertices joined in all possible ways to n other vertices, but in each of the two sets of vertices, no two vertices are joined.)

On the other hand, for any framework G(p) it is possible to create a pinned framework $G_0(q)$, with additional pinned vertices, whose infinitesimal rigidity is equivalent to the infinitesimal rigidity of G(p), and the only trivial infinitesimal flex of $G_0(q)$ is the 0 infinitesimal flex.

Choose any points $p_1, \ldots, p_d$ of G(p) whose affine span is (d - 1)-dimensional. For $G_0(q)$ fix p_1, and then add fixed joints and connect them to p_2 in such a way that infinitesimally p_2 is constrained to move in a fixed line through p_1. Similarly connect fixed joints to p_3 so that p_3 is constrained infinitesimally to move in a fixed plane through the previous line, etc. for all the points $p_4, \ldots, p_d$. We see this applied in Figure 2.23 below. See Whiteley (19xx) for more discussion of this idea.

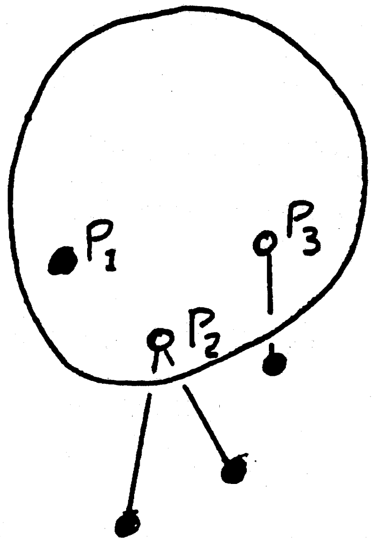


Figure 2.23

13. Reformulation of Infinitesimal Rigidity

When one has an infinitesimal flex p' of a framework $G'(p)$, it is often easier to test the change in distances, rather than whether p' is trivial directly. However, we must be careful when the affine span at p is not all of $\mathbb{R}^d$.

Example 2.28: Let $G(p)$ be any (tensegrity or bar) framework in $\mathbb{R}^d$ with v vertices and such that the dimension of the affine span of p is $n \leq v-2$. Let $p_1, \ldots, p_{n+1}$ be such that $< p_1, \ldots, p_{n+1} > = < p >$, the affine span of p. Define an infinitesimal flex for $G(p)$ by first setting $p_1' = p_2' = \ldots p_{n+1}' = \ldots p_{v-1}' = 0$. Define $p_v' \neq 0$ to be perpendicular to $< p >$. We will show $p' = (p_1', \ldots, p_v')$ is a non-trivial infinitesimal flex of $G(p)$. Clearly p' is an infinitesimal flex of $G(p)$.

Suppose, without loss of generality, that $p_{n+1} = 0$. Suppose S is any skew symmetric matrix, and $p_0' \in \mathbb{R}^d$. Suppose $Sp_i + p_0' = p_i'$, $i = 1,\dots,v$. We look for a contradiction. $Sp_i = -p_0'$, for $i = 1,\dots,v-1$, and $Sp_{n+1} = S0 = 0 = -p_0'$. Thus $Sp_i = 0$, for $i = 1,\dots,n+1$, and $\langle p_1,\dots,p_n \rangle = \langle p \rangle$ is a linear subspace of $\mathbb{R}^d$ and thus $Sq = 0$ for all $q \in \langle p \rangle$. Thus in particular $Sp_v = 0$. But $p_v' \neq 0$ and so p' cannot be a trivial infinitesimal flex of $G(p)$. Figure 2.24 shows this infinitesimal flex for $v = 4$, $d = 3$, $n = 2$.

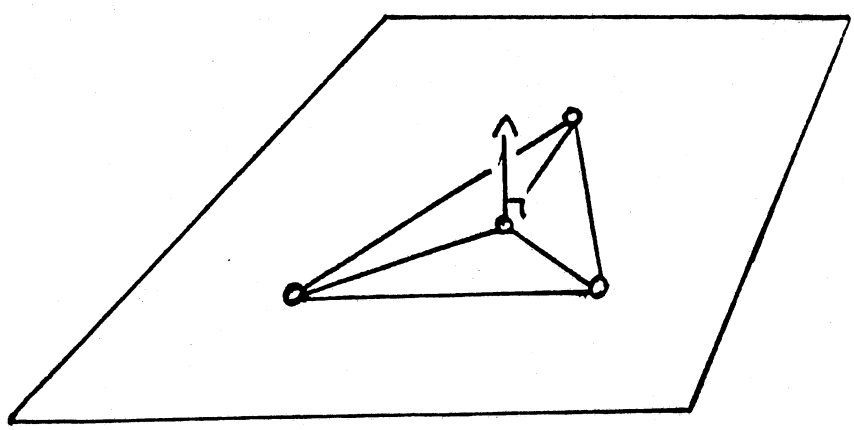


Figure 2.24

Thus when p has "too many" vertices and lies in some hyperplane of $\mathbb{R}^d$, then $G(p)$ can never be infinitesimally rigid, no matter what G is.

The next example shows what can happen even when the joints of p are affine independent. $G(p)$ may still not be infinitesimally rigid.

Example 2.29: Let $p = (p_1, \ldots, p_v)$ be affine independent in $\mathbb{R}^d$ (so $v \leq d+1$). Let $G(p)$ be any tensegrity framework *except* a bar simplex. Then $G(p)$ is not infinitesimally rigid in $\mathbb{R}^d$.

For some $\{i,j\} \subset \{1,2,\ldots, v\}$, $i \neq j$, $\{i,j\}$ is not a bar of G. In other words $\{i,j\}$ is either a cable, strut or not a member of G. Say $\{i,j\} = \{1,2\}$. Define $p_2' = \ldots = p_v' = 0$. Define $p_1' \neq 0$ to be perpendicular to the affine hyperplane $< p_1, p_3, \ldots, p_v >$, but chosen so that

$$p_1' \cdot (p_1 - p_2) \begin{cases} < 0 & \text{if } \{1,2\} \text{ is a cable} \\ > 0 & \text{if } \{1,2\} \text{ is a strut} \\ \neq 0 \text{ (either sign)} & \text{if } \{1,2\} \text{ is not a member of } G \end{cases}$$

Clearly $p' = (p_1', p_2', \ldots, p_v')$ is an infinitesimal flex of $G(p)$ and p' is non-trivial since $(p_1' - p_2') \cdot (p_1 - p_2) \neq 0$. See Figure 2.25.

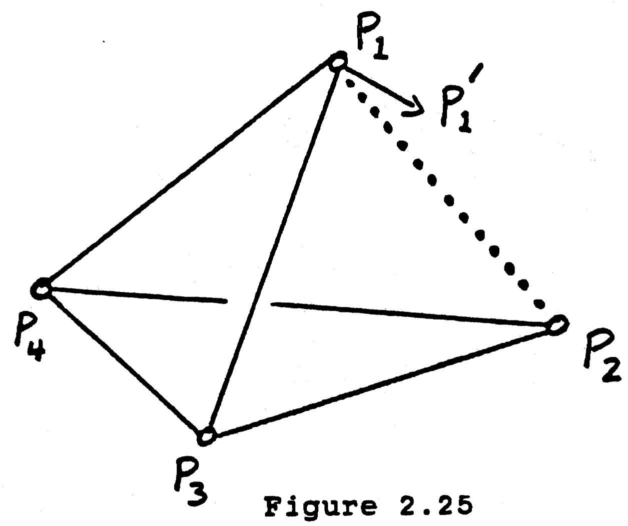


Figure 2.25

With these examples in mind we state the following.

Proposition 2.30: Any tensegrity framework $G(p)$ in $\mathbb{R}^d$ is infinitesimally rigid if and only if (i) or (ii) (or both) hold:

(i) $G(p)$ is a bar simplex.

(ii) The affine span of p is $\mathbb{R}^d$, and for all $\{i,j\} \subset \{1,\ldots, v\}$, $i \neq j$, (not just members of G), and for all infinitesimal flexes p' of $G(p)$

$$(2.2) \qquad (p_i - p_j)\cdot(p_i' - p_j') = 0.$$

Proof: If (i) holds then $G(p)$ is infinitesimally rigid by Proposition 2.20.

Suppose (ii) holds. By Corollary 2.24 there is a bar framework $H(p)$ which is infinitesimally rigid in $\mathbb{R}^d$. The Formula (2.2) implies that p' is an infinitesimal flex of $H(p)$, and thus p' is a trivial infinitesimal flex.

Conversely suppose neither (i) nor (ii) hold. If the affine span of p is not all of $\mathbb{R}^d$, then - using the assumption that $G(p)$ is not a bar simplex - either Example 2.28 or Example 2.29 imply that $G(p)$ is not infinitesimally rigid. If for some $i \neq j$, $\{i,j\} \subset \{1,\ldots,v\}$, Formula (2.2) does not hold, then by Proposition 2.9 p' is a non-trivial infinitesimal flex of $G(p)$ and thus $G(p)$ is not infinitesimally rigid in both cases.

14. The Rigidity Map

Recall from Section 8 that the set of all $p = (p_1, \ldots, p_v) \in \mathbb{R}^{vd}$, $p_i \in \mathbb{R}^d$, $i = 1, \ldots, v$, is called the <u>configuration space</u>.

Let H denote the (Lie) group of congruences of $\mathbb{R}^d$. H can be regarded concretely as

$$H = \{ (A, p_0) \mid A \text{ is an orthogonal } d \times d \text{ matrix}, p_0 \in \mathbb{R}^d \},$$

and the group operation is defined as

$$(A, p_0)(B, q_0) = (AB, Aq_0 + p_0),$$

where (A, p_0), $(B, q_0) \in H$. Note that $(A, p_0)p_i = Ap_i + p_0$, for $p_i \in \mathbb{R}^d$. Recall from Section 8 also that two configurations p, $q \in \mathbb{R}^{vd}$ are congruent if there is an $h \in H$, h regarded as a function, such that

$$\begin{bmatrix} h(p_1) \\ \vdots \\ h(p_0) \end{bmatrix} = \begin{bmatrix} q_1 \\ \vdots \\ q_0 \end{bmatrix},$$

and from Section 10 the orbit of p is

$$H_p = \{ q \mid q \sim p \}.$$

<u>Remark 2.31</u>: It turns out that H is a smooth manifold of dimension $d(d + 1)/2$, and if the affine span of p is d dimensional H_p is a smooth manifold of dimension $d(d + 1)/2$ as well.

Now suppose G is a fixed bar graph. We define a canonical differentiable map which will capture the rigidity information associated to any framework.

Let e be the number of edges of $G = (V;E)$. Similar to the configuration space, we define $\mathbb{R}^e$, called the space of metrics for G. Each coordinate of $\mathbb{R}^e$ will correspond to some member $\{i,j\} \in E$.

For any configuration $p = (p_1^T, \ldots, p_v^T)^T \in \mathbb{R}^{vd}$, we define a point in the space of metrics by

$$f_G(p) = f(p) = \begin{bmatrix} \vdots \\ (p_i - p_j)^2 \\ \vdots \end{bmatrix},$$

where $\{i,j\}$ is a typical member of G. We call $f_G = f: \mathbb{R}^{vd} \to \mathbb{R}^e$ the rigidity map associated to G.

We will reinterpret the rigidity properties of G in terms of its rigidity map f_G.

For any subset X of a Euclidean space and $\epsilon > 0$ define

$$N_\epsilon(X) = \{ y \mid |x - y| < \epsilon, x \in X\},$$

called the ϵ-neighborhood of X.

Proposition 2.32: Consider a configuration $p \in \mathbb{R}^{vd}$. A bar framework $G(p)$ in $\mathbb{R}^d$ is rigid if and only if there is an $\epsilon > 0$ such that

$$f_G^{-1}f_G(p) \cap N_\epsilon(H_p) \subset H_p.$$

Note $H_p \subset f_G^{-1}f_G(p) \cap N_\epsilon(H_p)$ always is true.

Proof: We use Definition 1 of rigidity, the topological definition. Suppose $G(p)$ is rigid by this definition. Let $\epsilon > 0$ be such that if $G(p) \sim G(q)$, then $p \sim q$, for $|p - q| < \epsilon$. But then clearly $G(p) \sim G(q)$ if and only if $q \in f_G^{-1}f_G(p)$. If $q \in N_\epsilon(H_p)$, then $|h(q) - p| < \epsilon$ for some congruence h. Since $G(p)$ is rigid $q \sim h(q) \sim p$. Thus $q \in H_p$.

Suppose $f_G^{-1}f_G(p) \cap N_\epsilon(H_p) \subset H_p$ for some $\epsilon > 0$. Suppose $G(p) \sim G(q)$ and $|p - q| < \epsilon$ for some $q \in \mathbb{R}^{vd}$. Then $q \in f_G^{-1}f_G(p) \cap N_\epsilon(H_p) \subset H_p$. So there is an $h \in H$ such that $q \subset h(p)$ and thus $q \sim p$. Thus $G(p)$ is rigid.

For any signed graph G and a configuration $p \in \mathbb{R}^{vd}$ with corresponding joints, consider any open set $U_p \subset \mathbb{R}^{vd}$ such that $p \in U_p$. If it is true that for all $q \in U_p$ such that $G(p) \gtrsim G(q)$ then $p \sim q$, then we say U_p is a rigidity neighborhood of p with respect to G. $G(p)$ is rigid if and only if p has some rigidity neighborhood U_p.

However, we must be careful to observe that if U_p is a rigidity neighborhood for p, it may not be a rigidity neighborhood for other configurations $q \in U_p$. For example,

Figure 2.17 shows a bar framework $G(p)$ where there are nearby configurations $G(q)$, $G(\bar{q})$ such that $G(q) \sim G(\bar{q})$, but $q \nsim \bar{q}$. Thus no rigidity neighborhood of p will be a rigidity neighborhood of q or $\bar{q}$. See Figure 2.26.

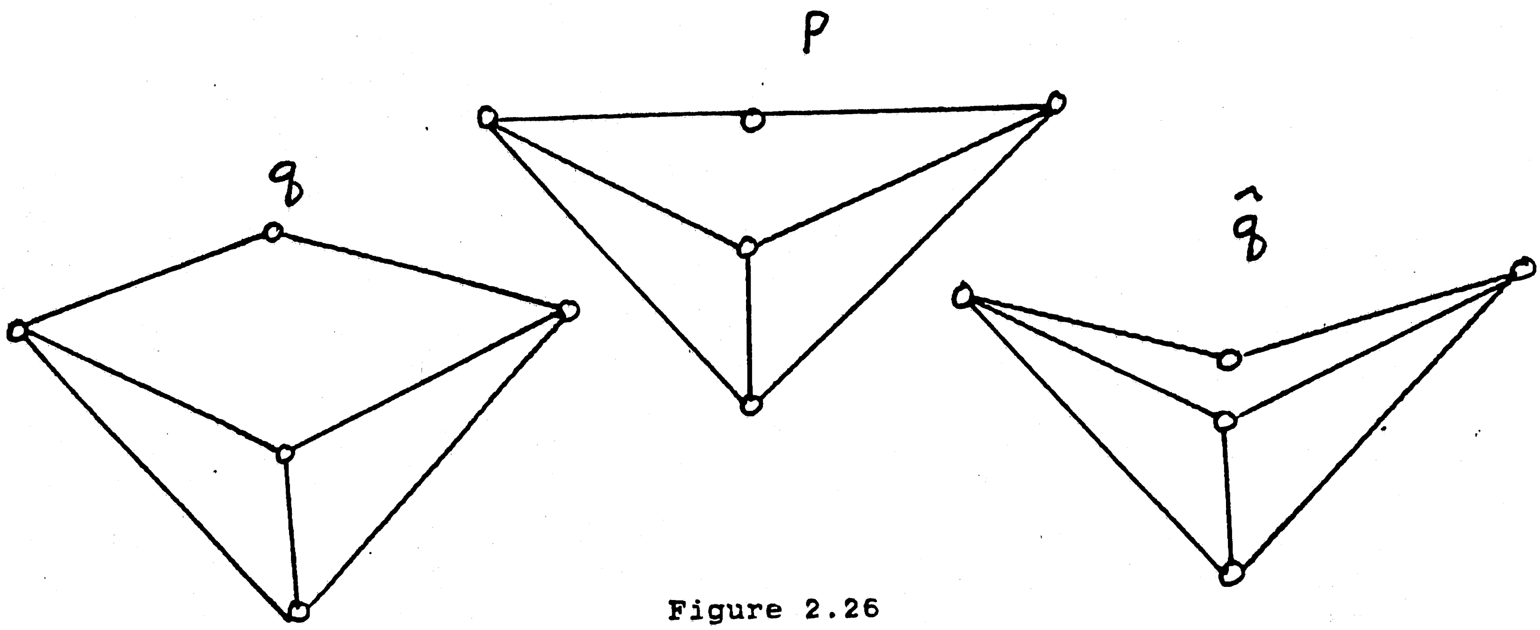


Figure 2.26

15. The Differential of the Rigidity Map

The infinitesimal flexes of a framework $G(p)$ in $\mathbb{R}^d$ naturally lie in the tangent space of the configuration space $\mathbb{R}^{vd}$; this tangent space (at a point $p \in \mathbb{R}^{vd}$) is of course naturally identified as $\mathbb{R}^{vd}$ again.

Since the rigidity map $f = f_G: \mathbb{R}^{vd} \to \mathbb{R}^e$ is differentiable, we can take the differential at a point p, $df_p: \mathbb{R}^{vd} \to \mathbb{R}^e$, where $\mathbb{R}^{vd}$ and $\mathbb{R}^e$ are properly regarded as the tangent spaces at p and $f(p)$ respectively.

We now wish to reinterpret the notion of infinitesimal rigidity in terms of df_p. Let p' be an infinitesimal flex of

$G(p)$. We now calculate the directional derivative of f in the direction p' at the point p. Thus we define

$$p(t) = p + tp',$$

for $0 \le t \le 1$. Note $p(0) = p$, and

$$\frac{dp(t)}{dt}\Big|_{t=0} = p' ,$$

$$f(p(t)) = \begin{bmatrix} \vdots \\ [p_i - p_j + t(p_i' - p_j')]^2 \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \vdots \\ (p_i - p_j)^2 + 2t(p_i - p_j)\cdot(p_i' - p_j') + t^2(p_i' - p_j')^2 \\ \vdots \end{bmatrix} .$$

So

$$df_p(p') = \frac{d}{dt}f(p(t))\Big|_{t=0} = \begin{bmatrix} \vdots \\ 2(p_i - p_j)\cdot(p_i' - p_j') \\ \vdots \end{bmatrix} .$$

The following is an immediate consequence of the above calculation.

Proposition 2.33: A vector p' in $\mathbb{R}^{vd}$ is an infinitesimal flex of the bar framework $G(p)$ if and only if p' is in the kernel of df_p.

Thus the bar framework $G(p)$ is infinitesimally rigid if and only if the kernel of df_p is contained in the trivial infinitesimal flexes of p.

We will denote the trivial infinitesimal flexes of $G(p)$ by T_p. Thus $T_p \subset \mathrm{kern}(df_p)$.

It is interesting to compute df_p as a matrix in terms of the standard basis of $\mathbb{R}^{vd}$. We see that the columns are grouped as v sets of d coordinates, and there are e rows corresponding to the bars of G. If $\{i,j\}$ is a bar of G, then the row $\{i,j\}$ has $2(p_i - p_j)^T$ as the i-th set of d coordinates and $2(p_j - p_i)^T$ as the j-th set of d coordinates. All the other entries of the row are zero.

$$df_p = \begin{bmatrix} & & \overset{i}{\vdots} & & & \overset{j}{} & \\ 0,\ldots, & 0, & 2(p_i - p_j)^T, & 0,\ldots, & 0, & 2(p_j - p_i)^T, & 0,\ldots,0 \\ & & \vdots & & & & \end{bmatrix}.$$

We check

$$df_p(p') = \begin{bmatrix} \vdots \\ 2(p_i - p_j)\cdot p_i' + 2(p_j - p_i)\cdot p_j' \\ \vdots \end{bmatrix}$$

$$= \begin{bmatrix} \vdots \\ 2(p_i - p_j)\cdot(p_i' - p_j') \\ \vdots \end{bmatrix},$$

so the above matrix is indeed the matrix of df_p. Because the factor 2 plays no essential role we define the <u>rigidity matrix</u> $R(p)$ for $G(p)$ as the matrix of $\frac{1}{2}df_p$.

Remark 2.34: Since the $\text{rank}(R(p)) + \dim \text{kernel}(R(p)) = vd$, we can use (ii) of Proposition 2.30 to show that if the affine span of p is $\mathbb{R}^d$, then $\text{rank}(R(p)) = vd - d(d + 1)/2$, or $\text{rank}(R(p)) \geq vd - d(d+1)/2$, or $\dim \text{kernel}(R(p)) \leq d(d + 1)/2$ are all equivalent to the infinitesimal rigidity of $G(p)$.

16. The Rank of the Rigidity Matrix.

Needless to say, the rank of the rigidity matrix is closely related to the rigidity of the bar framework $G(p)$.

Recall from Proposition 2.20 that any bar simplex in $\mathbb{R}^d$ is infinitesimally rigid, but from Example 2.28 and Example 2.29, if the dimension of the affine span of p is less than d, and $G(p)$ is not a bar simplex, then $G(p)$ is never infinitesimally rigid. With this in mind we state the relation between the rank of the rigidity matrix $R(p)$ - or equivalently the dimension of it null space - and the infinitesimal rigidity of $G(p)$.

Proposition 2.35: Let $G(p)$ be a bar framework in $\mathbb{R}^d$. Then $G(p)$ is infinitesimally rigid if and only if either (i) or (ii) (or both) hold.

(i) $G(p)$ is a bar simplex.

(ii) The affine space of p is $\mathbb{R}^d$ and

$$\dim(\text{kern } R(p)) = d(d + 1)/2.$$

Proof: By Proposition 2.33, to show $G(p)$ is infinitesimally rigid it is equivalent to show that the kernel of $R(p)$ is just the trivial infinitesimal flexes of p, T_p. The dimension of the linear space - a Lie algebra - of infinitesimal congruences of all of $\mathbb{R}^d$ by Corollary 2.11 is $d(d + 1)/2$, and we want to know when this is true for the trivial infinitesimal flexes at just p. So restriction to the points of p gives a linear map

$$\text{Infinitesimal congruences of } \mathbb{R}^d \longrightarrow T_p ,$$

which is clearly onto. If we regard $p' = 0$ as a trivial infinitesimal flex of p, Lemma 2.11 shows that the 0 infinitesimal flex of $\mathbb{R}^d$ is the only infinitesimal flex whose restriction is 0 on p, in case the affine span of p is $\mathbb{R}^d$. Thus the linear map above has 0 kernel, and both spaces have dimension $d(d + 1)/2$. Thus when (ii) holds, G(p) is infinitesimally rigid. When (i) holds, Proposition 2.19 implies that G(p) is infinitesimally rigid.

Conversely if neither (i) nor (ii) hold, then G(p) is not a bar simplex, and the affine span of p is not $\mathbb{R}^d$. Thus either Example 2.28 or Example 2.29 shows G(p) is not infinitesimally rigid, finishing the Proposition.

17. Counting

We make some simple observations about the relation between the number of vertices and edges of G, when G(p) is infinitesimally rigid.

Proposition 2.36: If $G(p)$ is an infinitesimally rigid bar framework with v vertices and e bars in $\mathbb{R}^d$ then

(2.3) $$vd - d(d + 1)/2 \leq e.$$

Proof: Note that if $G(p)$ is a bar simplex of dimension k, then $v = k + 1$, $e = \frac{k(k + 1)}{2}$, and for all integers k

$$vd - \frac{d(d + 1)}{2} = (k + 1)d - \frac{d(d + 1)}{2} \leq \frac{k(k + 1)}{2} = e.$$

(The above inequality holds since it is equality for $k = d$ and $k = d - 1$, it clearly holds for $k = 0$ say, and both sides are quadratic functions of k.)

For the case when the affine span of p is all of $\mathbb{R}^d$, we have at least two ways to finish the proof. The first way is to use Proposition 2.35. Since the rigidity matrix $R(p)$ has rank $vd - d(d + 1)/2$, the number of rows of $R(p)$ must be at least the rank. Thus $vd - d(d + 1)/2 \leq e$.

The second way is to simply observe that each of the e equations of (2.1), defining an infinitesimal flex p', is a linear equation in the vd variables of $p' \in \mathbb{R}^{vd}$. Since the trivial infinitesimal flexes define a linear subspace of dimension $d(d + 1)/2$ satisfying (2.1), we must have $vd - d(d + 1)/2 \leq e$ in order for the dimension of the solution space to be exactly $d(d + 1)/2$.

<u>Remark 2.37</u>: The inequality (2.3) comes up so often, it is worth mentioning two important special cases. For $d = 2$, (2.3) becomes

$$2v - 3 \leq e,$$

and for $d = 3$, (2.3) becomes

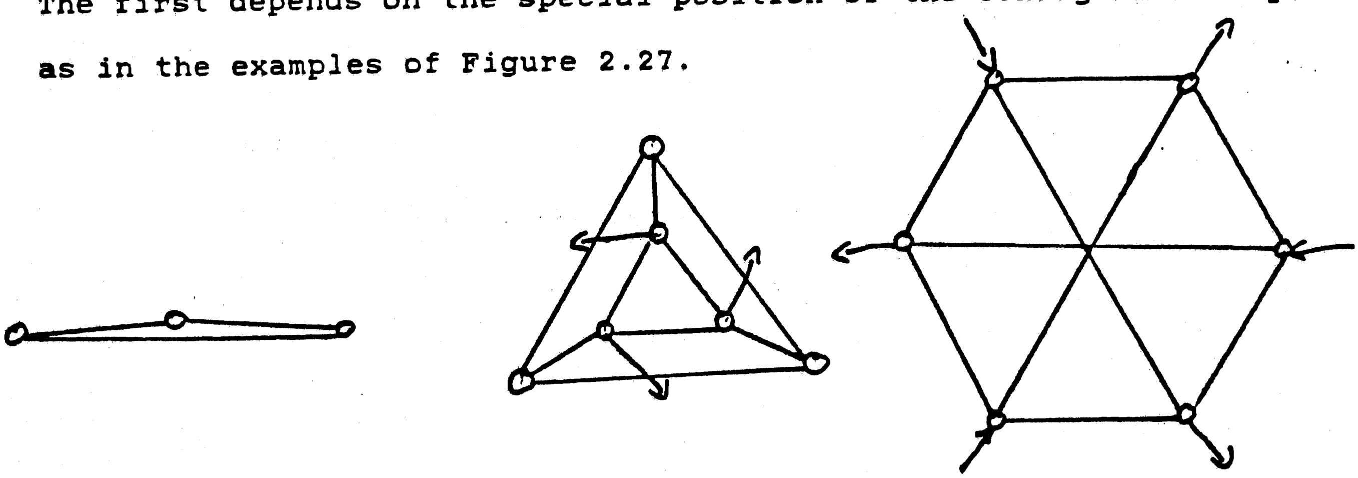
$$3v - 6 \leq e.$$

For $d = 1$ we get $v - 1 \leq e$. However, it is easy to check that if all the bars have non-zero length, a bar framework $G(p)$ in $\mathbb{R}^1$ is infinitesimally rigid if and only if G is connected.

Note that in any dimension, for any tensegrity framework $G(p)$, if any member has 0 length, then $G(p)$ is infinitesimally rigid if and only if $G(p)$ is infinitesimally rigid with that member removed.

<u>Remark 2.38</u>: We emphasize that Proposition 2.36 does <u>not</u> say that $vd - d(d + 1)/2 \leq e$ <u>guarantees</u> that $G(p)$ is infinitesimally rigid.

There are at least two possible kinds of examples where $vd - d(d + 1)/2 \leq e$ and yet $G(p)$ is not infinitesimally rigid. The first depends on the special position of the configuration p, as in the examples of Figure 2.27.



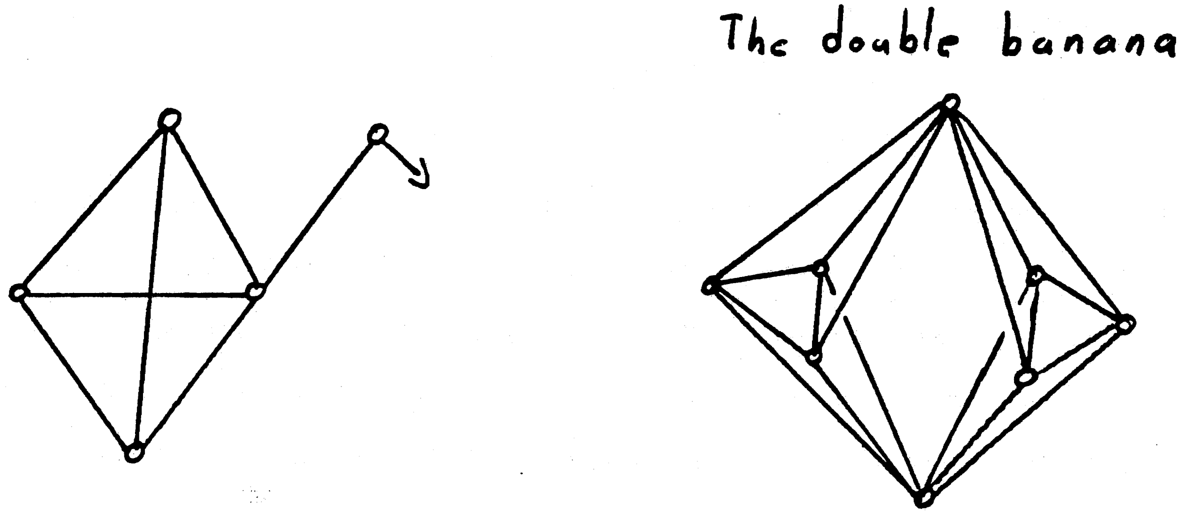
$v = 3,\ e = 3,\ d = 2$ $\quad$ $2v - 3 = 3 = e$

$v = 6,\ e = 9,\ d = 2$ $\quad$ $2v - 3 = 9 = e$

$v = 6,\ e = 9,\ d = 2$ $\quad$ $2v - 3 = 9 = e.$

Figure 2.27

The second type of example depends on G in that for any realization of G, $G(p)$ is not infinitesimally rigid, as in Figure 2.28.



$v = 5$, $e = 7$, $d = 2$

$2v - 3 = 7 = e$

$v = 8$, $e = 18$, $d = 3$

$3v - 6 = 18 = e$

Figure 2.28

<u>Remark 2.39</u>: It is easy to see that for the case when the affine span of p is all of $\mathbb{R}^d$, and $vd - d(d + 1)/2 < e$, then one can find a bar (in fact at least $vd - d(d + 1)/2 - e$ of them) such that when it is (or they are) removed the framework remains infinitesimally rigid.

For example in Figure 2.28 in $\mathbb{R}^2$ for the framework on the left $e = (2v - 3) + 1$, any bar can be deleted, and it will remain infinitesimally rigid. For the framework on the right $e = (2v - 3) + 1$ again but only one of the three bottom bars can be deleted to get an infinitesimally rigid framework.

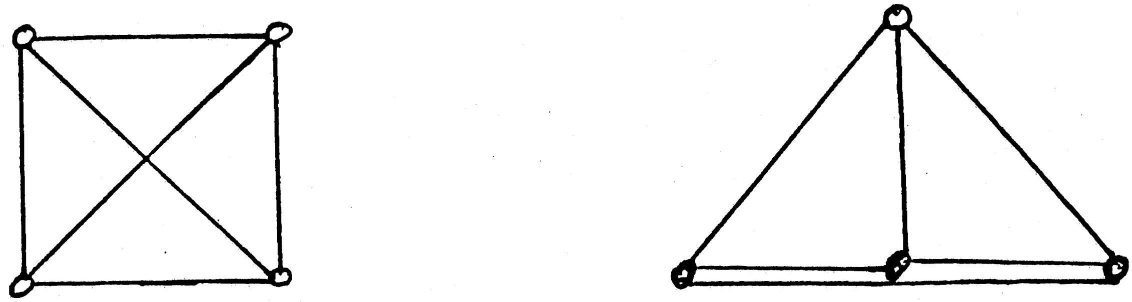


Figure 2.28

On the other hand, for a k simplex in $\mathbb{R}^d$, $k \leq d - 2$, and $vd - d(d + 1)/2 < e$. But by Example 2.29 such a framework with a bar deleted is never infinitesimally rigid.

Remark 2.30: For an infinitesimally rigid tensegrity framework $G(p)$ with at least one strut or cable, an argument similar to the argument in the second proof of Proposition 2.35 shows that

$$vd - d(d + 1)/2 + 1 \leq e,$$

where e is the total number of members of G. This depends on the fact that if the intersection of k half spaces through the origin in $\mathbb{R}^d$ is just the origin, then $k \geq d+1$. Each of the member inequality or equality constraints of Formula 2.1 can be regarded as defining at least a half space in the cordinates of p', where the coordinates of p are regarded as fixed.

We can also analyze those tensegrity frameworks where each member is needed for infinitesimal rigidity. Suppose we have k half spaces in $\mathbb{R}^d$ whose intersection is exactly the origin, and the collection of half spaces is minimal in that the intersection of any proper subcollection is not the origin. Then $k \leq 2d$.

Using this suppose that $G(p)$ is an infinitesimally rigid tensegrity framework in $\mathbb{R}^d$, with the affine span of p all of $\mathbb{R}^d$, with e_- cables, with e_+ struts, and with e_0 bars, such that the removal of any member makes the framework <u>not</u> infinitesimally rigid. Then

$$\frac{e_+ + e_-}{2} + e_0 \leq vd - \frac{d(d+1)}{2} .$$

For example, the framework of Figure 2.29 is infinitesimally rigid in $\mathbb{R}^3$, $e_- = 12$, $e_+ = 0$, $e_0 = 12$, $v = 8$, $d = 3$.

$$\frac{e_+ + e_-}{2} + e_0 = 18 = vd - \frac{d(d+1)}{2} .$$

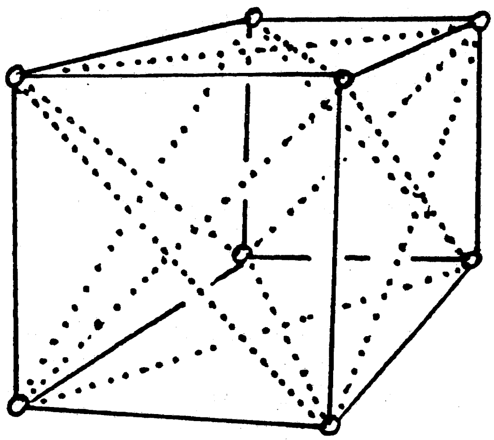


Figure 2.29

This is a cube with all edges as bars, and all facial diagonals as cables. The infinitesimal rigidity of this framework was proved in Connelly (1981) and Whiteley (19xx). See also Chapter xx. The removal of any member makes the framework infinitesimally flexible.

__Remark 2.31__: One must be careful not to confuse the notion of what the conditions of infinitesimal rigidity mean. A bar can "rigidify" only two vertices at a time. If three joints lie on a line with bars between them, it does not guarantee that any infinitesimal flex, when restricted to those three joints, is trivial. For example, in Figure 2.30 the framework on the left is not infinitesimally rigid.

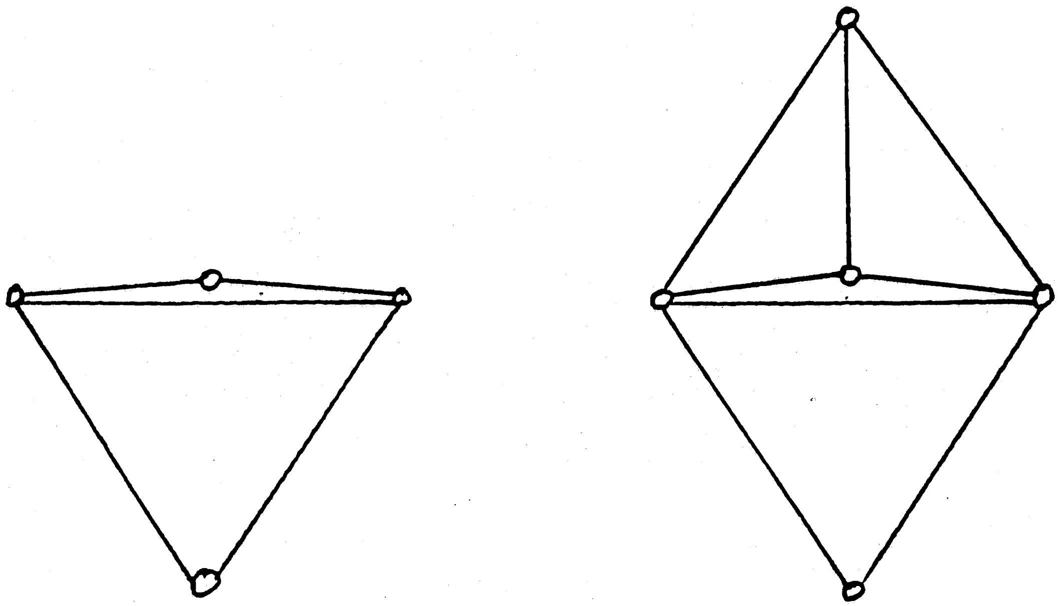


Figure 2.30

On the other hand, the framework on the right is infinitesimally rigid because of the additional bars. It is easy to construct any such rigidifying external structure, if that is what is desired.

Lastly we do our counting for pinned frameworks.

Proposition 2.32: Let $G(p)$ be an infinitesimally rigid bar framework in $\mathbb{R}^d$ with pinned joints whose affine span is d-dimensional or (d-1)-dimensional. Let v be the number of unpinned vertices of G, and e the number of bars of G. Then

$$vd \leq e.$$

If G is a tensegrity framework with at least one cable or strut but e members in all, then

$$vd + 1 \leq e.$$

Problems:

Problem 2.14: Suppose P is a convex polyhedron in $\mathbb{R}^3$. Let $p = (p_1,\ldots,p_v)$ be its points. Define a bar graph G on the vertices $v = \{1,\ldots,v\}$ by saying $\{i,j\}$ is a bar of G if $< p_i, p_j >$ is an edge of P. This defines a framework $G(p)$, in $\mathbb{R}^3$. If $G(p)$ is infinitesimally rigid in $\mathbb{R}^3$ show that all the faces of P are triangles.

Problem 2.15: If the dimension of the affine span of $p = (p_1,\ldots, p_v)$ is (d-1)-dimensional in $\mathbb{R}^d$, show that a bar framework $G(p)$ is infinitesimally rigid in $\mathbb{R}^3$ if and only if $\dim \operatorname{kern} R(p) = d(d + 1)/2$.

Problem 2.16: Suppose $G(p)$ is a bar framework with v vertices in $\mathbb{R}^d$, $\dim \operatorname{kern} R(p) = k \geq d(d + 1)/2$, and the affine space of p is all of $\mathbb{R}^d$. Consider $\mathbb{R}^d \subset \mathbb{R}^n$, $d < n$, and consider $G(p)$ as a bar framework in $\mathbb{R}^n$. For this new framework calculate the dimension of the kernel at $R(p)$.

Problem 2.17: For any finite graph let v_i be the number of vertices at degree i. (A vertex has degree i if it is joined to exactly i other vertices.) Define the average degree $\bar{v}$ as

$$\bar{v} = \frac{\sum_{i=1}^{\infty} i v_i}{\sum_{i=1}^{\infty} v_i} ,$$

where we note that $v = \sum_{i=1}^{\infty} v_i$ is the total number of vertices of G. If $G(p)$ is an infinitesimally rigid bar framework with v vertices, e bars in $\mathbb{R}^d$, with the affine span of p all of $\mathbb{R}^d$, show that

$$2d - \frac{d(d+1)}{v} \leq \bar{v} .$$

Problem 2.18: If $G(p)$ is an infinitesimally rigid bar framework in $\mathbb{R}^2$ with at least four vertices, show that $G(p)$ must have a vertex of degree four or more.

Problem 2.19: Suppose G is a _tree_, i.e. a connected graph with no cycles. Suppose also that the degree of at most one vertex is two, and G has v vertices. Let p be a configuration in $\mathbb{R}^2$ with v vertices such that no three points of p lie on an (affine) line. Show that $G(p)$ with all its _endpoints_ (i.e., vertices of degree one) pinned is infinitesimally rigid in $\mathbb{R}^2$.

18. The Averaging Method

There is a special technique that can be used to relate results about infinitesimal rigidity and rigidity itself. In the following $G(p)$ can be any tensegrity framework in $\mathbb{R}^d$.

<u>Proposition 2.41</u>: Let $p, q \in \mathbb{R}^d$ be two configurations in $\mathbb{R}^d$. Then

a. $G(p) \leq G(q)$ if and only if $p - q$ is an infinitesimal flex for $G(\frac{p + q}{2})$,

b. If $p - q$ is a trivial infinitesimal flex at $\frac{p + q}{2}$, then p is congruent to q,

c. If the affine span $< \frac{p + q}{2} >$ contains $p - q$ and p is congruent to q, then $p - q$ is a trivial infinitesimal flex at $\frac{p + q}{2}$.

<u>Proof</u>: Let $i, j \in V$ be any pair of vertices of G, not just those where $\{i,j\} \in E$. Then

$$[(p_i - q_i) - (p_j - q_j)]\cdot[(p_i + q_i)/2 - (p_j + q_j)/2]$$
$$= \frac{1}{2}[(p_i - p_j) - (q_i - q_j)]\cdot[(p_i - p_j) + (q_i - q_j)]$$
$$= \frac{1}{2}[(p_i - p_j)^2 - (q_i - q_j)^2].$$

When $\{i,j\}$ is a member of G, then the above equality implies a.

For part b, when $p - q$ is a trivial infinitesimal flex at $\frac{p + q}{2}$, then both sides of the equality are 0 for all $i \neq j$, and thus by Proposition 2.7 p is congruent to q.

For part c, if p is congruent to q, then the defining relations for an infinitesimal flex (2.1) are all 0. Since $< \frac{p + q}{2} >$ contains the infinitesimal flex $p - q$, Corollary 2.24 implies that $p - q$ is a trivial infinitesimal flex at $\frac{p + q}{2}$.

Remark 2.42: Proposition 2.41 can be restated where p is a fixed configuration and p' is a candidate for a infinitesimal flex. We replace p in the Proposition with $p + p'$ and q ith $p - p'$. The corresponding statements become the following:

a'. $G(p + p') \leq G(p - p')$ if and only if p' is an infinitesimal flex at p,

b'. If p' is a trivial infinitesimal flex at p, then $p + p'$ is congruent to $p + p'$,

c'. If the affine span $< p >$ contains p', and $p + p'$ is congruent to $p - p'$, then p' is a trivial infinitesimal flex at p.

This reformulation implies that every infinitesimally flexible framework $G(p)$ has a sequence of pairs of arbitrarily close realizations where $G(p - p')$ "snaps" to $G(p + p')$, since we can take p' to be arbitrarily small. This helps explain the empirical observation that infinitesimally non-rigid frameworks feel "shakey". In fact Wunderlich (xx) uses the word shakey for infinitesimally non-rigid frameworks.

Example 2.43: The following is an example of a framework $G(p)$ in $\mathbb{R}^2$ with a nontrivial infinitesimal flex p' where we see the two equivalant $G(p + p')$ and $G(p - p')$.

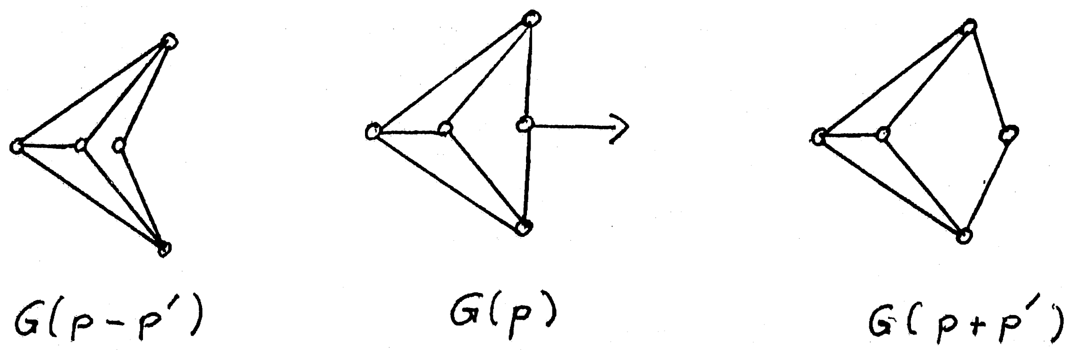


Figure 2.32

Remark 2.44: The spanning condition in part c is necassary. Consider the following congruent bar triangles $G(p)$ and $G(q)$ in $\mathbb{R}^2$. We obtain p by reflecting q about a line as in Figure 2.33 below.

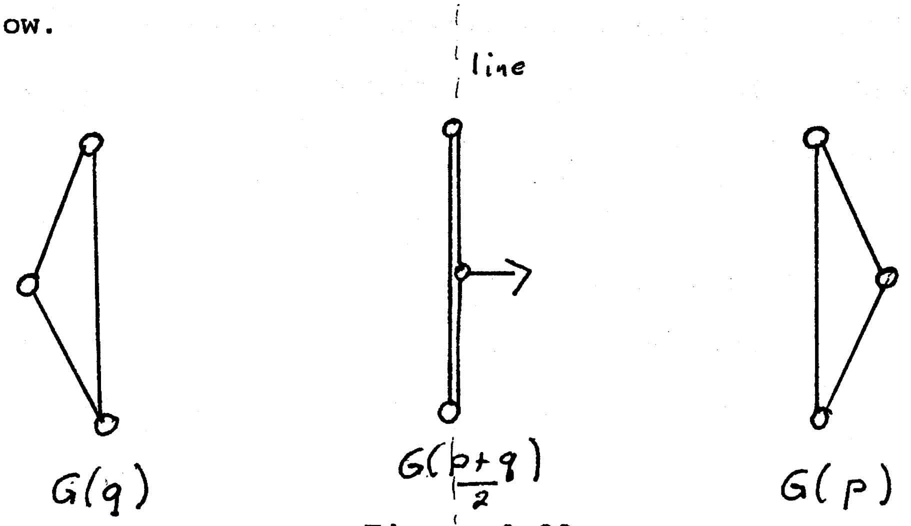


Figure 2.33

Here $p - q$ is a non-trivial infinitesimal flex of $G(\frac{p + q}{2})$ since $\frac{p + q}{2}$ lies on the line of reflection.

On the other hand it can turn out that even though $p - q$ is not contained in $< \frac{p + q}{2} >$ and p is congruent to q, $p - q$ will still be a trivial infinitesimal flex at $\frac{p + q}{2}$. The important point is whether the linear part of the congruence that takes p to q has any eigen values equal to -1. If not, then the conclusion of part c follows. This generalization of Proposition 2.41 is explored in the exercises.

Remark 2.45: For the case of a bar framework, we see that Proposition 2.41 gives us information about the rigidity map $f_G: \mathbb{R}^{vd} \to \mathbb{R}^e$. In particular, at a configuration $p \in \mathbb{R}^{vd}$, modulo congruences, f_G is a kind of folding map. For instance, modulo congruences, f_G cannot have an isolated singularity.

Similarly if $U \subset \mathbb{R}^{vd}$ is an open rigidity neighborhood for all $p \in U$, then each $G(p)$ for $p \in U$ must be infinitesimally rigid. For example, if G is the triangulation of a sphere, and

$G(\mathbf{p})$ is a strictly convex realization, then a classical Theorem of Cauchy (1813) implies that if $G(\mathbf{p})$ is equivalant to $G(\mathbf{q})$ and $G(\mathbf{q})$ is strictly convex as well, then $\mathbf{p}$ is congruent to $\mathbf{q}$. See Chapter xx. Thus we can take U to be the strictly convex realizations of G, and Proposition 2.41 implies that each such $G(\mathbf{p})$ is infinitesimally rigid, which is a Theorem of M. Dehn (19xx).

One could imagin trying to reverse the above argument as well. In other words suppose we know that all strictly convex realizations $G(\mathbf{p})$ of triangulations of a sphere are infinitesimally rigid. Let $G(\mathbf{p})$ be equivalant to $G(\mathbf{q})$. Then $\mathbf{p} - \mathbf{q}$ is an infinitesimal flex of $G(\frac{\mathbf{p} + \mathbf{q}}{2})$, but it is difficult to show, a priori, that $G(\frac{\mathbf{p} + \mathbf{q}}{2})$ is a strictly convex realization of G.

<u>Remark 2.46</u>: If G has some fixed vertices say $\mathbf{p}_1, \dots, \mathbf{p}_k$ such that the affine span $< \mathbf{p}_1, \dots, \mathbf{p}_k >$ includes $\mathbf{p}$ and $\mathbf{q}$, then b and c in Proposition 2.42 simply translate into the (trivial) statement that $\mathbf{p} - \mathbf{q}$ is the $\mathbf{0}$ infinitesimal flex if and only if $\mathbf{p} = \mathbf{q}$.

<u>Problems</u>:

<u>Problem 2.20</u>: If S is any skew symmetric matrix, show that $I - S$ is non-singular and $(I + S)(I - S)^{-1} = (I - S)^{-1}(I + S)$ is orthogonal.

<u>Problem 2.21</u>: Use Problem 2.20 directly to give another proof of b in Proposition 2.41.

<u>Problem 2.22</u>: Let A be an orthogonal matrix with no eigenvalues equal to -1. Show that $I + A$ is non-singular and that $(I + A)^{-1}(I - A) = (I - A)(I + A)^{-1}$ is skew symmetric.

Problem 2.23: Suppose that $\mathbf{p}$ and $\mathbf{q}$ are two congruent configurations in $\mathbb{R}^d$, where A is an orthogonal d by d matrix and $p_0 \in \mathbb{R}^d$ such that $q_i = Ap_i + p_0$ for each i. Use Problem 2.22 to show that $\mathbf{p} - \mathbf{q}$ is a trivial infinitesimal flex at $\frac{\mathbf{p}+\mathbf{q}}{2}$ if and only if A can be chosen to have no eigenvalues of -1. Show that this generalizes part c of Proposition 2.41.

19. Generic Rigidity and the Rigidity Map

We investigate some general facts about the topology of infinitesimally rigid realizations of tensegrity graphs.

Theorem 2.47: Let G be any tensegrity graph. Then

$$I = \{ \mathbf{p} \mid G(\mathbf{p}) \text{ is infinitesimally rigid in } \mathbb{R}^d \}$$

is an open subset of $\mathbb{R}^{vd}$.

Proof: For any configuration $\mathbf{p} \in \mathbb{R}^{vd}$ we define

$$T_{\mathbf{p}} = \{ \mathbf{p}' \in \mathbb{R}^{vd} \mid \mathbf{p}' \text{ is a trivial infinitesimal flex at } \mathbf{p} \},$$
$$E_{\mathbf{p}} = \{ \mathbf{q}' \in \mathbb{R}^{vd} \mid \mathbf{q}' \cdot \mathbf{p}' = 0 \text{ for all } \mathbf{p}' \in T_{\mathbf{p}} \} = T_{\mathbf{p}}^{\perp}$$
$$S^{vd-1} = \{ \mathbf{q} \in \mathbb{R}^{vd} \mid \mathbf{q} \cdot \mathbf{q} = 1 \}.$$

Note that the unit sphere S^{vd-1} is a compact subset of $\mathbb{R}^{vd}$. Later we will interpret the set $E_{\mathbf{p}}$ as the collection of "equilibrium forces" at $\mathbf{p}$, and we will explore its relation to infinitesimal rigidity. However, for now we only need that the following set is closed in $\mathbb{R}^{vd} \times \mathbb{R}^{vd}$. This will be proved in Section 21 without reference to Proposition 2.47 or its consequences.

$$E = \{ (\mathbf{p}, \mathbf{p}') \mid \mathbf{p}' \in E_{\mathbf{p}} \cap S^{vd-1} \}$$

So $(\mathbf{p}, \mathbf{p}') \in E$ if $\mathbf{p}'$ is a "normalized equilibrium force" at $\mathbf{p}$. Using that E is closed we will show that the following set

$$F = \{ \mathbf{p} \in \mathbb{R}^{vd} \mid G(\mathbf{p}) \text{ is infinitesimally flexible} \}$$

is closed in $\mathbb{R}^{vd}$. Since $I = \mathbb{R}^{vd} \setminus F$, we will have that I is open.

Let $p(1),\ldots,p(k),\ldots$ be a sequence of configurations in F converging to p. Let $p'(k)$ be a non-trivial infinitesimal flex of $G(p(k))$ for $k = 1, 2, \ldots$ .Let $q'(k) \in E_{p(k)} \cap S^{vd-1}$ be another infinitesimal flex of $G(p(k))$ obtained by subtracting the orthogonal projection of $p'(k)$ in $T_{p(k)}$ and then dividing the result by its length. Then $(p(k), q'(k)) \in E$ for all $k = 1, 2, \ldots$ .Since S^{vd-1} is compact and $q'(k) \in S^{vd-1}$, there is a subsequence which converges to a $q' \in S^{vd-1}$. Since E is closed $(p, q') \in E$, and q' is a non-trivial infinitesimal flex of $G(p)$. Thus $G(p)$ is infinitesimally flexible, F is closed, and I is open.

Remark 2.48: It is not true that the set

$$\{ (p, p') \in \mathbb{R}^{vd} \times \mathbb{R}^{vd} \mid p' \text{ is a trivial infinitesimal flex at } p \}$$

is closed. Consider a configuration p of three distinct points on a line in $\mathbb{R}^3$. Then there is an infinitesimal flex p' at p, shown in Figure 2.34, such that p' is non-trivial but p' is the limit of $p'(k)$, and p is the limit of $p(k)$, $k = 1, 2, \ldots$, but $p'(k)$ is a trivial infinitesimal flex at $p(k)$.

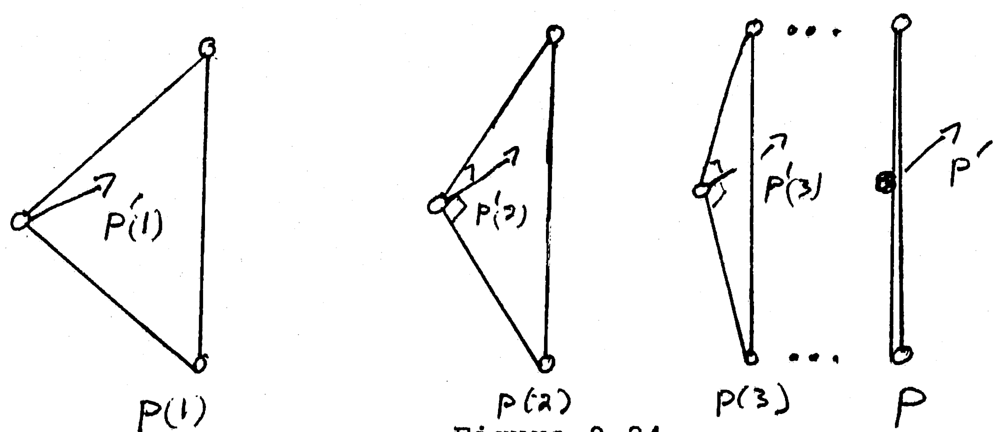


Figure 2.34

Notice that in the proof of Proposition 2.47 we need that there are only a finite number of vertices in G, and thus the unit sphere is compact. This is one of the key differences between the theory of rigidity developed here for frameworks and the theory of rigidity for smooth manifolds. In the next section we will see that infinitesimal rigidity implies rigidity, but such a result is not known even in the category of smooth two-manifolds in three-space. See Spivak (19xx) volume 5 for a good discussion of this kind of problem.

Corollary 2.49: Let G be any bar graph. Then I is either an open dense subset of $\mathbb{R}^{vd}$ or $I = \phi$.

Proof: Considering Proposition 2.47 we need only to show that I is dense in $\mathbb{R}^{vd}$, when $I \neq \phi$. Suppose G has all possible members as bars, and $v \leq d + 1$, where v is the number of vertices of G. Then

$$I = \{ (p_1, \dots, p_v) \mid p_1, \dots, p_v \text{ are affine independant } \}$$

is clearly dense. Otherwise by Proposition 2.35

$$I = \{ p \in \mathbb{R}^{vd} \mid \dim \ker df_p = d(d + 1)/2 \}$$

and

$$\mathbb{R}^{vd} \setminus I = \{ p \in \mathbb{R}^{vd} \mid \operatorname{rank} df_p \neq vd - d(d + 1)/2 \},$$

which is an algebraic set. That is, $\mathbb{R}^{vd} \setminus I$ is defined by polynomial equalities with variables the coordinates of p. Thus by basic results from Algebraic Geometry, see Harthshorn (19xx) page xx for instance, we see that I is dense in $\mathbb{R}^{vd}$.

With the above Corollary in mind, we call a bar graph G generically d-rigid or just d-rigid if the set of infinitesimally rigid realizations of G in $\mathbb{R}^d$ are dense. We leave off the d if the dimension is understood. By the Corollary G is generically rigid if and only if there is one $\mathbf{p}$ where $G(\mathbf{p})$ is infinitesimally rigid.

We have already seen many examples such as in Figure 2.15 and in Figure 2.16 when a configuration $\mathbf{p}$ corresponds to an infinitesimal $G(\mathbf{p})$. On the other hand, the graphs of the frameworks of Figure 2.27 are generically 2-rigid, but in the given configurations they are not infinitesimally rigid. When G is generically d-rigid, but $G(\mathbf{p})$ is not infinitesimally rigid in $\mathbb{R}^d$, we say that $G(\mathbf{p})$ is in <u>special position</u>. In Figure 2.28 the graphs are not even generically 2-rigid or 3-rigid respectively.

One point of view of the above is that to test whether G is generically rigid one simply takes a "generic" point $\mathbf{p}$ and asks whether $G(\mathbf{p})$ is infinitesimally rigid. Here generic means that the coordinates of $\mathbf{p}$ do not satisfy any non-zero algebraic polynomial equation with rational coeficients. This test for rigidity is easier said than done.

A similar point of view is to look at a <u>regular point</u> of f_G. This is a point $\mathbf{p}$, where the rank of $df_{\mathbf{p}}$ is maximal for a fixed graph G. From the proof of the Corollary we get the following. See Asimow and Roth (19xx).

<u>Corollary 2.50</u>: Let G be any bar graph. G is generically rigid if and only if $G(\mathbf{p})$ is infinitesimally rigid at a regular point $\mathbf{p}$ of f_G.

From the point of view of rigidity, a configuration $\mathbf{p}$ is a regular point for G if the space of infinitesimal flexes has minimal dimension for a given graph G. For instance, the frameworks of Figure 2.28 correspond to regular points even though their graphs are not generically rigid. The frameworks of Figure 2.34 are examples of configurations which do not correspond to regular points and their graphs are not generically rigid. Note that the framework on the right is rigid.

Figure 2.34

Remark 2.51: We must keep in mind that for tensegrity frameworks it may very well turn out that the infinitesimally rigid realizations are not dense. For example, the frameworks in Figure 2.35, if $\mathbf{p}_1$ is in the open triangle formed by $\mathbf{p}_2$, $\mathbf{p}_3$, $\mathbf{p}_4$, then $G(\mathbf{p})$ is infinitesimally rigid. Otherwise $G(\mathbf{p})$ has a non-trivial infinitesimal flex. Such infinitesimally flexible configurations have a non-empty interior.

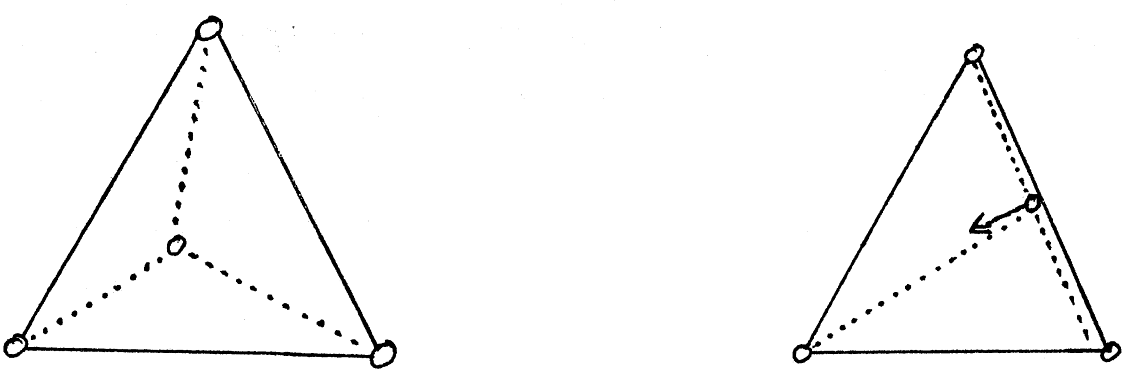


Figure 2.35

Remark 2.52: Generic d-rigidity is a combinatorial property of a graph, and it is of interest to find combinatorial characterizations of generically d-rigid graphs. For $d = 2$ a Theorem of Laman (19xx) does just that. See Chapter xx for a proof and more discussion.

Theorem 2.53: Let G be a bar graph with e edges and v vertices such that $e = 2v - 3$, and $e' \leq 2v' - 3$ for every subgraph with v' vertices and e' edges. Then G is generically 2-rigid.

Using this result Lovas and Yemeni (19xx) describe an efficient algorithm for combinatorially the 2-rigidity of any graph.

For dimensions greater than two, no Laman-type chacterization is known. And certainly there is no known efficient algorithm for determining even 3-rigidity for all graphs. The irony is that generic d-rigidity for any graph can be determined by choosing a "random" $\mathbf{p} \in \mathbb{R}^{vd}$ and computing the rank of the rigidity matrix. If the rank is $vd - d(d + 1)/2$, then we can say with certainty that $G(\mathbf{p})$ is infinitesimal rigid, and thus G is generically d-rigid. However, for many graphs G, it is very difficult to say, a priori, what configurations $\mathbf{p}$ correspond to those special positions where $G(\mathbf{p})$ is not infinitesimally rigid. Thus if $G(\mathbf{p})$ turns out to be infinitesimally flexible, we cannot tell with certainty (only with probability 1) that G in not generically d-rigid.

20. Rigidity and Infinitesimal Rigidity

We have seen several examples of rigid frameworks that are not infinitesimally rigid. We will now show the converse. Because of the central nature of this Theorem we will give three proofs, each having their advantages and disadvantages.

<u>Theorem 2.54</u>: Let $G(\mathbf{p})$ be any infinitesimally rigid tensegrity framework in $\mathbb{R}^d$. Then $G(\mathbf{p})$ is rigid in $\mathbb{R}^d$.

<u>Proof 1</u> (Alexandrov [19xx] and Gluck [19xx]): This proof only seems to work for the case when $G(\mathbf{p})$ is a bar framework but it has the advantage that some of the ideas apply to the case when the framework $G(\mathbf{p})$ is not infinitesimally rigid.

If $G(\mathbf{p})$ is a rod simplex the result follows easily from Proposition 2.7. Thus by Proposition 2.30 we can assume that the affine span of $\mathbf{p}$ is all of $\mathbb{R}^d$. We know that the orbit of $\mathbf{p}$ $H_{\mathbf{p}}$ ($H_{\mathbf{p}}$ is those configurations that are congruent to $\mathbf{p}$.) is contained in $f^{-1}f(\mathbf{p})$, where $f\colon \mathbb{R}^{vd} \to \mathbb{R}^d$ is the rigidity map for G as in Section 14. By Section 19 we know that $\mathbf{p}$ is a regular point of f. In other words (see Proposition 2.35) the dimension of the kernal of $df_{\mathbf{p}}$ is $d(d + 1)/2$. We now apply the implicit function theorem to conclude that $f^{-1}f(\mathbf{p})$ is a manifold of dimension $d(d + 1)/2$ in some neighborhood N of $H_{\mathbf{p}}$. Thus $f^{-1}f(\mathbf{p}) \cap N = H_{\mathbf{p}}$ and by Proposition 2.32 $G(\mathbf{p})$ is rigid.

Note that the above idea works even if $G(\mathbf{p})$ is not infinitesimally rigid. If $\mathbf{p}$ is a regular point for f, then the manifold $f^{-1}f(\mathbf{p})$, near $\mathbf{p}$, has dimension equal to the dimension of the infinitesimal flexes of $G(\mathbf{p})$. Thus at a regular point $\mathbf{p}$, $G(\mathbf{p})$ is rigid if and only if $G(\mathbf{p})$ is infinitesimally rigid. See Asimow and Roth [19xx].

Proof 2 (W. Whiteley): By Proposition 2.47 $I = \{ q \mid G(q)$ is infinitesimally rigid $\}$ is open in $\mathbb{R}^{vd}$. Let $q \in I$ be close enough to p such that $\frac{p + q}{2} \in I$. If $G(p) \leq G(q)$, then $p - q$ is an infinitesimal flex of $\frac{p + q}{2}$ by Proposition 2.41. Since $\frac{p + q}{2} \in I$, $p - q$ is a trivial infinitesimal flex at p, and thus p is congruent to q by Proposition 2.41 again. Thus $G(p)$ is infinitesimally rigid by Definition 1 of rigidity.

Proof 3 (R. Connelly [19xx]): We provide a only sketch of this proof. Suppose that $G(p)$ is not rigid. We will show that $G(p)$ is not infinitesimally rigid.

By Definition 3 of rigidity there is an analytic flex $p(t)$, $0 \leq t \leq 1$, such that $p(0) = p$, and $p(t)$ is not congruent to p for some $0 < t \leq 1$ (and thus for but all but a finite number of such t). As before we can assume that the affine span of p is all of $\mathbb{R}^d$. Then consider the smallest $k = 1, 2, 3, \ldots$ such that

$$\left(\frac{d}{dt}\right)^k (p_i(t) - p_j(t))^2 \Big|_{t=0} \neq 0,$$

for some $i \neq j$, $\{i,j\}$ not necessarily a member of G. Such a k exists since, if not, then since $p(t)$ is analytic all distances would be preserved and $p(t)$ would be a restriction of a congruence by Proposition 2.7.

It is then possible to find a trivial analytic flex $h_t: \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\left(\frac{d}{dt}\right)^n p_i(t) \Big|_{t=0} = \left(\frac{d}{dt}\right)^n h_t(t) \Big|_{t=0}, \text{ for } n = 1, \ldots, k - 1.$$

Then $q_i(t) = h_t^{-1} p_i(t)$ is a new analytic flex, replacing $p_i(t)$ such that

$$\left(\frac{d}{dt}\right)^n q_i(t) \Big|_{t=0} = 0, \text{ for } n = 1, \ldots, k-1,$$

and

$$(\frac{d}{dt})^k(q_i(t) - q_j(t))^2 = (\frac{d}{dt})^k(p_i(t) - p_j(t))^2, \text{ for } i \neq j.$$

Define

$$p_i' = (\frac{d}{dt})^k q_i(t)|_{t=0}.$$

Then

$$2(p_i - p_j)\cdot(p_i' - p_j') = (\frac{d}{dt})^k(q_i(t) - q_j(t))^2|_{t=0},$$

and thus p' is a non-trivial infinitesimal flex of $G(p)$. Thus $G(p)$ is not infinitesimally rigid as was to be shown.

Remark 2.55: Proof 2 has the virtue of being very much self contained and elementary using some special properties of the rigidity map. Proof 1 uses the implicit function theorem of course, but gives a very nice point of view of what rigidity means even if it only applies to bar frameworks. Proof 3 uses some non-trivial algebraic geometry, but has the virtue that it extends to show that second order rigidity implies rigidity. Proof 1 and Proof 2 do not seem to work in the context of second order flexes. See Connelly (19xx).