Estimates of Green function for some perturbations of subordinated Brownian motion

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July 22 2010

joint work with Kim and Song
Sharp estimates for the Green function $G_D^X(x,y)$ for the killed processes $X^D$.

1. Chen and Song (1998) Symmetric stable processes for $C^{1,1}$ domain.
5. Kim, Song and Vondraček (2010) Large class of subordinated Brownian motions for bounded $C^{1,1}$ open set.
We want to prove that the Green function $G_D^X(x, y)$ for the large class of SBM and the Green function of its perturbation $G_D^Y(x, y)$ are comparable for bounded connected $\kappa$ fat open set $D$. That is,

$$c^{-1}G_D^X(x, y) \leq G_D^Y(x, y) \leq cG_D^X(x, y)$$
Definition. A Levy process is a stochastic processes 

\( X = (X_t) \) in a probability space \((\Omega, \mathcal{F}, P)\) which satisfies,

1. \( X_0 = 0 \) \( P \)-a.s.
2. \( X_t \) has an independent increments.
3. \( X_t \) has a stationary increments.
4. \( X_t \) has a right continuous sample path with left limit \( P\)-a.s.
The Levy-Khintchine formula

\( \hat{\mu}(z) = \int e^{i(z,x)} \mu(dx) \). 

\[
\hat{\mu}(z) = \exp \left( -\frac{1}{2} (z, Az) + i(\gamma, z) + \int_{\mathbb{R}^d \setminus \{0\}} \left( e^{i(z,x)} - 1 - i(z,x)1_D(x) \right) \nu(dx) \right)
\]

\( A \) is a symmetric nonnegative definite \( d \times d \) matrix, \( \gamma \in \mathbb{R}^d \), 
\( D = \{|x| < 1\} \) and \( \nu \) is a measure in \( \mathbb{R}^d \) satisfying \( \nu(\{0\}) = 0 \), 
\( \int_{\mathbb{R}^d} (1 \wedge |x|^2) \nu(dx) < \infty \).
Killed processes

$D$ open set in $\mathbb{R}^d$.

$\tau_D = \inf\{ t > 0 | X_t \notin D \}$.

$$X_t^D := \begin{cases} X_t, & \text{if } t < \tau_D, \\ \partial, & \text{if } t \geq \tau_D. \end{cases}$$ (1)
**Definition** Let $W_t$ be a Brownian motion and $S_t$ a subordinator (increasing Levy process) which is independent to $W_t$.

$$X_t := W_{S_t}$$

is called a subordinated Brownian motion.
From now on, we will focus on a specific type of SBM. Let $S_t$ be a subordinator whose Laplace transform is given by

$$\mathbb{E} \left[ \exp(-\lambda S_t) \right] = \exp(-t\phi(\lambda))$$

where $\phi(\lambda)$ is a complete Bernstein function such that

$$\phi(\lambda) = \lambda^{\alpha/2} \ell(\lambda),$$

where $\ell$ is a slowly varying function at $\infty$. ($\ell(x)$ is a slowly varying function at $\infty$ if $\lim_{x \to \infty} \frac{\ell(ax)}{\ell(x)}$ for any $a > 0$.)

Then, $\mathbb{E} \left[ \exp(i\langle \xi, X_t \rangle) \right] = \exp(-t\Phi(\xi))$, where

$$\Phi(\xi) = \phi(|\xi|^2) = |\xi|^\alpha \ell(|\xi|^2), \quad \xi \in \mathbb{R}^d.$$  

In particular, $p^X(t, x, y)$ exists and $C^\infty$. 

Kim, Song and Vondraček (2009)
Under additional assumption on the slowly varying function $\ell$,

$$G^X(x, y) \sim \frac{\alpha \Gamma((d - \alpha)/2)}{2^{\alpha+1} \pi^{d/2} \Gamma(1 + \alpha/2)} \frac{1}{|x - y|^{d-\alpha} \ell(|x - y|^{-2})}, |x - y| \to 0.$$ 

$$\nu^X(x) \sim \frac{\alpha \Gamma((d + \alpha)/2)}{2^{1-\alpha} \pi^{d/2} \Gamma(1 - \alpha/2)} \frac{\ell(|x|^{-2})}{|x|^{d+\alpha}}, |x| \to 0.$$
\( \kappa \) fat domain

\( \kappa \) fat domain \( \exists \kappa \in (0, 1/2] \) and \( \exists \ R > 0 \) such that each \( Q \in \partial D \) and \( r \in (0, R), \ B(A_r(Q), \kappa r) \subset D \cap B(Q, r) \).

\( C^{1,1} \) domain \( \subseteq \) Lipschitz domain \( \subseteq \) \( \kappa \) fat domain.
Suppose $X$ is a subordinated Brownian motion. Suppose $Y$ is a Levy process satisfying following conditions.

1. $\sigma(x) = \nu^X(x) - \nu^Y(x) \geq 0$, 
2. $\sigma$ is a finite measure, 
3. $|\sigma(x)| \leq c|x|^{-d+\rho}, \quad \rho > 0, \quad |x| \leq 1$.

**Decomposition** $X = Y + T$, where $T$ is a compound Poisson process, independent to $Y$. 
Theorem

Let $D$ be a bounded connected $\kappa$ fat open set in $\mathbb{R}^d$, $d \geq 2$. Then there exists a constant $c = c(D, d, X, Y)$ such that,

$$c^{-1} G^X_D(x, y) \leq G^Y_D(x, y) \leq c G^X_D(x, y).$$
Let $D$ be a bounded connected set in $\mathbb{R}^d$.

**Property A** There is a constant $c$ such that,

$$\mathbb{E}_x \tau_D \mathbb{E}_y \tau_D \leq c G_D(x, y)$$

Property A holds if there is a $r > 0$ such that,

$$\inf_{x \in B(0, r)} \nu(x) > 0.$$

$\Rightarrow X, Y$ has property A.
Theorem (Grzywny, Ryznar 2007)

If either $X$ or $Y$ satisfies property A, then for all $x, y \in D$,

$$G_Y^D(x, y) \leq cG_X^D(x, y)$$

for some constant $c$. 
Green function comparability \( \geq \) side

From now on, \( D \) is a bounded, connected, \( \kappa \) fat open set.

When \( |x - y| \geq \theta \), ( \( \theta \) will be determined later.)

\[
x_0 \in \{ x \in D | \delta_D(x) \geq r_0/2 \}, \quad |x_1 - x_0| = r_0/4
\]

\[
\phi_D(x) := G_D^X(x, x_0) \land 1
\]

\[
r = r(x, y) := \delta_D(x) \lor \delta_D(y) \lor |x - y|
\]

\[
B(x, y) := \begin{cases} 
\{ A \in D | B(A, \kappa r) \subset D \cap B(x, 3r) \cap B(y, 3r) \}, & \text{if } r \leq r_0/32, \\
\{ x_1 \}, & \text{if } r > r_0/32. 
\end{cases}
\]

(2)
Green function comparibility ≥ side

Theorem (Kim, Song and Vondraček 2010)

There exists \( c = c(D, \alpha) \) such that for every \( x, y \in D \),
\[
c^{-1} \frac{\phi_D(x) \phi_D(y)}{\phi_D(A)^2 |x-y|^{d-\alpha} \ell(|x-y|^{-2})} \leq G_D^X(x, y) \leq c \frac{\phi_D(x) \phi_D(y)}{\phi_D(A)^2 |x-y|^{d-\alpha} \ell(|x-y|^{-2})},
\]
where \( A \in B(x, y) \).

Lemma

\[
\phi_D(x) \asymp \mathbb{E}_x[\tau_D^X]
\]

Lemma

When \( |x - y| \geq \theta \),
\[
\phi_D(A) \asymp \mathbb{E}_A[\tau_D] \geq \mathbb{E}_A[\tau_B(A, r_\theta)] \geq c \frac{r_\theta^\alpha}{\ell(r_\theta^{-2})}
\]
Green function comparability $\geq side, |x - y| \geq \theta$

Proposition ($|x - y| \geq \theta$)

When $|x - y| \geq \theta$,

$$G_D^X(x, y) \leq c_1 \mathbb{E}_x^X[\tau_D] \mathbb{E}_y^X[\tau_D]$$
$$\leq c_2 \mathbb{E}_x^Y[\tau_D] \mathbb{E}_y^Y[\tau_D]$$
$$\leq c_3 G_D^Y(x, y) \text{ since } Y \text{ satisfies property A.}$$
Green function comparibility $\geq \text{side, } |x - y| < \theta$

**Theorem (Generalized 3G theorem for $X$)**

Suppose that $D$ is a bounded $\kappa$-fat open set. Then there exist positive constants $c = c(D, \alpha)$ and $\gamma < \alpha$ such that for every $x, y, z, w \in D$,

$$
\frac{G^X_D(x, y)G^X_D(z, w)}{G^X_D(x, w)} \leq c \left[ \frac{\ell(|x - y|^{-2})}{|x - y|^{\gamma}} \left( \frac{|x - w|^{\gamma}}{\ell(|x - w|^{-2})} \wedge \frac{|y - z|^{\gamma}}{\ell(|y - z|^{-2})} \right) \vee 1 \right]
$$

$$
\times \left[ \frac{\ell(|z - w|^{-2})}{|z - w|^{\gamma}} \left( \frac{|x - w|^{\gamma}}{\ell(|x - w|^{-2})} \wedge \frac{|y - z|^{\gamma}}{\ell(|y - z|^{-2})} \right) \vee 1 \right]
$$

$$
\times \frac{\ell(|x - w|^{-2})}{\ell(|x - y|^{-2})\ell(|z - w|^{-2})} \frac{|x - w|^{d-\alpha}}{|x - y|^{d-\alpha} |z - w|^{d-\alpha}}
$$
Proof when $|x - y| < \theta$

$$G_D^X(x, y) \leq G_D^Y(x, y) + \int_D \int_D G_D^X(x, z)\sigma(z - w)G_D^X(w, y)dzdw$$

**Case 1**

$$\frac{G_D^X(x, y)G_D^X(z, w)}{G_D^X(x, w)} \leq c \frac{\ell(|x - w|^2)}{\ell(|x - y|^{-2})\ell(|z - w|^2)} \frac{|x - w|^{d - \alpha}}{|x - y|^{d - \alpha} |z - w|^{d - \alpha}}$$

**RHS**

$$\leq c_1 \int_D \int_D G_D^X(x, y)\sigma(y - z)G_D^X(z, w)dydz$$

$$\leq c_1 \int_D \int_D \frac{|x - y|^{-d + \alpha}}{\ell(|x - y|^{-2})} \cdot |y - z|^{-d + \rho} \cdot \frac{|z - w|^{-d + \alpha}}{\ell(|z - w|^{-2})} dydz$$

$$\times |x - w|^{d - \alpha} \ell(|x - w|^{-2}) G_D^X(x, w)$$

$$\leq c_1 \int_D \int_D \frac{|x - y|^{-d + \alpha - \epsilon_1}}{|x - y|^{-\epsilon_1} \ell(|x - y|^{-2})} \cdot |y - z|^{-d + \rho} \cdot \frac{|z - w|^{-d + \alpha - \epsilon_1}}{|z - w|^{-\epsilon_1} \ell(|z - w|^{-2})}$$

$$\times |x - w|^{d - \alpha} \ell(|x - w|^{-2}) G_D^X(x, w).$$
Proof

Note that

\[
\sup_{y \in D} \frac{1}{|x - y|^{-\epsilon \ell (|x - y|^{-2})}} \leq \sup_{z \in 2D} \frac{1}{|z|^{-\epsilon \ell (|z|^{-2})}} \equiv c_2
\]

\[
RHS \leq c_1 c_2^2 |x - w|^{d - \alpha \ell (|x - w|^{-2})} G_D^X(x, w)
\times \int_D \int_D |x - y|^{-d + \alpha - \epsilon_1} |y - z|^{-d + \rho} |z - w|^{-d + \alpha - \epsilon_1} dydz
\leq c_1 c_2^2 |x - w|^{d - \alpha \ell (|x - w|^{-2})} G_D^X(x, w)
\times |x - w|^{-d + \rho + 2(\alpha - \epsilon_1)}, \quad \text{when} \ 2\alpha + \rho - d < 0, 2\epsilon_1 < \rho + \alpha
\leq c_3 \cdot |x - w|^{\zeta_1} D^{\zeta_2 \ell (|x - w|^{-2})} G_D^X(x, w), \quad \zeta_1 > 0, \quad \zeta_2 \geq 0,
\text{when} \ 2\alpha + \rho - d < 0, 2\epsilon_1 < \rho + \alpha.
For the bounded connected $C^{1,1}$ domain $D$, there is a constant $c = c(D, d, X, Y)$ such that the Green function $G^Y_D(x, y)$ satisfies the following estimates:

$$
\begin{align*}
&c^{-1} \left(1 \wedge \frac{\delta_D(x)^{\frac{\alpha}{2}} \delta_D(y)^{\frac{\alpha}{2}} \ell(|x-y|^{-2})}{(\ell(\delta_D(x)^{-2}))^{1/2} (\ell(\delta_D(y)^{-2}))^{1/2} |x-y|^\alpha} \right) \frac{1}{\ell(|x-y|^{-2}) |x-y|^{d-\alpha}} \leq \\
&G^Y_D(x, y) \leq \\
&c \left(1 \wedge \frac{\delta_D(x)^{\frac{\alpha}{2}} \delta_D(y)^{\frac{\alpha}{2}} \ell(|x-y|^{-2})}{(\ell(\delta_D(x)^{-2}))^{1/2} (\ell(\delta_D(y)^{-2}))^{1/2} |x-y|^\alpha} \right) \frac{1}{\ell(|x-y|^{-2}) |x-y|^{d-\alpha}}
\end{align*}
$$
References


P. Kim, R. Song, Z. Vondraček, Two-sided Green function estimates for killed subordinate Brownian motions *To be appeared*


Thank You!