Limit shapes outside the percolation cone

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July 14, 2011
We consider **first-passage percolation** on the square lattice $\mathbb{Z}^2$.

- We place i.i.d. non-negative *passage times* $(\tau_e)$ on the edges of the square lattice.
- The passage time of a path $\gamma$ is

$$\tau(\gamma) = \sum_{e \in \gamma} \tau_e.$$
We consider first-passage percolation on the square lattice $\mathbb{Z}^2$.

- The passage time between two vertices $x$ and $y$ is

$$\tau(x, y) = \min_{\gamma: x \rightarrow y} \tau(\gamma).$$

- If $\tau_e \neq 0$ then $\tau$ is a metric.
We consider first-passage percolation on the square lattice $\mathbb{Z}^2$.

- The ball

$$B(t) = \{ x : \tau(0, x) \leq t \}$$

grows as $t$ increases.

- Richardson and Cox-Durrett proved a (very strong) law of large numbers for $B(t)$.
Theorem (Cox-Durrett)

Let $\mu$ have finite mean and be such that $\mu(\{0\}) < p_c$, the critical probability for bond percolation on $\mathbb{Z}^2$. There exists a deterministic compact, convex set $B_\mu$ with non-empty interior such that for each $\varepsilon > 0$,

$$\mathbb{P}\left((1 - \varepsilon)B_\mu \subseteq \frac{1}{t}B(t) \subseteq (1 + \varepsilon)B_\mu \text{ for all large } t\right) = 1.$$
QUESTIONS:

(1) Which convex, compact sets $B$ are realizable as $B_\mu$’s?
   - Solved by Häggstrom-Meester in the case of stationary FPP.
   - Some insight in i.i.d. case by Durrett-Liggett, Marchand.
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(2) What is the relationship between $\mu$ and $B_\mu$? How exactly does $B_\mu$ depend on $\mu$?
   - Unlike in other lattice models, the lattice matters!
QUESTIONS:

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(3) What are the fluctuations of $B(t)$ about $tB_\mu$?

We will focus on questions 1 and 3, though our technique sheds some light on question 2.
To study question 1, let $sides(C)$ be the number of extreme points of a compact convex set $C$.

- One might think that $B_\mu$ is a disk (forgets the lattice). This is not true.
- It is believed that at least $sides(B_\mu) = \infty$ for all but the most degenerate cases (e.g., $\mu = \delta_a$ for $a > 0$).
- Previous results:
  - By symmetry, $sides(B_\mu) \geq 4$.
  - Marchand (2002) proved that for a class of $\mu$'s, $sides(B_\mu) \geq 8$.
  - With M. Hochman, we showed there are $\mu$'s such that $sides(B_\mu) = \infty$.
    (Proof in *Examples of non-polygonal limit shapes for i.i.d. first-passage percolation and infinite coexistence in spatial growth models* on arXiv.)
We consider a subclass $\mathcal{M}_p$ of measures $\mu$ such that

$$\mu(\{1\}) = p > 0 \text{ and } \text{supp}(\mu) \subseteq [1, \infty).$$

- In 1981, Durrett and Liggett showed that for $\mu \in \mathcal{M}_p$ (for $p$ large), $B_\mu$ has some “flat edges.”
- In 2002, Marchand found the exact location of these edges.
Imagine that $p$ is large: there are many 1-edges ($e$'s with $\tau_e = 1$).

- For $n$ large, consider the line segment

$$L_n = \{(x, y) \in \mathbb{Z}^2 : x+y = n\} \cap [0, \infty)^2$$

in the first quadrant.

- If $z \in L_n$ then we want to estimate the passage time $\tau(0, z)$. 
Imagine that $p$ is large: there are many 1-edges ($e$'s with $\tau_e = 1$).

- Every path $P$ from 0 to $z$ has at least $n$ edges, so

$\tau(P) \geq \left[ \inf \supp(\mu) \right] |P| = |P| \geq n$.

Therefore $\tau(0, z) \geq n$. 
To examine the other inequality, we consider oriented percolation. For $p \in [0, 1]$ called each edge $e$ open with probability $p$ and closed with probability $1 - p$, independently.

- Two vertices $x$ and $y$ are connected ($x \rightarrow y$) if there is an oriented path from $x$ to $y$.
- An oriented path only moves up or right.
To examine the other inequality, we consider oriented percolation. For $p \in [0, 1]$ called each edge $e$ open with probability $p$ and closed with probability $1 - p$, independently.

- For $p = 0$, the origin cannot be part of an infinite open path; when $p = 1$ it always is.
- There is a critical probability:

$$\hat{p}_c = \sup \{ p : \mathbb{P}_p(0 \to \infty) = 0 \}.$$
When $p > \bar{p}_c$, not only does $\{0 \to \infty\}$ have positive probability, on that event we have many more connections.

- If $P = 0$, $x_1, x_2, \ldots$ is a path of vertices, $P$ has *direction* $\theta$ if

  $$\arg x_n \to \theta \text{ as } n \to \infty.$$
When $p > \bar{p}_c$, not only does $\{0 \to \infty\}$ have positive probability, on that event we have many more connections.

- For $p > \bar{p}_c$, there exists an interval $I_p = (\pi/4 - a(p), \pi/4 + a(p))$ such that for all $\theta \in I_p$, there is an infinite oriented path from 0 with direction $\theta$.
- This is true with conditional probability 1.
- This forms the *percolation cone*.
If $\mu \in \mathcal{M}_p$ and $p > \tilde{p}_c$ then with positive probability, 1-edges percolate.

- For any $\theta$ in the percolation cone, take an oriented path $P$ with only 1-edges.
- It intersects $L_n$ after only using $n$ edges.
- If $z$ is the intersection point, then

$$\tau(0, z) \leq \tau(P) = n.$$
If $\mu \in \mathcal{M}_p$ and $p > \bar{p}_c$ then with positive probability, 1-edges percolate.

- Therefore $\tau(0, z) = n$.
- Using these ideas, Durrett and Liggett showed $B_\mu$ has a flat edge in the percolation cone.
- Marchand made this more explicit:
Let $\vec{p}_c$ be the critical value for oriented bond percolation. Let $B_1$ be the closed $\ell^1$ unit ball and $Q_1$ the first quadrant. Let $\mu \in \mathcal{M}_p$.

**Theorem (Marchand)**

1. $B_\mu \subseteq B_1$
2. If $p = \vec{p}_c$, then 
   \[ B_\mu \cap \partial B_1 \cap Q_1 = \{(1/2, 1/2)\} \]
3. If $p > \vec{p}_c$, then 
   \[ B_\mu \cap \partial B_1 \cap Q_1 = [w_p(1), w_p(2)] \]
Differentiability

What happens at the corner of the flat edge?

- For $0 \leq \theta < \theta_p$ the point $v_\theta \in \partial B_\mu$ (in direction $\theta$) is strictly inside $B_\mu$.

- Two possibilities can occur at $w_p(2)$:
  
  ![Diagram](image)

  Both derivatives are different.  
  
  They are the same.
Main Theorem

Theorem (Auffinger, D.)

Let $\mu \in \mathcal{M}_p$ for $p \geq \bar{p}_c$ with finite mean. The boundary $\partial B_\mu$ is differentiable at $w_p(2)$.

- Proof is given in Limit shapes outside the percolation cone. on arXiv.
Consequences

(1) From Marchand’s theorem, all $B_\mu$’s for $\mu \in \mathcal{M}_p$ are non-polygonal.

(2) Solves questions of Häggstrom-Pemantle about growth models:
   - Infinite coexistence for Richardson-type growth models.
   - Graph of infection has infinitely many ends.

(3) Verifies predictions of Newman-Piza and Zhang:
   - New logarithmic lower bounds for fluctuations of passage time outside the percolation cone.
   - Completes the picture of convergence/divergence of shape fluctuations in 2d.