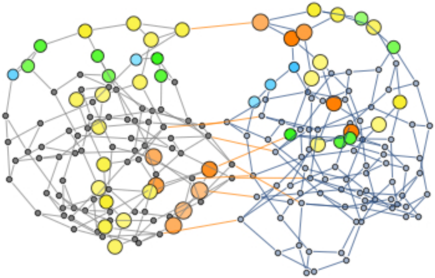
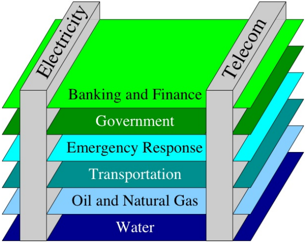
Cascades on interdependent networks



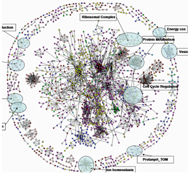


Raissa M. D'Souza
University of California, Davis
Dept of CS, Dept of Mech. and Aero. Eng.
Complexity Sciences Center
External Professor, **Santa Fe Institute**

A collection of interacting , dynamic networks form the core of modern society

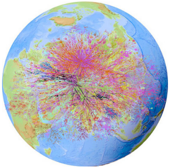
Networks:

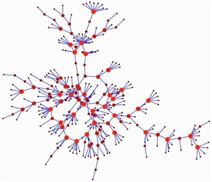
Transportation Networks/ Power grid
(distribution/ collection networks)



Biological networks
- protein interaction
- genetic regulation
- drug design

Computer networks

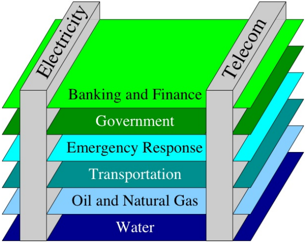


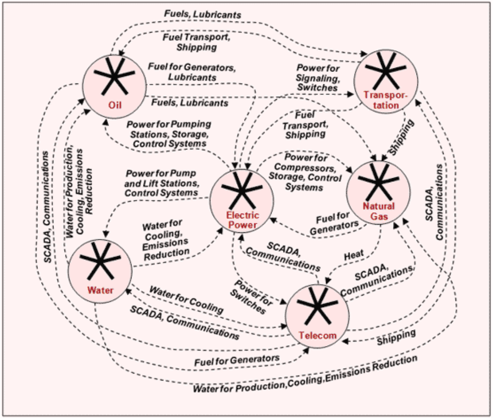


Social networks
- Immunology
- Information
- Commerce

- E-commerce $\rightarrow$ WWW $\rightarrow$ Internet $\rightarrow$ Power grid $\rightarrow$ River networks.
- Biological virus $\rightarrow$ Social contact network $\rightarrow$ Transportation nets $\rightarrow$ Communication nets $\rightarrow$ Power grid $\rightarrow$ River networks.

Critical infrastructure





From Peerenboom et. al.

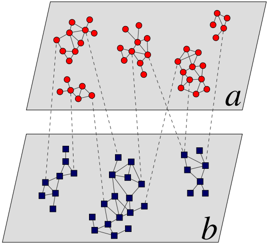
Moving to systems of interdependent networks
What are the simplest, useful, **abstracted** models?

- What are the **emergent new properties**?
  – Host-pathogen interactions
  – Phase transition thresholds
- What features confer resilience in one network while **introducing vulnerabilities** in others?
- How do **demands** in one system shape the performance of the others? (e.g., demand informed by social patterns of communication)
- How do **constraints** on one system manifest in others? (e.g., River networks shape placement of power plants)
- **Coupling of scales across space and time** / co-evolution.

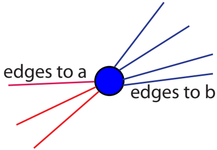
Configuration model for interacting networks

(E. Leicht and R. D'Souza, arXiv:0907.0894)

System of two networks



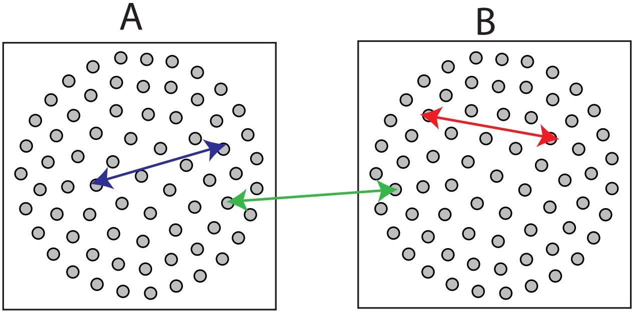
Connectivity for an individual node



- Degree distribution for nodes in network a: $p^a_{k_a k_b}$
- For the the system: $\{p^a_{k_a k_b}, p^b_{k_a k_b}\}$
- Generating functions to calculate properties of the ensemble of such networks.

Modular Erdős-Rényi

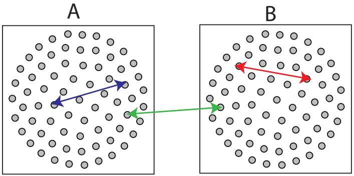
- Divide nodes initially into two groups (A and B):

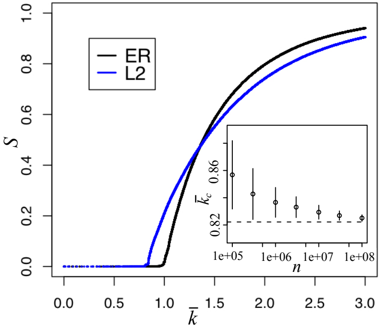


- Add internal a-a edges with rate λ.
- Add internal b-b edges with rate λ/r_1, with $r_1 > 1$.
- Add intra-group a-b edges with rate λ/r_2, with $r_2 > 1$, $r_2 \neq r_1$.

What happens? (Anything different?)

Wiring which respects group structures percolates earlier!



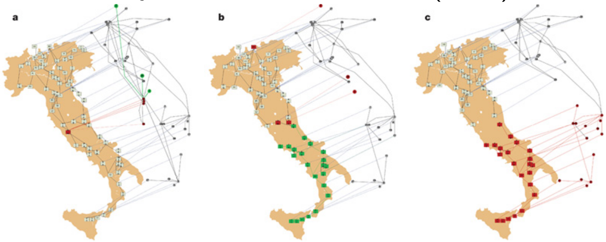


(Also tradeoffs between sparser and denser subnetworks.)

- Probability distribution for node degrees: $\{p^a_{k_a k_b}, p^b_{k_a k_b}\}$
- Generating functions to calculate properties of the ensemble of such networks.

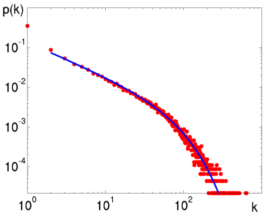
The flip side: "Catastrophic cascade of failures in interdependent networks"

Buldyrev et. al. *Nature* 464 (2010).

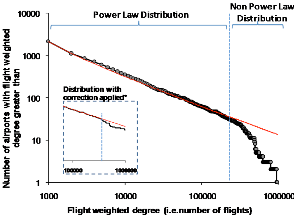


- Consider two coupled random graphs.
- Nodes fail (removed either in a targeted or random manner).
- Following an **iterative removal process**, small failures can lead to massive cascades of failure of the networks themselves.
- **Surprising**: What confers resilience to individual network (broad-scale degree distribution) may be a weakness for randomly coupled networks.

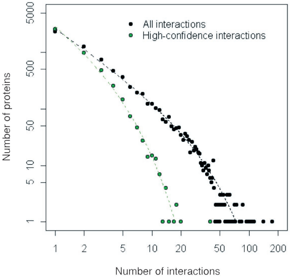
Single networks – broad scale degree distribution.



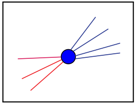
Social contacts
Szendröi and Csányi



Airport traffic
Bounova 2009



Protein interactions
Giot et al Science 2003



(node degree)

Approximated as power law $P_k \propto k^{-\gamma}$

$P_k \sim k^{-\gamma}$, the first two moments

(Note: $\gamma > 1$ required for $\sum_k P_k = 1$)

- First moment (Mean degree):

$$\langle k \rangle = \sum_{k=1}^{\infty} k p_k \approx \int_{k=1}^{\infty} k p_k dk$$

Diverges (i.e., $\langle k \rangle \to \infty$) if $\gamma \leq 2$.

- Second moment:

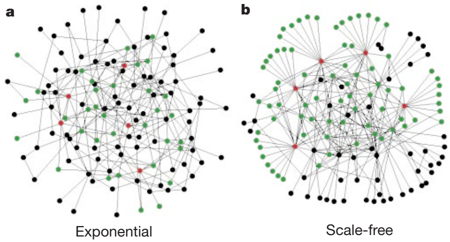
$$\langle k^2 \rangle = \sum_{k=1}^{\infty} k^2 p_k \approx \int_{k=1}^{\infty} k^2 p_k dk$$

Diverges (i.e., $\langle k^2 \rangle \to \infty$) if $\gamma \leq 3$.

- Many results follow for $2 < \gamma < 3$ since $\langle k \rangle / \langle k^2 \rangle \to 0$

Consequences of $p(k) \sim k^{-\gamma}$ for networks

- Most nodes are leaves (degree 1): *Network connectivity very robust to random node removal.*
- High degree nodes are hubs: *Network connectivity very fragile to targeted node removal.*



- Epidemic spreading on the network (contact process): if $2 < \gamma < 3$, then $\langle k \rangle / \langle k^2 \rangle \to 0$ and $\lim_{n \to 0}$ epidemic threshold $\to 0$.

(Buldyrev et al find broad scale more fragile for their particular cascade dynamics)

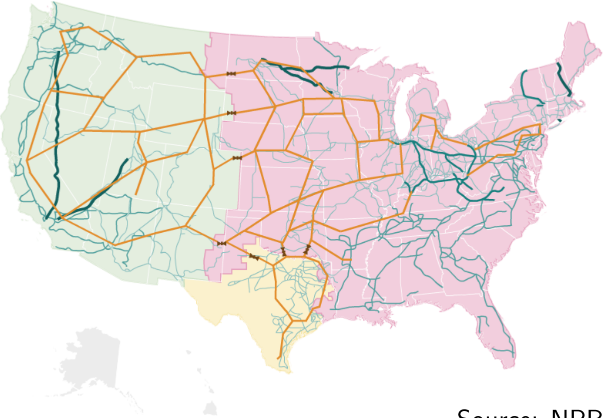
Dynamical processes on interdependent networks
Motivation: interconnected power grids

C. Brummitt, R. M. D'Souza and E. A. Leicht *PNAS* 109 (12), 2012.

Power grid: a collection of interdependent grids. (Interconnections built originally for emergencies.)

Blackouts cascade from one grid to another (in a non-local manner).

Building more interconnections (Fig: planned wind transmission).
Increasingly distributed

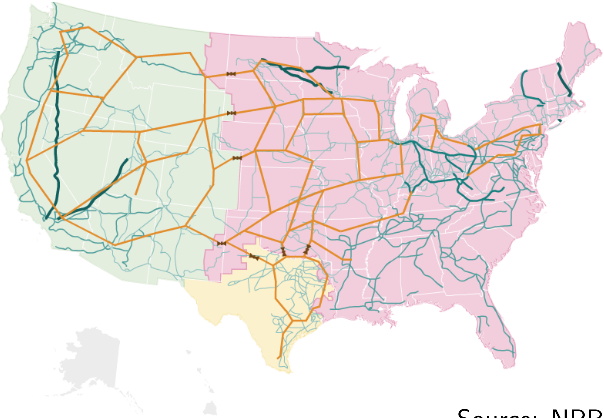


Source: NPR

Motivation cont.: interconnected power grids

What is the effect of interdependence on cascades?

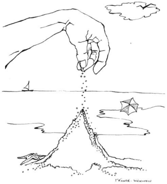
It is thought power grids organize to a "critical" state
– power law distribution of black out sizes
– maximize profits while fearing large cascades



Source: NPR

Sandpile models: "Self-organized criticality"

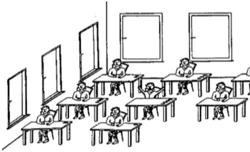
- Drop grains of sand ("load") randomly on nodes.
- Each node has a threshold for sand.
- Load $>$ threshold $\rightsquigarrow$ node topples $=$ sheds sand to neighbors.
- These neighbors may topple. And their neighbors. And so on.
- Cascades of load/stress on a system.



The classic Bak-Tang-Wiesenfeld sandpile model:

(Neuronal avalanches, banking cascades, earthquakes, landslides, forest fires, blackouts...)

- Finite square lattice in $\mathbb{Z}^2$
- Thresholds 4
- Open boundaries prevent inundation



Avalance size follows power law distribution $P(s) \sim s^{-3/2}$

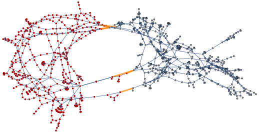
Sandpile model on arbitrary networks

Sandpile model on arbitrary networks:

- Thresholds = degrees (shed one grain per neighbor)
- Boundaries: shedded sand are deleted independently with probability f (:$\approx 10/N$)
- Mean-field behavior ($P(s) \propto s^{-3/2}$) robust. (Goh et al. PRL 03, Phys. A 2004/2005, PRE 2005. PLRGs with $2 < \gamma < 3$ not mean-field.)

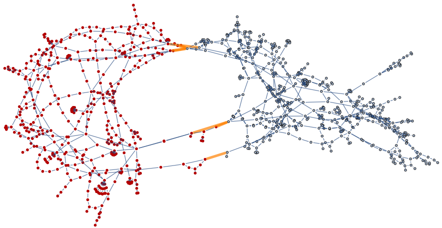
Sandpiles on interacting networks:

- Sparse connections between random graphs.
- Configuration model with multi-type degree distribution.



Sparsely coupled networks

Two-type network: a and b.



Degree distributions: $p_a(k_a, k_b), p_b(k_a, k_b)$

$p_a(k_a, k_b) =$ *fraction of a-nodes with* k_a, k_b *neighbors in* a, b.

Configuration model: create degree sequences until valid (even total intra-degree, same number of inter-edge stubs), then connect edge stubs at random.

Measures of avalanche size

- **Topplings:**

Drop a grain of sand. How many nodes eventually topple?

Avalanche size distributions: $s_a(t_a, t_b), s_b(t_a, t_b)$

e.g., $s_a(t_a, t_b) =$ *chance an avalanche begun in a topples t_a many a-nodes, t_b many b-nodes.*

To study this, we need a more basic distribution...

- **Sheddings:** Drop a grain of sand. How many grains are eventually shed from one network to another?

Shedding size distributions: $\rho_{od}(r_{aa}, r_{ab}, r_{ba}, r_{bb})$
$=$ *chance a grain shed from network o to d eventually causes $r_{aa}, r_{ab}, r_{ba}, r_{bb}$ many grains to be shed from* $a \to a, a \to b, b \to a, b \to b$

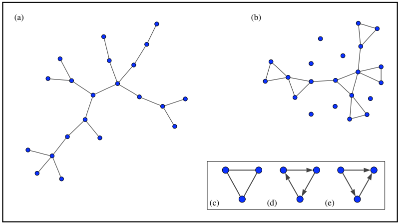
*Approximate shedding and toppling as multi-type branching processes.

Branching process approximation

Cascades in networks ≈ branching processes if they're *tree-like*.

Power grids are fairly tree-like:

	clustering coefficient
Power grid in SE USA	0.01
Similar Erdős-Rényi graph	0.001



Sandpile cascades on interacting networks ≈
a multitype branching process.

Overview of the calculations

From degree distribution to avalanche size distribution:

Input: degree distributions $p_a(k_a, k_b), p_b(k_a, k_b)$

$\Downarrow$ *compute*

shedding branching distributions $q_{aa}, q_{ab}, q_{ba}, q_{bb}$

$\Downarrow$ *compute*

toppling branching distributions u_a, u_b

$\Downarrow$ *plug in*

toppling branching generating functions $\mathcal{U}_a, \mathcal{U}_b$

$\Downarrow$ *plug in*

equations for avalanche size generating functions $\mathcal{S}_a, \mathcal{S}_b$

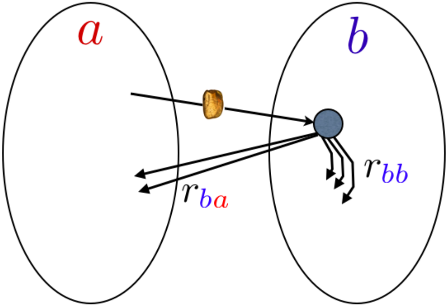
$\Downarrow$ *solve numerically, asymptotically*

Output: avalanche size distributions s_a, s_b

Shedding branch distribution, q_{od}

Example:

$q_{ab}(r_{ba}, r_{bb}) :=$ the branch (children) distribution for an *ab-shedding*.



Probability a single grain shed from a to b results in r_{ba} a-sheddings and r_{bb} b-sheddings.

Shedding branch distributions q_{od}

The crux of the derivation

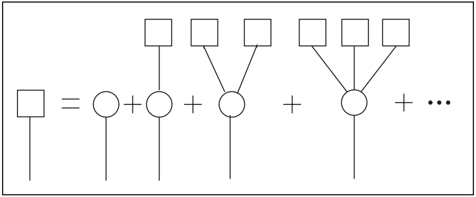
$q_{od}(r_{da}, r_{db}) :=$ chance a grain of sand shed from network o to d topples that node, sending r_{da}, r_{db} many grains to networks a, b.

$$q_{od}(r_{da}, r_{db}) = \underbrace{\frac{r_{do} p_d(r_{da}, r_{db})}{\langle k_{do} \rangle}}_{\text{I}} \underbrace{\frac{1}{r_{da} + r_{db}}}_{\text{II}} \qquad \text{for } r_{da} + r_{db} > 0.$$

- I: chance the grain lands on a node with degree $p_d(r_{da}, r_{db})$ (Edge following: r_{do} edges leading from network o.)
- II: empirically, sand on nodes is $\sim$ Uniform$\{0, ..., k-1\}$
- Chance of no children $= q_{od}(0,0) := 1 - \sum_{r_{da}+r_{db}>0} q_{od}(r_{da}, r_{db})$ (Probability a neighbor of any degree sheds, properly weighted.)
- Chance at least one child $= 1 - q_{od}(0,0)$.

I. Edge following probability: single network

- Degree distribution, P_k, with G.F. $G_0(x) = \sum_k P_k x^k$.
- Probability of following a random edge to a node of degree k: $q_k = kP_k / \sum_k kP_k$, with G.F. $G_1(x) = \sum_k q_k x^k$.
- ("Contact immunization" strategy used by CDC.)
- Generating function "self consistency" construction.
  $H_1(x)$: G.F. for dist in comp size following random edge



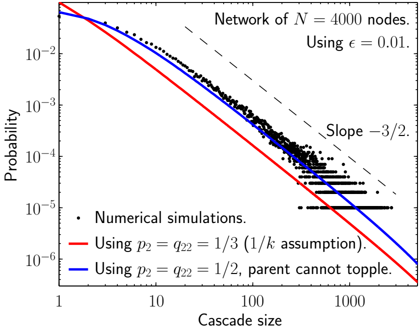
$$H_1(x) = xq_0 + xq_1 H_1(x) + xq_2[H_1(x)]^2 + xq_3[H_1(x)]^3 \cdots$$
$$= xG_1(H_1(x))$$

(c.f. Newman, Strogatz, Watts *PRE* 2001.)

II. Revisiting the "1/k" assumption

Pierre-Andre Noël, C. Brummitt, R. D'Souza
in progress

A node that just toppled is actually less likely to topple on the next time step.
(prob zero sand $\neq 1/k$)



Toppling branch distributions u_a, u_b

Key: a node topples iff it sheds at least one grain of sand.

Probability an o to d shedding leads to at least one other shedding: $1 - q_{od}(0,0)$. Probability a single shedding from an a-node yields t_a, t_b topplings:

$$u_a(t_a, t_b) = \sum_{k_a=t_a, k_b=t_b}^{\infty} p_a(k_a, k_b) Binomial[t_a; k_a, 1 - q_{aa}(0,0)] \cdot \cdot Binomial[t_b; k_b, 1 - q_{ab}(0,0)].$$

(e.g., k_a neighbors, t_a of them topple, each topples with prob $1 - q_{aa}(0,0)$.)

Associated generating functions: $\mathcal{U}_a(\tau_a, \tau_b), \mathcal{U}_b(\tau_a, \tau_b)$.

Summary of distributions and their generating functions

	distribution	generating function
degree	$p_a(k_a, k_b), p_b(k_a, k_b)$	$G_a(\omega_a, \omega_b), G_b(\omega_a, \omega_b)$
shedding branch	$q_{od}(r_{da}, r_{db})$	
toppling branch	$u_a(t_a, t_b), u_b(t_a, t_b)$	$\mathcal{U}_a(\tau_a, \tau_b), \mathcal{U}_b(\tau_a, \tau_b)$
toppling size	$s_a(t_a, t_b), s_b(t_a, t_b)$	$\mathcal{S}_a(\tau_a, \tau_b), \mathcal{S}_b(\tau_a, \tau_b)$

Self-consistency equations:

$$\mathcal{S}_a = \tau_a \mathcal{U}_a(\mathcal{S}_a, \mathcal{S}_b), \tag{1}$$

$$\mathcal{S}_b = \tau_b \mathcal{U}_b(\mathcal{S}_a, \mathcal{S}_b). \tag{2}$$

Want to solve (1), (2) for $\mathcal{S}_a(\tau_a, \tau_b), \mathcal{S}_b(\tau_a, \tau_b)$.
Coefficients of $\mathcal{S}_a, \mathcal{S}_b$ = avalanche size distributions s_a, s_b.

In practice, Eqs. (1), (2) are transcendental and difficult to invert.

Numerically solving $\vec{\mathcal{S}}(\vec{\tau}) = \vec{\tau} \cdot \vec{\mathcal{U}}(\vec{\mathcal{S}}(\vec{\tau}))$

Methods for computing s_a, s_b for small avalanche size:

Method 1: Iterate starting from $\mathcal{S}_a = \mathcal{S}_b = 1$; expand.
Method 2: Iterate symbolically; use Cauchy's integration formula

$$s_a(t_a, t_b) = \frac{1}{(2\pi i)^2} \iint_D \frac{\mathcal{S}_a(\tau_a, \tau_b)}{\tau_a^{t_a+1} \tau_b^{t_b+1}} \, d\tau_a d\tau_b,$$

where $D \subset \mathbb{C}^2$ encloses the origin and no poles of $\mathcal{S}_a$.
Method 3: Multidimensional Lagrange inversion (IJ Good 1960):

$$\mathcal{S}_a = \sum_{m_a, m_b = 0}^{\infty} \frac{\tau_a^{m_a} \tau_b^{m_b}}{m_a! m_b!} \left[\frac{\partial^{m_a+m_b}}{\partial \kappa_a^{m_a} \partial \kappa_b^{m_b}} \left\{ h(\vec{\kappa}) \mathcal{U}_a(\vec{\kappa})^{m_a} \mathcal{U}_b(\vec{\kappa})^{m_b} \left\| \delta_\mu^\nu - \frac{\kappa_\mu}{\mathcal{U}_\mu} \frac{\partial \mathcal{U}_\mu}{\partial \kappa_\mu} \right\| \right\} \right]_{\vec{\kappa}=0},$$

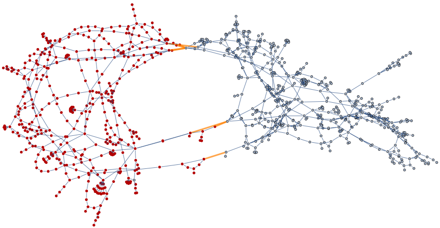
if the types $\mu, \nu \in \{a, b\}$ have a positive chance of no children.

- Unfortunately for large avalanches need to use simulation.

(Asymptotic approximations used for isolated networks do not apply.)

Plugging in degree distributions: A real world example

Two geographically nearby power grids in the southeastern US.

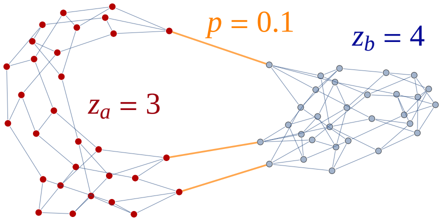


	Grid c	Grid d
# nodes	439	504
$\langle k_{int} \rangle$	2.4	2.9
$\langle k_{ext} \rangle$	0.02	0.01
clustering	0.01	0.08

8 links between these two distinct grids.
Different average internal degree $\langle k_{int} \rangle$. Long paths.
(Low clustering – approximately locally tree-like.)

A canonical idealization: Random regular graphs

Two random z_a-, z_b-regular graphs with "Bernoulli coupling": each node gets an external link independently with probability p. These ≈ power grids.



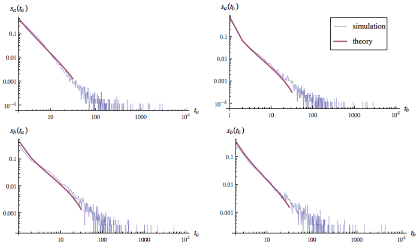
$$\mathcal{U}_a(\tau_a, \tau_b) = \frac{(p - p\tau_a + (z_a + 1)(\tau_a + z_a - 1))^{z_a}(1 + p(\tau_b - 1) + z_b)}{(z_a + 1)^{z_a} z_a^{z_a}(z_b + 1)}$$

Matching theory and simulation (for small'ish avalanches)

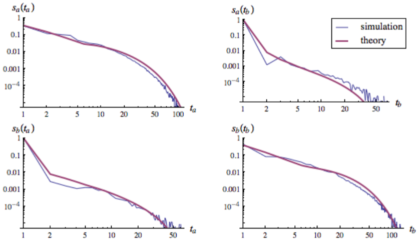
Plot marginalized avalanche size distributions

$$s_a(t_a) \equiv \sum_{t_b \geq 0} s_a(t_a, t_b), \quad s_a(t_b) \equiv \sum_{t_a \geq 0} s_a(t_a, t_b), \quad etc.$$

in simulations, branching process.

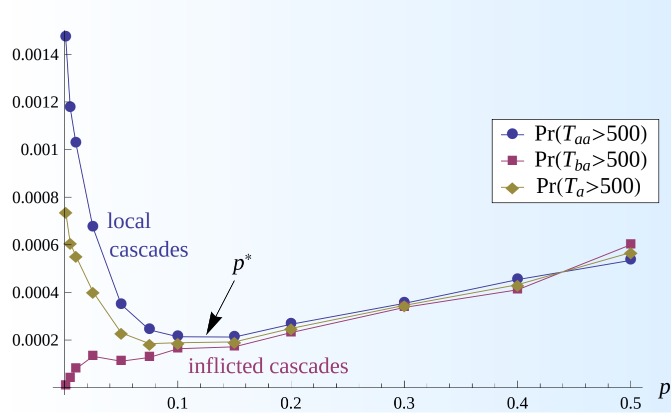


Regular(3)-Bernoulli(p)-Regular(10)



Power grids c, d.

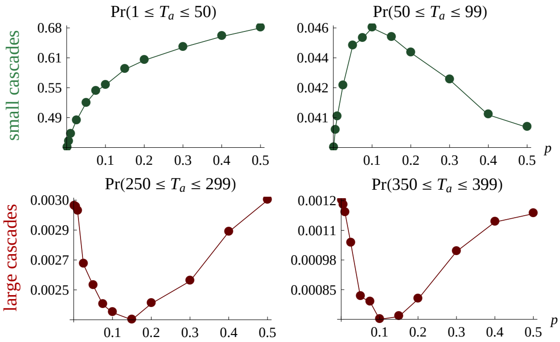
Main findings: For an individual network, optimal p^*



- (Blue curve) Initially increasing p *decreases* the largest cascades started in that network (second network is reservoir for load).
- (Red curve) Increasing p *increases* the largest cascades inflicted from the second network (two reasons: new channels and greater capacity).
- (Gold curve) Neglecting the origin of the cascade, the effects balance at a stable critical point, $p^* \approx 0.1$. (Reduced by 75% from p=0.001 to p=0.1)

Main findings: Individual network, "Yellowstone effect"

Supressing largest cascades amplifies small and intermediate ones!
(Supressing smallest amplifies largest (Yellowstone and Power Grids*))



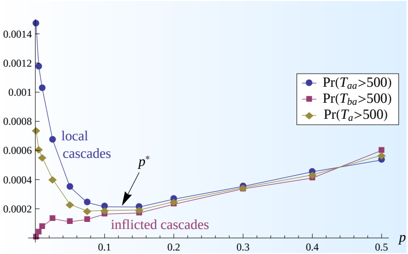
- To suppress smallest, isolation $p = 0$.
- To suppress intermediate (10% of system size) either $p = 0$ or $p = 1$.
- To suppress cascades $> 25\%$ of system size then $p = p* \approx 0.11$.

*Dobson I, Carreras BA, Lynch VE, Newman DE *Chaos*, (2007).

Main findings: System as a whole

More interconnections fuel larger system-wide cascades.

• Each new interconnection adds capacity and load to the system (Here capacity is a node's degree, interconnections increase degree)



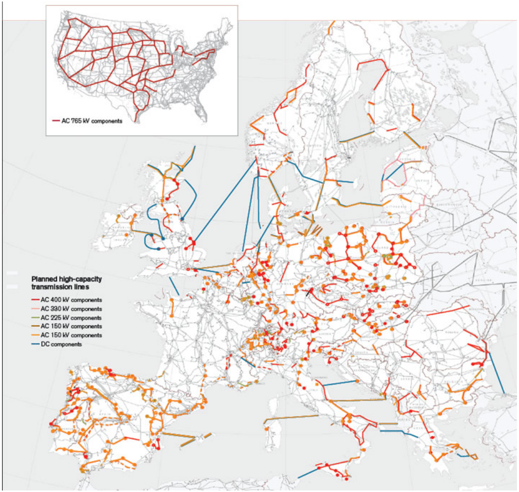
• Test this on coupled random-regular graphs by rewiring internal edges to be spanning edges (increase interconnectivity with out increasing degree). No increase in the largest cascades.

• Inflicted cascades (Red curve) increase mostly due to increased capacity.

• So an individual operator adding edges to achieve p^* may inadvertantly cause larger global cascades.

Larger cascades from increased interconections: A warning sign?

- Financial markets
- Energy transmission systems



Source:
Technology Review, "Joining the Dots", Jan/Feb (2011).

Main findings, continued: Frustrated equilibrium

Unless the coupled grids are identical, only one will be able to acheive it's p^*.

- Coupled $z_a \neq z_b$ regular random graphs (brancing process and simulation).

$$\frac{\langle s_a \rangle_b}{\langle s_b \rangle_a} = \frac{1 + z_a}{1 + z_b}$$

If $z_b > z_a$ inflicted cascades from b to a larger than those from a to b.

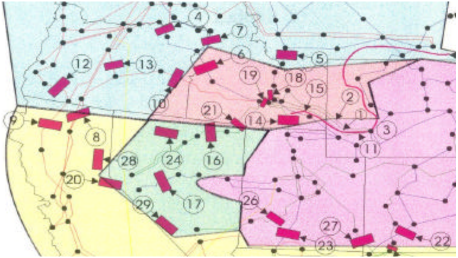
(An arm's race for capacity?)

Summary: Sandpile cascades on interacting networks

- Some interconnectivity can be *beneficial*, but too much is *detrimental*. Stable optimal levels are possible.
- From perspective of isolated network, seek optimal interconnectivity p^*.
- This equilibrium will be frustrated if the two networks differ in their load or propensity to cascade.
- Tuning p to suppress large cascades amplifies to occurrence of small ones. (Likewise, suppressing small, amplifies large.)
- Additional capacity and overall load from new interconnections fuels larger cascades in the system as a whole.
- What might be good for an individual operator (adding edges to achieve p^*), may be bad for society.

Possible extensions – Real power grids

- Expand multi-type processes to encode for different types of nodes (buses, transformers, generators)
- Linearized power flow equations – cascades in real power grids are non-local: e.g. fig: 3 to 4, 7 to 8
- Game theoretic/ economic consideration (we assume adding connections is cost-free)



(1996 Western blackout NERC report)

(Power grids as "critical" – Balancing profit and fear of outages)

Possible extensions

Teams and social networks

- Tasks (sand) arriving on people (nodes)
- Each person has a capacity for tasks: sheds once overloaded
- Coupling to a second social network (team) can reduce large cascades

Amplifying cascades

- Encourage adoption of new products
- Snowball sampling

Airline networks

- Different carriers accepting load (bumped passengers)

Other types of cascades, not just than sandpiles

- Watt's threshold model: "topple" is some **fraction** ϕ of your neighbors have "toppled" (rather than "toppling", Watt's think of cascades in adopting a new product).
  – Harder to "topple" nodes of high degree.

- Kleinberg: rather than thresholds, diminishing returns (concave / sub-modular utility)

References and Acknowledgements

- C. Brummitt, R. M. D'Souza and E. A. Leicht, "Suppressing cascades of load in interdependent networks", it PNAS 109 (12) 2012.

- Note **Author Summary** for high-level overview.