Renormalization of hierarchically interacting Lambda-Cannings processes

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Plan

- **Introduce** a (class of) models for *evolution of spatially structured populations*.
- **Analyze** the large (space-time) scale behavior of the models and their *ergodic behavior*.
Universality and renormalization

- Substantial literature on renormalization of \textit{diffusive spatially interacting models}.

\textbf{This talk:}

- Universality for a class of \textit{non-diffusive} spatio-temporal models with \textbf{jumps}. 
Spatial multi-scale $\wedge$-Cannings process

- Structured population evolving in **time** and **space**.
- Geographically scattered colonies of individuals.
- **Reproduction: Cannings process** – progeny of size comparable to the size of the whole population ($\Rightarrow$ skewed offspring distribution):
  - Genetic diversity observed in population genetics data is less than the one predicted by diffusive models.
  - Highly selective environment, selective sweeps.
  - Population size fluctuations/bottlenecks.
  - ...
- **Migration**.
- **Occasional global catastrophes $\Rightarrow$ reshuffling-resampling.**
Single-colony $\Lambda$-Cannings process

- Single colony of $M$ individuals: $I := \{1, \ldots, M\}$.
- Genetic types: $T(i, t) \in E, t \in \mathbb{R}_+, i \in I$.
- Type-space: $E$ compact Polish.
- Initially: $T(i, 0) \sim \theta, \theta \in \mathcal{M}_1(E)$.
- Resampling: fixed-$M$ reproduction.
- Poisson point process $\Pi$ with $dt \otimes \Lambda^*(dr)$.
- $\Lambda^*(dr) \in \mathcal{M}([0, 1])$.
- $(t, r) \in \Pi$ encodes a resampling event at time $t$:
  - Mark the individuals for resampling independently with success prob $r$.
  - Randomly choose a parent individual (among the marked ones).
  - Change the types of the marked individuals to the type of the parent.
- Progeny comparable to the size of population: $\approx Mr$. 
**Single-colony \( \Lambda \)-Cannings process**

\[
\begin{array}{cccccc}
1 & 0 & 0 & 1 & 0 & 1 \\
\end{array}
\]

**Figure:** Cannings resampling event in a colony of \( M = 8 \) individuals of two types.

- **Q:** Which \( \Lambda^*(dr) \)?
- **A:** \( \Lambda^*(dr) := \Lambda(dr)/r^2 \), where \( \Lambda \in \mathcal{M}_f([0, 1]) \).
- \[
\Rightarrow \frac{1}{2}N(N-1)\int_0^1 \Lambda^*(dr)r^2 < \infty
\]
- **Assumptions:** \( \Lambda(\{0\}) = 0 \) (no diffusive part)
- **Genetic types distribution:** \( X(t) := \frac{1}{M} \sum_{i=1}^{M} \delta_{T(i,t)} \in \mathcal{M}_1(E) \).
- Let \( M \to \infty \).
Multi-colony $\Lambda$-Cannings process

Assume:

- Individuals **migrate** between colonies.
- Symmetric (rate $c$) random walk on the **full graph** of $N$ vertices.

**Figure**: Possible immediate migration steps between $N = 4$ colonies with $M = 3$ individuals of two types in the mean-field version.
Hierarchical geographic space

- **Geographical space:** hierarchical group (regular tree):
  \[ \Omega_N = \left\{ \eta = (\eta^l)_{l \in \mathbb{N}_0} \in \{0, 1, \ldots, N-1\}^{\mathbb{N}_0} \mid \sum_{l \in \mathbb{N}_0} \eta^l < \infty \right\} \text{ of order } N. \]

- **Ultrametric:**
  \[ d(\eta, \zeta) = \min\{k \in \mathbb{N}_0 \mid \eta^l = \zeta^l, \text{ for all } l \geq k\}, \quad \eta, \zeta \in \Omega_N. \]

- **Topology:** blocks
  \[ B_k(\eta) = \{ \zeta \in \Omega_N : d(\eta, \zeta) \leq k \}, \quad \eta \in \Omega_N, k \in \mathbb{N}_0 \]

- **Migration between colonies:** hierarchical random walk:
  - **Migration rates:**
    \[ c := (c_k)_{k \in \mathbb{N}_0} \in (0, \infty)^{\mathbb{N}_0} \]
  - Each indiv. at \( \eta \in \Omega_N \) jumps unif. in \( B_k(\eta) \) at rate \( c_{k-1}/N^{k-1} \)
Transience vs. recurrence: Comparison with SRW on $\mathbb{Z}^d$

N.B. Dawson, Gorostiza and Wakolbinger (2005), the HRW is

- **recurrent** $\iff \sum_{k \in \mathbb{N}_0} (1/c_k) = \infty$.
- **transient** $\iff \sum_{k \in \mathbb{N}_0} (1/c_k) < \infty$

**Example.** $c_k = c^k$. Can associate (potential theoretic) **dimension** $d = d(c, N)$ to HRW. Then:

- **recurrent** $\iff c < 1 \iff$ SRW $d < 2$
- **critically recurrent** $\iff c = 1 \iff$ SRW $d = 2$
- **transient** $\iff c > \infty \iff$ SRW $d > 2$
Hierarchically interacting $\Lambda$-Cannings process

Colonies on $\Omega_N$.

- **Migration**: hierarchical random walk.
- **Non-local reshuffling-resampling**:
  - **Resampling measures**: $\Delta := (\Lambda_k)_{k \in \mathbb{N}_0} \in \mathcal{M}_f([0,1])^\mathbb{N}_0$
  - For each $\eta$, for each $\Lambda_k$, resample-reshuffle macro-colony $B_k(\eta)$ at rate $\frac{1}{N^{2k}}$
  - **Reshuffling**:

Figure: Random reshuffling in a 1-block on the hierarchical lattice of order $N = 3$ with $M = 3$ individuals of two types per colony.
Summary (so far)

Hierarchically interacting \((c, \Lambda)\)-Cannings process

\[
X^{(\Omega_N)} = \left( X^{(\Omega_N)}(t) \right)_{t \geq 0} \quad \text{with} \quad X^{(\Omega_N)}(t) = \{ X^{(\Omega_N)}_{\eta}(t) \}_{\eta \in \Omega_N} \in \mathcal{M}_1(E)^{\Omega_N}.
\]

Competition between:

- **Migration** \(c = (c_k)_{k \in \mathbb{Z}_+}\) (spatial movement) vs. **Resampling** \(\Lambda = (\Lambda_k)_{k \in \mathbb{Z}_+}\) (reproduction under constrained resources).
  
  plus

- (Hierarchy of) **time scales**.

**N.B.** Important features:

- **Non-diffusive behavior**: jumps.
- **Strongly correlated global updates**: non-local reshuffling-resampling.

\[
Q: \mathcal{L} \left[ X^{(\Omega_N)}(t) \right] \xrightarrow{N \to +\infty} ? \quad t \to +\infty
\]
Large space-time scale analysis: renormalization

- “Separate” slow and fast time scales.
- Renormalize.
- Macroscopic observables:

\[ Y_{\eta,k}^{(N)}(tN^k) = \frac{1}{N^k} \sum_{\zeta \in B_k(\eta)} X_{\zeta}^{(\Omega_N)}(tN^k), \quad \eta \in \Omega_N, \ k \in \mathbb{Z}_+ \]

(block averages of order \( k \in \mathbb{Z}_+ \)).

- **Single scale** (mean-field) \( \rightsquigarrow \) propagation of chaos \( \rightsquigarrow \) McKean-Vlasov process.
- **Multiple scales** simultaneously: \( \rightsquigarrow \) Markov interaction chain.
- **Hierarchical mean-field limit:**

\[ \Omega_N \uparrow \Omega_\infty, \quad N \to +\infty. \]
McKean-Vlasov limiting object

Algebra of test functions: \( \mathcal{B} \subseteq C_b(\mathcal{M}_1(E), \mathbb{R}) \) with \( G \in \mathcal{B} \):

\[
G(x) = \int_{E^n} x^\otimes n (du) \varphi(u), \quad x \in \mathcal{M}_1(E), n \in \mathbb{N}, \varphi \in C_b(E^n, \mathbb{R}).
\]

Generator: \( L^{c,d,\Lambda}_\theta : \mathcal{B} \to C_b(\mathcal{M}_1(E), \mathbb{R}) \)

\[
(L^{c,d,\Lambda}_\theta G)(x) = c \int_E (\theta - x) (da) \frac{\partial G(x)}{\partial x} [\delta_a]
+ d \int_E \int_E Q_x(du, dv) \frac{\partial^2 G(x)}{\partial x \partial x} [\delta_u, \delta_v]
+ \int_{[0,1]} \Lambda^* (dr) \int_E x(da) \left[ G((1 - r)x + r\delta_a) - G(x) \right], \quad G \in \mathcal{B},
\]

where

\[
Q_x(du, dv) = x(du) \delta_u(dv) - x(du) x(dv).
\]

\( \Lambda^\Lambda \)-processes with immigration-emigration:

\[
Z^{c,d,\Lambda}_\theta = (Z^{c,d,\Lambda}_\theta(t))_{t \geq 0}, \quad Z^{c,d,\Lambda}_\theta(0) = \theta.
\]
Asymptotic behavior of the macroscopic observables

- **Volatility constants:** $d = (d_k)_{k \in \mathbb{Z}_+}$,

  $d_0 = 0, \quad d_{k+1} = \frac{c_k(\lambda_k/2 + d_k)}{c_k + (\lambda_k/2 + d_k)}, \quad k \in \mathbb{Z}_+$,

  where $\lambda_k = \Lambda_k([0; 1])$.

- **N.B.** (inhomogeneous) **Möbius transformation.**

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**Theorem (behaviour of the macroscopic observables)**

For every $k \in \mathbb{Z}_+$, uniformly in $\eta \in \Omega_\infty$,

$$\mathcal{L} \left[ \left( Y_{\eta,k}^{(N)}(tN^k) \right)_{t \geq 0} \right] \xrightarrow{N \to +\infty} \mathcal{L} \left[ \left( Z_{\theta}^{c_k,d_k,\Lambda_k}(t) \right)_{t \geq 0} \right].$$
Ergodic behavior of $X^{(N)}$, $N < \infty$

Set

$$m_k := \frac{\lambda_k/2 + d_k}{c_k}.$$ 

Theorem (Clustering vs. coexistence criterion)

- **[Clustering]** (= formation of large mono-type regions), if $\sum_{k \in \mathbb{Z}_+} m_k = \infty$
  vs.

- **[Local coexistence]** (= convergence to multi-type equilibria), if $\sum_{k \in \mathbb{Z}_+} m_k < \infty$.

N.B. $\sum_{k \in \mathbb{Z}_+} m_k = \infty$ vs. $< \infty \iff \sum_{k \in \mathbb{N}_0} (1/c_k) \sum_{l=0}^k \lambda_l = \infty$ vs. $< \infty$.

- Recurrent migration $\rightsquigarrow$ clustering.

- $\exists$ transient migrations and strong enough reshuffling-resampling $\sum_{l \in \mathbb{N}_0} \lambda_l = \infty \rightsquigarrow$ clustering.
Theorem (Clustering vs. coexistence criterion)

The following dichotomy holds:

(a) **[Local coexistence]** If \( \sum_{k \in \mathbb{Z}^+} m_k < \infty \), then for every \( \theta \in \mathcal{P}(E) \) and every \( X^{(\Omega_N)}(0) \) whose law is stationary and ergodic w.r.t. translations in \( \Omega_N \) and has a single-site mean \( \theta \),

\[
\mathcal{L} \left[ X^{(\Omega_N)}(t) \right] \xrightarrow{t \to +\infty} \nu^{(\Omega_N),c,\lambda}_{\theta} \in \mathcal{P}(\mathcal{P}(E)^{\Omega_N})
\]

for some unique law \( \nu^{(\Omega_N),c,\lambda}_{\theta} \) that is stationary and ergodic w.r.t. translations in \( \Omega_N \) and has single-site mean \( \theta \).

(b) **[Clustering]** If \( \sum_{k \in \mathbb{Z}^+} m_k = \infty \), then, for every \( \theta \in \mathcal{P}(E) \),

\[
\mathcal{L} \left[ X^{(\Omega_N)}(t) \right] \xrightarrow{t \to +\infty} \int_0^1 \theta(du)\delta_{(\delta_u)^{\Omega_N}} \in \mathcal{P}(\mathcal{P}(E)^{\Omega_N})
\]