Renormalization of hierarchically interacting Lambda-Cannings processes

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EURANDOM
Eindhoven University of Technology

July 24, 2012, Cornell University

Plan

- **Introduce** a (class of) models for **evolution of spatially structured populations.**
- **Analyze** the large (space-time) scale behavior of the models and their **ergodic behavior**.

Universality and renormalization

- Substantial literature on renormalization of **diffusive spatially interacting models**.
- Finite and infinite systems of interacting Fisher-Wright/Feller diffusions and Fleming-Viot/Dawson-Watanabe measure-valued diffusions.
- T. Cox, D. Dawson, A. Greven, F. den Hollander, R. Sun, J. Swart, J. Vaillancourt, et al.

This talk:

- Universality for a class of **non-diffusive** spatio-temporal models with **jumps**.

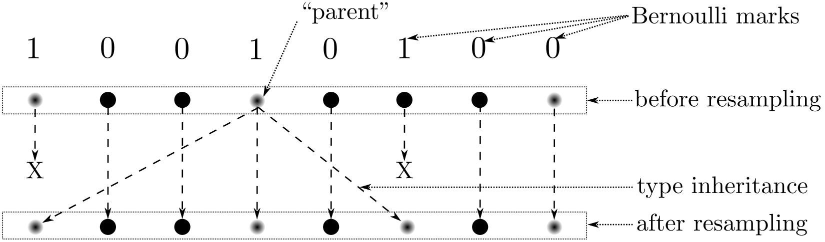
Spatial multi-scale Λ-Cannings process

- Structured population evolving in **time** and **space**.
- Geographically scattered colonies of individuals.
- **Reproduction: Cannings process –** progeny of size comparable to the size of the whole population ($\rightsquigarrow$ skewed offspring distribution):
  - Genetic diversity observed in population genetics data is less than the one predicted by diffusive models.
  - Highly selective environment, selective sweeps.
  - Population size fluctuations/bottlenecks.
  - ...
- **Migration**.
- **Occasional global catastrophes $\rightsquigarrow$ reshuffling-resampling.**

Single-colony Λ-Cannings process

- Single colony of M **individuals**: $I := \{1, \dots, M\}$.
- Genetic **types**: $T(i,t) \in E$, $t \in \mathbb{R}_+$, $i \in I$.
- **Type-space**: E compact Polish.
- Initially: $T(i,0) \sim \theta$, $\theta \in \mathscr{M}_1(E)$.
- **Resampling**: fixed-M reproduction.
- Poisson point process Π with $\mathrm{d}t \otimes \Lambda^*(\mathrm{d}r)$.
- $\Lambda^*(\mathrm{d}r) \in \mathscr{M}([0,1])$.
- $(t,r) \in \Pi$ encodes a **resampling** event at time t:
  - **Mark** the individuals for resampling independently with success prob r.
  - Randomly choose a **parent** individual (among the marked ones).
  - **Change** the types of the marked individuals to the type of the parent.
- Progeny comparable to the size of population: $\approx Mr$.

Single-colony Λ-Cannings process



$$E = \{\bullet, \bullet\}.$$

Figure: Cannings resampling event in a colony of $M = 8$ individuals of two types.

- **Q:** Which $\Lambda^*(\mathrm{d}r)$**?**
- **A:** $\Lambda^*(\mathrm{d}r) := \Lambda(\mathrm{d}r)/r^2$, where $\Lambda \in \mathscr{M}_\mathrm{f}([0,1])$.
- $\Rightarrow \frac{1}{2}N(N-1)\int_0^1 \Lambda^*(\mathrm{d}r)r^2 < \infty$
- Assumptions: $\Lambda(\{0\}) = 0$ (no diffusive part)
- **Genetic types distribution:** $X(t) := \frac{1}{M}\sum_{i=1}^M \delta_{T(i,t)} \in \mathscr{M}_1(E)$.
- Let $M \to \infty$.

Multi-colony Λ-Cannings process

Assume:

- Individuals **migrate** between colonies.
- Symmetric (rate c) random walk on the **full graph** of N vertices.

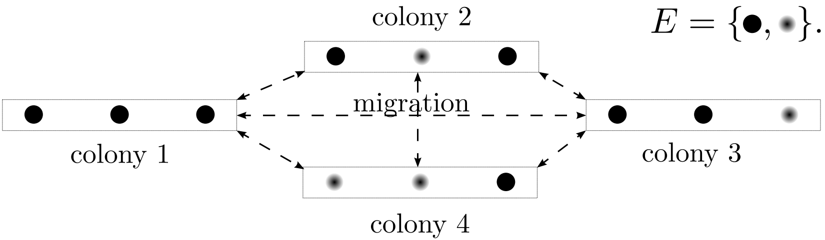
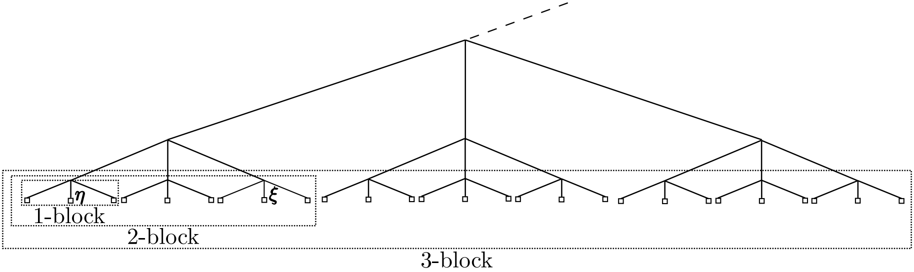


Figure: Possible immediate migration steps between $N = 4$ colonies with $M = 3$ individuals of two types in the mean-field version.

Hierarchical geographic space



- **Geographical space**: **hierarchical group** (regular tree):
  $\Omega_N = \left\{ \eta = (\eta^l)_{l \in \mathbb{N}_0} \in \{0, 1, \ldots, N-1\}^{\mathbb{N}_0} \mid \sum_{l \in \mathbb{N}_0} \eta^l < \infty \right\}$ of order N.
- **Ultrametric:** $d(\eta, \zeta) = \min\{k \in \mathbb{N}_0 \mid \eta^l = \zeta^l,$ for all $l \geq k\}$, $\eta, \zeta \in \Omega_N$.
- **Topology: blocks** $B_k(\eta) = \{\zeta \in \Omega_N \colon\ d(\eta, \zeta) \leq k\}$, $\eta \in \Omega_N$, $k \in \mathbb{N}_0$
- **Migration** between colonies: **hierarchical random walk**:
  - **Migration rates:** $\underline{c} := (c_k)_{k \in \mathbb{N}_0} \in (0, \infty)^{\mathbb{N}_0}$
  - each indiv. at $\eta \in \Omega_N$ jumps unif. in $B_k(\eta)$ at rate c_{k-1}/N^{k-1}

Transience vs. recurrence: Comparison with SRW on $\mathbb{Z}^d$

N.B. Dawson, Gorostiza and Wakolbinger (2005), the **HRW** is

- **recurrent** $\Leftrightarrow \sum_{k\in\mathbb{N}_0}(1/c_k) = \infty$.
- **transient** $\Leftrightarrow \sum_{k\in\mathbb{N}_0}(1/c_k) < \infty$

Example. $c_k = c^k$. Can associate (potential theoretic) **dimension** $d = d(c,N)$ to HRW. Then:

- **recurrent** $\Leftrightarrow c < 1 \Leftrightarrow$ SRW $d < 2$
- **critically recurrent** $\Leftrightarrow c = 1 \Leftrightarrow$ SRW $d = 2$
- **transient** $\Leftrightarrow c > \infty \Leftrightarrow$ SRW $d > 2$

Hierarchically interacting Λ-Cannings process

Colonies on Ω_N.

- **Migration**: hierarchical random walk.
- **Non-local reshuffling-resampling**:
  - **Resampling measures**: $\underline{\Lambda} := \left(\Lambda_k\right)_{k\in\mathbb{N}_0} \in \mathscr{M}_{\mathrm{f}}([0,1])^{\mathbb{N}_0}$
  - For each η, for each Λ_k, resample-reshuffle macro-colony $B_k(\eta)$ at rate $\frac{1}{N^{2k}}$
  - **Reshuffling:**

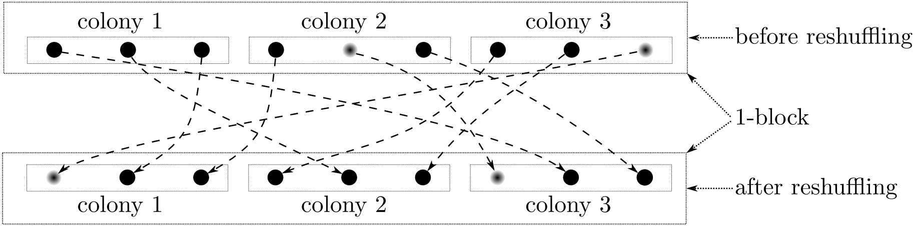


Figure: Random reshuffling in a 1-block on the hierarchical lattice of order $N = 3$ with $M = 3$ individuals of two types per colony.

Summary (so far)

Hierarchically interacting $(\underline{c}, \underline{\Lambda})$-Cannings process

$$X^{(\Omega_N)} = \left(X^{(\Omega_N)}(t)\right)_{t \geq 0} \quad \text{with} \quad X^{(\Omega_N)}(t) = \left\{X_\eta^{(\Omega_N)}(t)\right\}_{\eta \in \Omega_N} \in \mathscr{M}_1(E)^{\Omega_N}.$$

Competition between:

- **Migration** $\underline{c} = (c_k)_{k \in \mathbb{Z}_+}$ (spatial movement)
  vs.
  Resampling $\underline{\Lambda} = (\Lambda_k)_{k \in \mathbb{Z}_+}$ (reproduction under constrained resources).
  plus
- (Hierarchy of) **time scales**.

N.B. Important features:

- **Non-diffusive behavior**: jumps.
- **Strongly correlated global updates:** non-local reshuffling-resampling.

Q: $\mathscr{L}\left[X^{(\Omega_N)}(t)\right] \xrightarrow[\substack{N \to +\infty \\ t \to +\infty}]{} \text{?}$

Large space-time scale analysis: renormalization

- **“Separate" slow and fast time scales.**
- **Renormalize.**
- **Macroscopic observables:**

$$Y^{(N)}_{\eta,k}(tN^k) = \frac{1}{N^k} \sum_{\zeta \in B_k(\eta)} X^{(\Omega_N)}_{\zeta}(tN^k), \qquad \eta \in \Omega_N, k \in \mathbb{Z}_+$$

(block averages of order $k \in \mathbb{Z}_+$).

- **Single scale** (mean-field) $\rightsquigarrow$ **propagation of chaos** $\rightsquigarrow$ **McKean-Vlasov process.**
- **Multiple scales** simultaneously: $\rightsquigarrow$ Markov **interaction chain**.
- **Hierarchical mean-field limit:**

$$\Omega_N \uparrow \Omega_\infty, \quad N \to +\infty.$$

McKean-Vlasov limiting object

Algebra of test functions: $\mathscr{B} \subseteq C_b(\mathscr{M}_1(E), \mathbb{R})$ with $G \in \mathscr{B}$:

$$G(x) = \int_{E^n} x^{\otimes n}(\mathrm{d}u)\,\varphi(u), \qquad x \in \mathscr{M}_1(E), n \in \mathbb{N}, \varphi \in C_\mathrm{b}(E^n, \mathbb{R}).$$

Generator: $L_\theta^{c,d,\Lambda} \colon \mathscr{B} \to C_\mathrm{b}(\mathscr{M}_1(E), \mathbb{R})$

$$(L_\theta^{c,d,\Lambda} G)(x) = c \int_E (\theta - x)\,(\mathrm{d}a)\,\frac{\partial G(x)}{\partial x}[\delta_a]$$
$$+ d \int_E \int_E Q_x(\mathrm{d}u, \mathrm{d}v)\,\frac{\partial^2 G(x)}{\partial x \partial x}[\delta_u, \delta_v]$$
$$+ \int_{[0,1]} \Lambda^*(\mathrm{d}r) \int_E x(\mathrm{d}a)\left[G\big((1-r)x + r\delta_a\big) - G(x)\right], \quad G \in \mathscr{B},$$

where

$$Q_x(\mathrm{d}u, \mathrm{d}v) = x(\mathrm{d}u)\,\delta_u(\mathrm{d}v) - x(\mathrm{d}u)\,x(\mathrm{d}v).$$

C^Λ-processes with immigration-emigration:

$$Z_\theta^{c,d,\Lambda} = \big(Z_\theta^{c,d,\Lambda}(t)\big)_{t \geq 0}, \quad Z_\theta^{c,d,\Lambda}(0) = \theta.$$

Asymptotic behavior of the macroscopic observables

- **Volatility constants:** $\underline{d} = (d_k)_{k\in\mathbb{Z}_+}$,

$$d_0 = 0, \qquad d_{k+1} = \frac{c_k(\lambda_k/2 + d_k)}{c_k + (\lambda_k/2 + d_k)}, \quad k \in \mathbb{Z}_+,$$

  where $\lambda_k = \Lambda_k([0;1])$.
- **N.B.** (inhomogeneous) **Möbius transformation**.

Theorem (behaviour of the macroscopic observables)

For every $k \in \mathbb{Z}_+$, uniformly in $\eta \in \Omega_\infty$,

$$\mathscr{L}\left[\left(Y^{(N)}_{\eta,k}(tN^k)\right)_{t\geq 0}\right] \xrightarrow[N\to+\infty]{} \mathscr{L}\left[\left(Z^{c_k,d_k,\Lambda_k}_\theta(t)\right)_{t\geq 0}\right].$$

Ergodic behavior of $X^{(N)}$, $N < \infty$

Set

$$m_k := \frac{\lambda_k/2 + d_k}{c_k}.$$

Theorem (Clustering vs. coexistence criterion)

- **[Clustering]** *(= formation of large mono-type regions), if* $\sum_{k\in\mathbb{Z}_+} m_k = \infty$ ***vs.***
- **[Local coexistence]** *(= convergence to multi-type equilibria), if* $\sum_{k\in\mathbb{Z}_+} m_k < \infty$.

N.B. $\sum_{k\in\mathbb{Z}_+} m_k = \infty$ vs. $< \infty \Leftrightarrow \sum_{k\in\mathbb{N}_0} (1/c_k) \sum_{l=0}^{k} \lambda_l = \infty$ vs. $< \infty$.

- Recurrent migration $\rightsquigarrow$ clustering.
- $\exists$ transient migrations and strong enough reshuffling-resampling $\sum_{l\in\mathbb{N}_0} \lambda_l = \infty \rightsquigarrow$ clustering.

Theorem (Clustering vs. coexistence criterion)

The following dichotomy holds:

(a) **[Local coexistence]** *If* $\sum_{k\in\mathbb{Z}_+} m_k < \infty$*, then for every* $\theta \in \mathscr{P}(E)$ *and every* $X^{(\Omega_N)}(0)$ *whose law is stationary and ergodic w.r.t. translations in* Ω_N *and has a single-site mean* θ*,*

$$\mathscr{L}\left[X^{(\Omega_N)}(t)\right] \underset{t\to+\infty}{\Longrightarrow} \nu_\theta^{(\Omega_N),\underline{c},\underline{\lambda}} \in \mathscr{P}(\mathscr{P}(E)^{\Omega_N})$$

for some unique law $\nu_\theta^{(\Omega_N),\underline{c},\underline{\lambda}}$ *that is stationary and ergodic w.r.t. translations in* Ω_N *and has single-site mean* θ*.*

(b) **[Clustering]** *If* $\sum_{k\in\mathbb{Z}_+} m_k = \infty$*, then, for every* $\theta \in \mathscr{P}(E)$*,*

$$\mathscr{L}\left[X^{(\Omega_N)}(t)\right] \underset{t\to+\infty}{\Longrightarrow} \int_0^1 \theta(\mathrm{d}u)\,\delta_{(\delta_u)^{\Omega_N}} \in \mathscr{P}(\mathscr{P}(E)^{\Omega_N}).$$