Fluctuations of Outliers of Finite Rank Perturbations to Random Matrices

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Joint work with A. Soshnikov, A. Pizzo, S. O’Rourke
Consider a random $N \times N$ Wigner real symmetric matrix

$$X_N = \frac{1}{\sqrt{N}} W_N = \frac{1}{\sqrt{N}} \begin{pmatrix} W_{11} & W_{12} & \cdots \\ W_{12} & W_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$W_{ij}$ are i.i.d. for $1 \leq i < j \leq N$ with

$$E[W_{12}] = 0, \quad E[W_{12}^2] = \sigma^2, \quad E[W_{12}^4] < \infty$$

$W_{ii}$ are i.i.d. for $1 \leq i \leq N$ with

$$E[W_{11}] = 0, \quad E[W_{11}^2] < \infty$$

If $W_{12} \overset{d}{=} \frac{1}{\sqrt{2}} W_{11}$ is Gaussian, then the matrix is said to be from the Gaussian Orthogonal Ensemble (GOE).
The most fundamental result for Wigner matrices is the Wigner semi-circle law.

A real symmetric matrix with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_N$ induces a measure, called the empirical spectral distribution (ESD), on the real line, given by $\frac{1}{N} \sum \delta_{\lambda_i}$.

The ESD of $X_N$ converges a.s. in distribution to $\mu_{sc}$ where

$$
\frac{d\mu_{sc}(x)}{dx} = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - x^2} \mathbf{1}_{[-2\sigma,2\sigma]},
$$

and the largest (smallest) eigenvalue converges to $2\sigma$ ($-2\sigma$).
Student Version of MATLAB
Z. Füredi and J. Komlós ('81) first studied deformed Wigner matrices.

They assumed the distribution on the entries of the random matrix have a common non-zero mean, $c$.

This can be viewed as

$$W_N + C$$

where $(C)_{ij} = c$ is a constant matrix.

The largest eigenvalue is $Nc + \sigma^2/c$ with Gaussian fluctuations.
Deformed Wigner Matrices

- These results were extended to $W_N/\sqrt{N} + C/N$.
- The largest eigenvalue and the edge of the semicircle are both of constant order.
- First done with Gaussian Matrices by S. Peché (’06).
- Then for Wigner matrices by S. Peché and D. Féral (’07).
- A phase transition is observed depending on the value of $c$. 
M. Capitaine, C. Donati-Martin and D. Féral (’09,’12) consider different forms of the perturbation and higher rank perturbations.

Assume the distribution is symmetric and satisfies a Poincaré Inequality:

\[ \nabla [f(x)] \leq \mathbb{E} [ |\nabla f(x)|^2 ] \]

Large eigenvalues converge similarly to the rank one case. They also show the fluctuations are non-universal for several special perturbations.
Concurrent with our research Knowles and Yin also consider finite rank perturbations.

Under more restrictive assumptions, they give finer details on the spectrum.
In this research we consider deformed random matrices given by

\[ M_N = X_N + A_N \]

\[ A_N = U_N^* \Theta U_N \]

has a fixed finite rank and eigenvalues \( \{\theta_j\}_{j=1}^r \).

By the interlacing theorem \( N - r \) eigenvalues converge to the semi-circle.

We are interested in the locations and fluctuations of the remaining \( r \) eigenvalues.
Theorem (Pizzo, R., Soshnikov)

Let $J_{+\sigma}$ (resp. $J_{-\sigma}$) be the number of $j$'s such that $\theta_j > \sigma$ (resp., $\theta_j < -\sigma$) and let

$$\rho_j := \theta_j + \frac{\sigma^2}{\theta_j}$$

then:

(a) For $1 \leq j \leq J_{+\sigma}$, $1 \leq i \leq k_j$, $\lambda_{k_1 + \ldots + k_{j-1} + i} \to \rho_j$

(b) $\lambda_{k_1 + \ldots + k_{J_{+\sigma}} + 1} \to 2\sigma$

(c) $\lambda_{k_1 + \ldots + k_{J_{-\sigma}}} \to -2\sigma$

(d) For $j \geq J - J_{-\sigma} + 1$, $1 \leq i \leq k_j$, $\lambda_{k_1 + \ldots + k_{j-1} + i} \to \rho_j$

the convergence is in probability.
Theorem (Delocalized case - Pizzo, R., Soshnikov)

\[ c_{\theta_j} \sqrt{N} (\lambda_1 + \ldots + \lambda_{j-1} + \rho_j) \], \quad i = 1, \ldots, k_j

converges in distribution to the (ordered) eigenvalues of a 
k_j \times k_j \text{ GOE matrix with the variance of the matrix entries given by}

\[ \frac{\theta_j^2 \sigma^2}{\theta_j^2 - \sigma^2} \]

plus a deterministic matrix with entries \(lp^{th}\) entry given by

\[ \frac{\theta^2 - \sigma^2}{\theta^4} \sum_{i \neq j} u_i^l u_j^p. \]
The Stieltjes Transform, \( g \), of a measure, \( \mu \), is given by:

\[
g(z) = \int \frac{d\mu(x)}{z - x}.
\]

If \( \mu \) is an ESD of a matrix, \( M \), then its Stieltjes Transform can be written as

\[
\frac{1}{N} \text{Tr}(zI - M)^{-1} =: \text{tr}_N(R(z)).
\]

The Stieltjes Transform of \( \mu_{sc} \) satisfies the equation

\[
\sigma^2 g^2_\sigma(z) - zg_\sigma(z) + 1 = 0.
\]
Characterization of outlying eigenvalues

- If $z$ is an eigenvalue of $M_N$

$$\det(zI_N - X_N - A_N) = 0$$

if additionally it is not an eigenvalue of $X_N$ then

$$\det(z - X_N - A_N) = \det(z - X_N)\det(I + R_N(z)U_N^*\Theta U_N)$$

$$= \det(z - X_N)\det(I + \Theta U_N R_N(z)U_N^*)$$

$$= \det(z - X_N)\det(\Theta)\det(\Theta^{-1} + U_N R_N(z)U_N^*)$$

- Using that

$$\det(I + AB) = \det(I + BA)$$
We begin with the resolvent identity:

\[ zR_N(z) = I_N + X_N R_N(z) \]

\[ z\mathbb{E}[R_{ij}(z)] = \delta_{ij} + \sum_l \mathbb{E}[X_{il}R_{lj}(z)] \]

and use decoupling formula

\[ \mathbb{E}(\xi \phi(\xi)) = \sum_{a=0}^{p} \frac{\kappa a + 1}{a!} \mathbb{E}(\phi^{(a)}(\xi)) + \epsilon \]
• This shows the diagonal elements approximately solve the equation for $g_{\sigma}$

$$\mathbb{E}[R_{ii}(z)] = g_{\sigma}(z) + O(N^{-1})$$

• On the off-diagonal this implies

$$\mathbb{E}[R_{ij}(z)] = \frac{\kappa_{3,ij}}{N^{3/2}} g_{\sigma}^4(z) + o(N^{3/2})$$

• Similarly, we can bound the variance of quadratic forms

$$\nabla[u_N^* R_N(z) v_N] = O(N^{-1})$$
Resolvent

Let \( u_N, v_N \) be a sequence of \( N \) dimensional unit vectors.

\[
\sqrt{N} \mathbb{E}[u_N^* R_N(z) v_N] - \frac{1}{N} g_\sigma^4(z) u_N^* M_3 v_N = \sqrt{N} g_\sigma(z) u_N^* v_N + o(1)
\]

where \( M_3 = (1 - \delta_{ij}) \kappa_{3,ij} \).

Furthermore, if \( \|u_N\|_1 \) or \( \|v_N\|_1 \) is \( o(\sqrt{N}) \) then the second term on the left side is \( o(1) \).
The eigenvalues are $z$ such that
\[
\det(\Theta^{-1} - U_N^* R(z) U_N) = 0.
\]

By the variance estimates and Markov’s Inequality
\[
\|U_N^* R_N(z) U_N - g_\sigma(z) I_r\| = O(N^{-1/2})
\]
with probability going to one.

Then the eigenvalues converge to
\[
g_\sigma^{-1}(1/\theta_k) + O(N^{-1/2}) = \theta_k + \sigma^2/\theta_k + O(N^{-1/2}).
\]

The fluctuations are given by
\[
(g'_\sigma(\rho_1) + o(1))(\lambda_i - \rho_i) = -1/\sqrt{Ny_i} + o(N^{-1/2})
\]
Fluctuations - delocalized perturbations
We can deduce the fluctuations of the outliers from the joint distribution of

$$\sqrt{N} \left( \sum_{ij} R_{ij}(\rho) \bar{u}_i u_j^p - \mathbb{E} \left[ \sum_{ij} R_{ij}(\rho) \bar{u}_i u_j^p \right] \right)$$

for $1 \leq l \leq p \leq k_j$.

Previous we had shown finitely many resolvent entries converge to independent random variables given by

$$W_{ij} + G_{ij}(\rho)$$

with $G_{ij}(\rho)$ a Gaussian with variance

$$(1 + \delta_{ij})\sigma^2 g'_{\sigma}(\rho) + \delta_{ij}\kappa_4 g^2_{\sigma}(\rho)$$
Delocalized perturbations

- If \( \| u^i_N \|_\infty \to 0 \) for all eigenvectors then the fluctuations are universal.

\[
(G_N(z))_{lp} := \sqrt{N}(u^*_N R_N(z) u^p_N - \mathbb{E}[u^*_N R_N(z) u^p_N]).
\]

Converges in finite dimensional distributions to \( \Gamma(z) \) with independent, centered, Gaussian entries with covariance given by:

\[
\frac{2}{2 - \delta_{lp}} \left( -g_\sigma(z)g_\sigma(w) + \frac{g_\sigma(z)g_\sigma(w)}{1 - \sigma^2 g_\sigma(z)g_\sigma(w)} \right)
\]

for \( l \leq p \) and \( \Gamma_{lp}(z) = \Gamma_{pl}(\overline{z}) \) for \( l > p \).
Delocalized perturbations

- Decompose into a Martingale Difference Sequence.

\[
\sqrt{N}(u_N^* R_N(z) u_N^p - \mathbb{E}[u_N^* R_N(z) u_N^p]) = \sqrt{N} \sum_{k} (E_k - E_{k-1}) u_N^* R_N(z) u_N^p
\]

- Apply Martingale central limit theorem.
- Done by Bai and Pan ('12), we extend to non-vanishing third moment and joint distribution of several vectors.
Delocalized perturbations

\[ \sum_{k=1}^{N} \frac{\sigma^2 g_\sigma(z_1) g_\sigma(z_2) |u_k^p|^2 \sum_{j>k} |u_j^q|^2}{1 - \sigma^2 \frac{N-k}{N} g_\sigma(z_1) g_\sigma(z_2)} \]

\[ \to \int_{0}^{1} \frac{\sigma^2 (1-t) g_\sigma(z_1) g_\sigma(z_2)}{1 - \sigma^2 (1-t) g_\sigma(z_1) g_\sigma(z_2)} dt \]

**Theorem (Steinitz ’13)**

Let \( \{v_i\}_{i=1}^{N} \) be a finite family of vectors in \( \mathbb{R}^m \) of size \( N \). Assume that \( v_i^l \leq c \) for all \( i, l \) and \( \sum_{i=1}^{N} v_i = 0 \). Then there exist a permutation \( \pi \in S_N \) and some universal constant \( K_m \) that only depends on \( m \), such that

\[ \left\| \frac{\lfloor Nt \rfloor}{\sum_{i=1}^{N} v_{\pi_i}} \right\|_\infty \leq cK_m \]

for all \( 0 \leq t \leq 1 \).
The difference between

\[
(c_{\theta_j} \sqrt{N}(\lambda_{k_1+\ldots+k_{j-1}+i} - \rho_j), \ i = 1, \ldots, k_j)
\]

and the vector formed by the (ordered) eigenvalues of a \(k_j \times k_j\) GUE (GOE) matrix with the variance of the matrix entries given by

\[
\frac{\theta_j^2 \sigma^2}{\theta_j^2 - \sigma^2}
\]

plus a deterministic matrix with entries given by

\[
\frac{\theta^2 - \sigma^2}{\theta^4} u_N^l M_3 u_p
\]

converges in probability to zero.
Thank you