Probability Models of Information Exchange on Networks
Lecture 2

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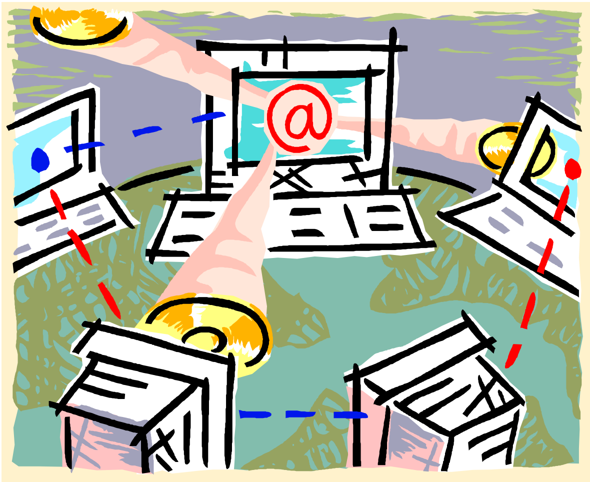

Lecture Plan

• We will see the first simple models of information exchange on networks.

• Assumptions:

• $P[S = +] = P[S = -] = 0.5$.

• N independent signals S_i with $P[S_i = S \mid S] = p > 0.5$.

. Agents update their opinions in various ways …

Lecture Plan

• Basic questions:

• <u>Agreement</u>: Do all agents converge to the same belief/action?

• <u>Learning</u>: Do the agents converge to the correct state of the world (with high probability)?

• Compare to the Jury Theorem where we have agreement (the majority value) and learning (by the Jury Theorem)

The DeGroot Model

- DeGroot model: Repeated averaging of probabilities of self and neighbors.

- Formal definition: n individuals denoted 1,…,n.
- At time 0 some initial beliefs: $s(i) = s(i,0)$ for $i \in [n]$.
- Averaging weights: $w(i,j) \geq 0$ satisfy $\sum_j w(i,j) = 1$
- Update rule: $s(i,t+1) = \sum_j w(i,j)\, s(j,t)$
- In matrix notation: $s(t+1) = W\, s(t)$

- Linear model, linear updates …
- Introduced by De Groot (Penn Statistician, 1931-1989) in 1974.

The DeGroot Model and Markov Chains

- In matrix notation: $s(t+1) = W\, s(t)$

- Let $X(i,t)$ be the Markov chain defined by W started at i and run for t steps.

- Claim: $s(i,t) = E[\, s(X(i,t))\,]$

- Recall that W is called ergodic if there is a power t such that all entries of W^t are positive.

Markov (wikipedia)

Kolmogorv (wikipedia)

Agreement in the DeGroot Model

- Corollary: If W is ergodic with stationary distribution π then $s(i,t) \rightarrow_{t \rightarrow \infty} \sum_j \pi(j)\, s(j)$. (all agents converge to same s)

- Interpretation: $\pi(j)$ is the centrality of node j (it's PageRank)

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- Network interpretation: Suppose w(i,i) > 0 for all i. Let G = ([n],E) be the directed graph where $(i,j) \in E$ iff w(i,j) > 0. Then if G is strongly connected then W has a unique stationary distribution so agreement is reached.

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- Network interpretation: Suppose w(i,i) > 0 for all i. Let G = ([n],E) be the directed graph where $(i,j) \in E$ iff w(i,j) > 0. Then if G is strongly connected then P has a unique stationary distribution so agreement is reached.

- Exercise: infinite graphs ??

- Next question: Do the agents also learn?

Learning in the DeGroot Model

- Def: Consider the DeGroot model W corresponding to a chain with a unique stationary distribution π and signal of bias p. Assume the initial signals S(1),..,S(n) are conditionally i.i.d. with P[S(i) = S | S] = p > ½. Then:

- The probability of learning in the model is:
- P_p[Learning in W] := $P_p[S = \mathrm{sgn}(\sum \pi_j S(j))]$

- Note – this definition is somewhat arbitrary. Ideally we would like the definition to also measure how convinced the agents are.

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- Def 2: Given a sequence of models W^n and a bias p, we say that learning holds if P_p[Learning in W^n] $\to 1$ as $n \to \infty$

- Note: Learning depends only on stationary distribution(s).

- Claim: Learning holds for all p>1/2 for the models W^n if and only if $\limsup_n \max_j \pi^n(j) = 0$.

Learning in the DeGroot Model

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- Pf: If $\limsup_n \max_j \pi^n(j) = 0$ apply the weak law of large numbers.

- Other direction: exercise!

- Q: Examples of sequences of graphs where:
- 1. There is learning
- 2. There is no learning.
- 3. There is learning for some values of p but not for others.

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- Other direction: exercise!

- Example 1: Consider $w = 1$ for all edges (including loops). Then learning for the DeGroot model holds for any sequence of connected bounded degree graphs G_n with $|V_n| \to \infty$.
- This is true since $\max \pi^n(i) \leq \max \deg / (|V_n|-1) \to 0$

- Example 2: No learning for (G_n) = (star on n vertices)

Rate of Convergence in the DeGroot Model

- Next we briefly discuss the rate of convergence in the Degroot model.

- Assume that the chain corresponding to W has a unique stationary distribution.

- Def: The total variation distance between two distributions P,Q is:

- $|P-Q|_1 = 0.5 \sum_x |P(x)-Q(x)|$

Rate of Convergence

- In the theory of M.C. there are many ways to measure the rate of convergence of a chain.
- We can directly use any of them to obtain bounds on the rate of convergence. For example:

- <u>Claim:</u> For all $s \in [1/2-\delta, 1/2+\delta]^n$ it holds that
- $| s(i,t) - E_\pi s| = | E s (X(i,t)) - E_\pi s| \leq 2 \delta |P(i,t) - \pi|_1$

- $P(i,t)$ is the distribution of $X(i,t)$.
- Long after mixing time, agents have almost converged.

- Some of the techniques in the theory of Markov chain include: Spectral techniques and conductance, coupling, log sobolev inequalities etc.

Rate of Convergence

- <u>Condoctance bounds:</u>
- $\pi(\partial A) := \sum \{w(x,y)\, \pi(x) + w(y,x)\, \pi(y) : (x,y) \in (A,A^c)\}$
- $I = \min \{Q(\partial A)/ \pi(A) : \pi(A) \leq ½\}$.
- Then gap $\geq I^2/8$ (Cheeger's inequality) and therefore
- By time $t = 8\, s\, (1+\max_i \ln(1/\pi(i)))/I^2$ we have:
- $|\, P(i,t) - \pi|_1 \leq \exp(-s)$ for all i.

• This means if there are no "isolated communities" then convergence is quick.

- In the minimum suffices to look at connected sets.

Cheating in the DeGroot Model

- A basic question:
- What happens if somebody cheats? Can they convince the rest of the group to reach whatever value they want?

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- <u>Claim:</u> If the chain is ergodic and there is one cheater by repeatedly stating a fixed value all opinions will converge to that value.

Cheating in the DeGroot Model

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- Claim: If the chain is ergodic and there is one cheater by repeatedly stating a fixed value all opinions will converge to that value.

- Pf: Follows from MC interpretation:
- Let the cheater play v. Let A(i,t) be the event that X(i,0),..,X(i,t) hit the cheater by time t then:
- $p(t,i) = E[1(A(i,t)) \times v + (1-1(A(i,t)) \times X(i,t)]$

- Exercise / Research Question: Multiple cheaters?

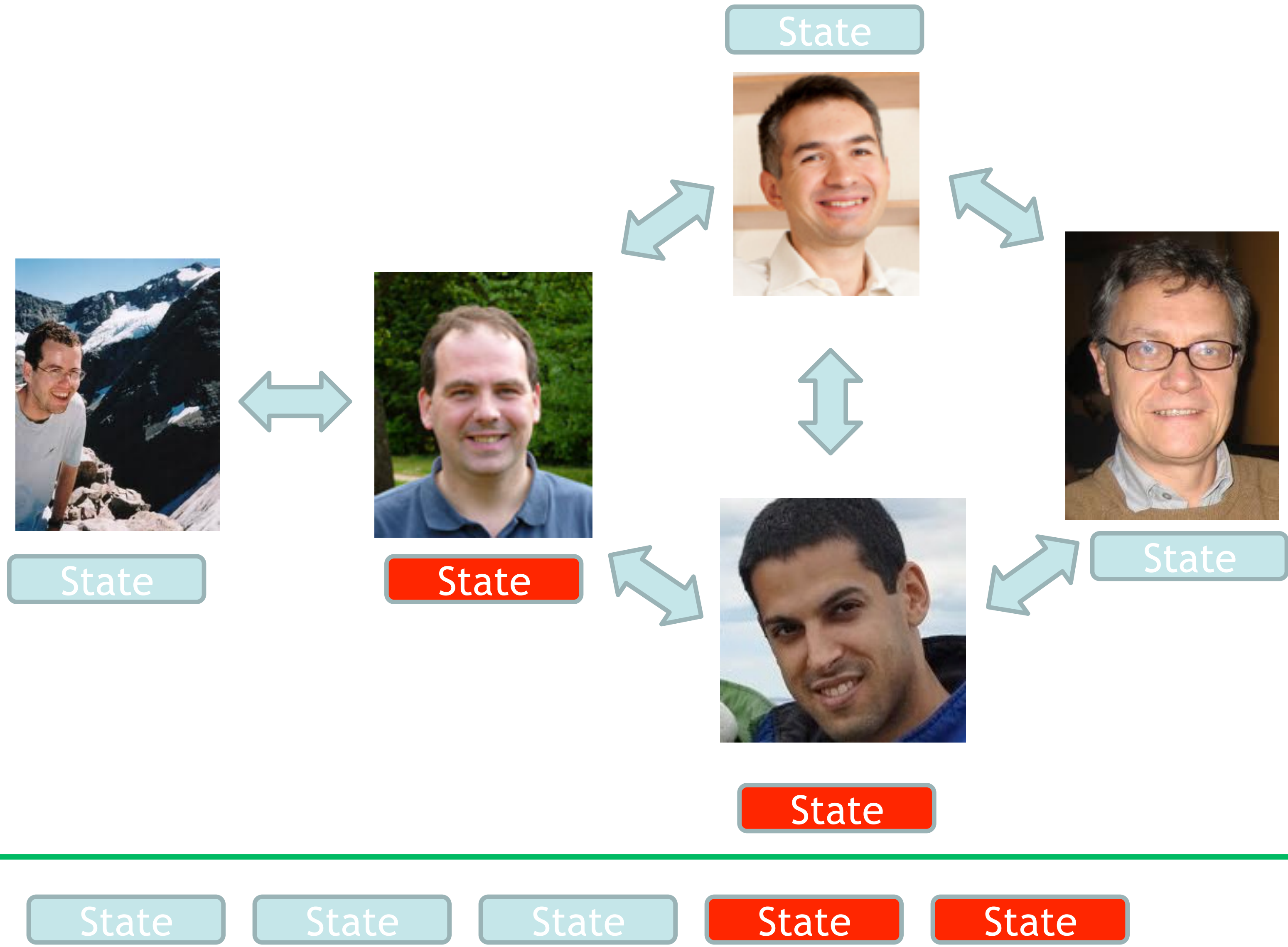
The Voter Model

- The Voter model: Repeated sampling of opinions of neighbors.

- Formal definition: n individuals denoted 1,…,n.
- At time 0 some initial -/+ beliefs: s(i) = for i $\in$ [n].
- Averaging weights: w(i,j) $\geq$ 0 satisfy $\sum_j$ w(i,j) =1
- Update rule: P[s(i,t+1) = s(j,t)] = w(i,j)

- Introduced in the 1970s by P. Cliford and A. Sudury and studied by Holley and Liggett

Voter Model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Agreement in the Voter Model

- <u>Claim:</u> Let G be the directed graph defined by letting i -> j $\in$ E iff w(i,j) > 0. Assume that G is strongly connected and w(i,i)>0 for all i. Then the voter model on G converges a.s. to all + or all -.

Agreement in the Voter Model

- Claim: Let G be the directed graph defined by letting $i \to j \in E$ iff $w(i,j) > 0$. Assume that G is strongly connected and $w(i,i)>0$ for all i. Then the voter model on G converges a.s. to all + or all -.

- Pf Sketch:
- The voter model is a Markov chain.
- All + and all – are fixed point of the chain.
- Under the conditions from any other state it is possible to reach the all + or all – states.

(Non) Learning in the Voter Model

• Setup: Assume self loops and that G is strongly connected.

• Claim: Suppose $p < P[S(i) = S] < q$. Let r be the probability that the voter model converges to the correct state then $p < r < q$ (no learning)

(Non) Learning in the Voter Model

- Setup: Assume self loops and that G is strongly connected.

- Claim: Suppose $p < P[S(i) = S] < q$. Let r be the probability that the voter model converges to the correct action then $p < r < q$ (no learning)

- Claim: For all S, $P[S(i,t) = 1] = DG(i,t)$ where $DG(i,t)$ is the value of node i at iteration t in the DeGroot model with the same starting conditions S.

Convergence Rate of the Voter Model

- There are numerous ways of analyzing the convergence rate of the voter model.

- Coalescence of Random walks:
- S(i,t) = S(X(i,t))
- If we can couple X(i,t) so that by time T we have X(i,T) = X(j,T) for all i and j, then agreement must have occurred by time T!

- See Liggett, Durrett, …

Convergence Rate of the Voter Model

- Martingale arguments:

- Example: suppose G is undirected and connected.

- Claim: $M(t) = \sum_{\iota} d(i)\, S(i,t)$ is a martingale.

Convergence Rate of the Voter Model

- Martingale arguments:

- Example: suppose G is undirected and connected.

- Claim: $M(t) = \sum_{\iota} d(i)\, S(i,t)$ is a martingale.

- Pf Sketch: For each edge e, the contribution of that edge is a martingale.

- Then: use Wald's 2^{nd} equation to bound the convergence time.

- Exercise: Do it: show that converges in time $O(|V|^2)$
- Get better bounds when the graph expands.

Bribery in the Voter Model

- <u>Martingale arguments:</u>

- Example: suppose G is undirected and connected.

- $M(t) = \sum_{\iota} d(i)\, S(i,t)$ is a martingale.

- <u>Exercise:</u> Suppose you can bribe k nodes who are – and change their value to +. Which nodes will you bribe?

(Non) Learning in the Voter Model

- Recall that we saw that if the original signals satisfy
- Suppose $p < P[S(i) = S] < q$ then the probability r of convergence to the correct action also satisfies $p < r < q$ (no learning)

- Question: is there a dynamic "like" the voter model where learning holds?

- Stronger question: is there a dynamic "like" the voter model which converges to the majority of the original signals?

- <u>Claim:</u> No local dynamics without additional memory.

Strong Weak Voter Model

- It’s possible with one extra bit of memory in a non-synchronous model:

Strong Weak Voter Model

- It's possible with one extra bit of memory in a non-synchronous model:
- Edges are chosen according to Poisson process.
- All Voters have **opinion** (red/blue) and **strength** of opinion (STRONG/weak). Originally all strong.
- When they meet,
  - Update color:
    - STRONG influence weak
    - Otherwise voter model
  - Update Strengths:
    - Two STRONGS of different colors cancel to weak
    - Otherwise stay the same
    - Two sides swap location with probability 0.5.

Recent Research Topic

- What can local dynamics compute on networks?
- Motivation from:
- Experimental Sociology Latane L' Herrou (96)
- Sensor networks (Aspens and collaborators 04-…)
- Biological computing (Winfree …, 2005-)
- See also recent preprint by M-Parkash-Valiant

A 3rd type of dynamics

- We've seen two networks dynamics:
- 1st: Average your neighbors (DeGroot)
- 2nd :Sample one of your neighbors (voter)

A 3rd type of dynamics

- We've seen two networks dynamics:
- 1st: Average your neighbors (DeGroot)
- 2nd :Sample on of your neighbors (voter)
- 3rd: Take a majority of your neighbors.

Majority Dynamics

- 3^{rd}: Take a majority of your neighbors.
- Formal definition: n individuals denoted 1,…,n.
- At time 0 some initial -/+ beliefs: s(i) = for $i \in [n]$.
- Weights: $w(i,j) \geq 0$ satisfy $\sum_j w(i,j) = 1$
- Update rule: $s(i,t+1) = \mathrm{sgn}(\sum w(i,j)\, s(j,t))$
- Much harder to analyze
- Not all nodes converge to same value
- Very few tools available – we'll see a few.
- Go to Omer's Talk!

The effect of a voter

Def: $E_p[f_i]$ is called the influence of voter i, where $f_i(x) = f(1,x_{-i})-f(-1,x_{-i})$

Theorem (Talagrand 94):

- Let f be a monotone function.
- If $\delta = \max_r \max_i E_r[f_i]$ and $p < q$ then
- $E_p[f \mid s = +] (1-E_q[f \mid s=+]) \leq \exp(c \ln \delta (q-p))$
- for some fixed constant c>0.

- In particular: if f is fair and monotone, taking p=0.5:

- $E_q[f \text{ is correct}] \geq 1- \exp(c \ln \delta (q-0.5))$

The effect of a voter

. <u>Theorem (Talagrand 94):</u>

- Let f be a monotone function.
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- In particular: if f is fair and monotone, taking $p=0.5$:

- $E_q[f \text{ is correct}] \geq 1- \exp(c \ln \delta (q-0.5))$

- This means that if each voter has a small influence then the function aggregates well!

An important case

Def: A function $f: \{-,+\}^n \rightarrow \{-,+\}$ is transitive if there exists a

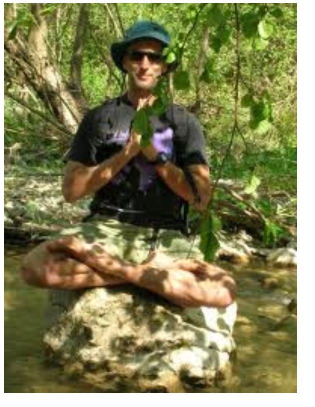
- group G acting transitively on [n] s.t.
- for every $x \in \{-,+\}^n$ and any $\sigma \in G$ it holds that $f(x_\sigma) = f(x)$,

where

- $x_\sigma(i) = x(\sigma(i))$

Thm (Friedgut-Kalai-96) :

- If f is transitive and monotone and
- $E_p[f \mid s= +] > \varepsilon$ then
- $E_q[f \mid s = +] > 1-\varepsilon$ for $q=p+c \log(1/2\varepsilon)/ \log n$



Note: If f is fair transitive and monotone we obtain

$E_q[f \text{ is correct}] > 1-\varepsilon$ for $q=0.5+c \log(1/2\varepsilon)/ \log n$

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Back to Majority Dynamics

- Note: If f is fair transitive and monotone we obtain E_q[f is correct] > $1-\varepsilon$ for $q=0.5+c \log(1/2\varepsilon)/ \log n$

- Can be applied to: f = Majority of signals at the end of majority dynamics on transitive graphs to show that

- E_q[majority is correct] > $1-\varepsilon$ for $q=0.5+c \log(1/2\varepsilon)/ \log n$

- So information is retained.

- See M-Neeman-Tamuz for generalization to quasi-transitive and pluralities instead majorities.