Probability Models of Information Exchange on Networks
Lecture 5

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Informational Framework

- Each agent receives a **private signal** X_i which depends on S.
- **Conditioned** on S, private signals are **i.i.d.**
- **Example.** Gaussian private signals.

$$\mathbb{P}\left[X_i = \cdot | S = 0\right] \sim N(0, \sigma^2) \qquad \mathbb{P}\left[X_i = \cdot | S = 1\right] \sim N(1, \sigma^2)$$

- **Example.** Bernoulli private signals.

$$\mathbb{P}\left[X_i = S\right] = \tfrac{1}{2} + \delta \qquad \mathbb{P}\left[X_i = 1 - S\right] = \tfrac{1}{2} - \delta$$

- Private signals suffice to estimate S correctly w.h.p. as $|V| \to \infty$.

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- $X_1(1) = X_2(1) = \mathbb{P}[S = + \mid 0.4, 0.7]$.
- $$\frac{\mathbb{P}[0.4, 0.7 \mid S = +]}{\mathbb{P}[0.4, 0.7 \mid S = -]} = \frac{0.8 * 1.4}{1.2 * 0.6} = \frac{14}{9}.$$

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- $$\frac{\mathbb{P}[0.4, \geq 0.5 \mid S = +]}{\mathbb{P}[0.4, \geq 0.5 \mid S = -]} = \frac{0.8 * 3/4}{1.2 * 1/4} = 2 \implies A_1(1) = +.$$
  $$\frac{\mathbb{P}[0.7, \leq 0.5 \mid S = +]}{\mathbb{P}[0.7, \leq 0.5 \mid S = -]} = \frac{1.4 * 1/4}{0.6 * 3/4} = \frac{7}{9} \implies A_2(1) = -.$$

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- etc.

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- Stronger version (M-Sly-Tamuz-12): Under the non-atomic beliefs three possible limiting actions are possible
  1. For all i, $A_i(t) \to 1$ and $X_i(\infty) > \frac{1}{2}$.
  2. For all i, $A_i(t) \to -1$ and $X_i(\infty) < \frac{1}{2}$.
  3. For all i, $A_i(t)$ does not converge and $X_i(\infty) = \frac{1}{2}$.

Agreement in actions models

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- $\implies$ if $i \to j$ then $p(i) \geq p(j)$.
- In strongly connected graphs $p(i) = p(j)$ for all i, j.
- Moreover, $\mathbb{P}\left[A_i = S \mid \mathcal{F}_i(\infty)\right] \geq \mathbb{P}\left[A_j = S \mid \mathcal{F}_i(\infty)\right]$

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- $\implies$ if $i \to j$ then $p(i) \geq p(j)$.
- In strongly connected graphs $p(i) = p(j)$ for all i, j.
- Moreover, $\mathbb{P}[A_i = S \mid \mathcal{F}_i(\infty)] \geq \mathbb{P}[A_j = S \mid \mathcal{F}_i(\infty)]$
- $\implies$ $\mathbb{P}[A_i = S \mid \mathcal{F}_i(\infty)] = \mathbb{P}[A_j = S \mid \mathcal{F}_i(\infty)]$ a.s.
- So $X(i, \infty) > 1/2$ but $A_j(t) = 0$ i.o. impossible.

Why do economists care about information?

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"An Economy Professor and a farmer were waiting for a bus in New Hampshire. To pass the time the farmer suggested that they played a game ..."

.

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What kind of game would you like to play?

.

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How about this. I will ask a question and if you can't answer my question you'll give me one dollar. Then you ask me a question and if I can't answer you question I will give you one dollar.   .

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This sounds attractive, but I must warn you. I am not just an ordinary person. I am a professor of economics

.

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O - in that case we should change the rules. Tell you what. If you can't answer my question you still give me a dollar. But if I can't answer yours, I only have to give you fifty cents. .

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Yes - that sounds like a fair agreement.

.

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Okay - here's my question. What goes up the hill on seven legs and down on three legs? .

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Gosh - I don't know. What goes up the hill on seven legs and down on three legs?

.

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Well I don't know either. But if you give me your dollar, I'll give you my fifty cents.

The learning challenge

- It is known that all agents converge to same belief / action.
- But what is this belief?
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  - "different motivation is simply technical expediency" (Ellison and Fundenberg)
  - "to keep the model mathematically tractable... this possibility [fully Bayesian agents] is precluded in our model... simplifying the belief revision process considerably" (Bala and Goyal)

The Action Learning Theorem

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- In the revealed actions model with non-atomic beliefs and $F_+ \neq F_-$ there exists a sequence $q(n, F_+, F_-) \to 1$ such for all connected graphs G of size n

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- False: without non-atomic assumption, on directed graphs, w.o independence.

The strategic setup

- **Common prior.** $\mathbb{P}[S=1] = \mathbb{P}[S=0] = \frac{1}{2}$.
- **Actions.** $A_i(t) \in \{0,1\}$.
- **Utilities.** $U_i(t) = \mathbf{1}_{A_i(t)=S}$.
- $\mathcal{F}_i(t) = \{$ private signal X_i, neighbors' previous actions. $\}$
- **Myopic agents.** Maximize $\mathbb{E}[U_i(t)]$.

$$A_i(t) = \underset{a\in\{0,1\}}{\operatorname{argmax}} \mathbb{P}[S = a|\mathcal{F}_i(t)].$$

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$$A_i(t) = \operatorname*{argmax}_{a \in \{0,1\}} \mathbb{P}[S = a | \mathcal{F}_i(t)] .$$

- **Strategic agents.** Discount by $0 < \lambda < 1$.

$$u_i = \mathbb{E}\left[\sum_t \lambda^t U_i(t)\right] = \sum_t \lambda^t \mathbb{P}[A_i(t) = S] .$$

- $A_i(t)$ is a function of $\mathcal{F}_i(t)$.
- Our results hold in **every equilibrium**!

Learning for revealed actions

Action Learning Theorem (M., Sly and Tamuz (2012))
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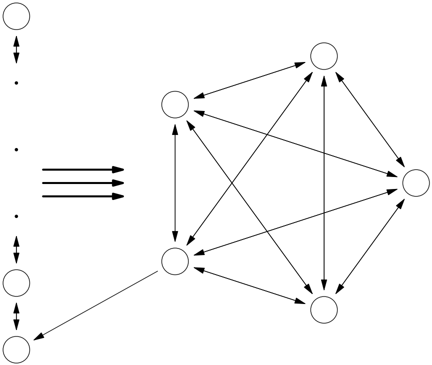
False in general without non-atomic assumption and on directed graphs, independence is also needed.

The Royal Family: When Agents Fail to Learn

- **Royal family.**
- **Bounded private signals.** E.g., Bernoulli.
- The combined signal of the royal family is very strong.
- Everyone follows it after one step if they happen to agree.
- **But it still may be wrong.**



Abstract Proof Approach

- Dynamics are very complicated. Abstract approach needed: Assume by contradiction there is a sequence $G_n = (V_n, E_n)$ of graphs with $|V_n| \to \infty$ and $\limsup \mathbb{P}[\text{ Learning in } G_n] < 1$.

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- Imitation principle: If $i -> j$ then $p_i(t+1) \geq p_j(t)$.
- $\Longrightarrow$ For G (strongly) connected there exists $p(G) = sup_t p_i(t)$ which does not depend on i.

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- $\implies$ can assume G_n are bounded degree.

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- $d(G_1, \rho_1), (G_2, \rho_2) = 2^{-r}$, where r is the maximum radius such that $B_{G_1}(\rho_1, r)$ is isomorphic to $B_{G_2}(\rho_2, r)$.

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- This topology is a bit simpler than the Benjamini-Schramm topology.

Local limits of graphs and Agreement Probabilities

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- Let

$$p^* = \inf_{G \text{ infinite}} p(G).$$

Limiting Actions and Transitive Graphs

- Recall Claim: Under the non-atomic beliefs three possible limiting actions are possible
  1. For all i, $A_i(t) \to 1$ and $X_i(\infty) > \frac{1}{2}$.
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- More work needed for general graphs.

Informative times on general graphs

- Using induction we find a vertex i and times $t_1 < \ldots < t_k$ such that $\mathbb{P}\left[A_i(t_\ell) = S\right] \approx p^*$ and the $A_i(t_\ell)$ are almost independent.

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- Note that the case $k = 1$ follows from the definition of p^*.
- The induction claim implies the theorem by taking the majority of the Y_ℓ. This identifies S with probability better than p^* unless $p^* = 1$.

Proof Sketch of Main Induction Step

- Note for any i and any $\epsilon' > 0$, there exists t' and an $\mathcal{F}_{t'}$-measurable $\tilde{A}^*$ such that $\mathbb{P}\left[A^* = \tilde{A}^*\right] \geq 1 - \epsilon'$.

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- But eventually agent j will learn A^* too giving j another informative time. This completes the induction.

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- Like many finite to infinite proofs, no rate.

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Thanks for listening

Questions?